

On the strength of PFA I*[†]

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Abstract

Building on the work of Schimmerling ([11]) and Steel ([17]), we show that the failure of square principle at a singular strong limit cardinal implies that there is a non-tame mouse. This is the first step towards getting a model of $AD_{\mathbb{R}} + “\Theta$ is regular” from PFA via the *core model induction*.

One of the wholly grails of inner model program has been determining the exact consistency strength of forcing axioms such as *Proper Forcing Axiom* (PFA) and *Martin’s Maximum* (MM). As the consistency of PFA, MM and other similar axioms have been established relative to one supercompact cardinal, it is natural to conjecture that the exact consistency strength of such forcing axioms is one supercompact cardinal.

Conjecture 0.1 (The PFA Conjecture) *PFA is equiconsistent with one supercompact cardinal.*

Recently, in [21], Viale and Weiss showed that any of the known methods for forcing PFA requires a strongly compact cardinal and if the forcing used is proper then it requires a supercompact cardinal. This result suggest that The PFA Conjecture must indeed be true.

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Many constructions in inner model theory have been motivated by The PFA Conjecture or by its sister conjectures. The *core model induction*, a method for establishing reversals pioneered by Woodin and further developed by Schimmerling, Schindler, Steel and others, has been a very successful method for attacking The PFA Conjecture. The known partial results, however, do not use the full force of PFA, but only a consequence of it, namely that \square principle fails at some singular strong limit cardinal. Letting Lim be the set of limit ordinals, recall that \square_κ is the statement: there is a sequence $\langle C_\alpha : \alpha \in Lim \cap \kappa^+ \rangle$ such that

1. $C_\alpha \subseteq \alpha$ is a club subset of α ,
2. if $\alpha \in C_\beta \cap Lim$ then $C_\alpha = C_\beta \cap \alpha$,
3. $o.t.(C_\alpha) \leq \kappa$.

Theorem 0.2 (Todorćevic, [2]) *PFA implies $\neg \square_\kappa$ for all $\kappa \geq \omega_1$.*

Building on an earlier work of Schimmerling and Woodin, Steel, in [17], established the following theorem.

Theorem 0.3 (Steel, [17]) *Assume $\neg \square_\kappa$ for some singular strong limit cardinal κ . Then AD holds in $L(\mathbb{R})$. In particular, there is an inner model with infinitely many Woodin cardinals.*

The failure of the square principle is an anti inner model hypothesis as it usually implies V isn't an inner model. For instance, Jensen showed that $L \models \forall \kappa \geq \omega_1 \square_\kappa$. More generally, Schimmerling and Zeman showed that if \mathcal{M} is a mouse then for $\kappa \geq \omega_1^{\mathcal{M}}$, $\mathcal{M} \models \square_\kappa$ iff $\mathcal{M} \models$ “ κ isn't subcompact” (see [12]). Subcompactness is a large cardinal notion reminiscent of supercompactness. It is much stronger than Woodin cardinals but it is weaker than supercompact cardinals. Unlike supercompactness, subcompact cardinals can exist in *short extender mice*. The theory of such mice has been fully developed (see [5] or [20]). Theorem 2.1 is the exact consequence of the aforementioned Schimmerling-Zeman result that we will need in this paper.

Arranging the failure of \square_κ for regular cardinal κ is relatively easy. In fact, Velickovic showed that if κ is a regular cardinal and $\lambda > \kappa$ is a Mahlo cardinal then after Levy collapsing λ to κ^+ , \square_κ fails (see Theorem 1.3 of [11]). Arranging the failure of \square_κ for a singular cardinal κ is much harder and the known constructions use a subcompact cardinal. Reversing this result has been another wholly grail of the inner model program.

Conjecture 0.4 *Suppose κ is a singular cardinal such that $\neg\Box_\kappa$ holds. Then there is an inner model with a subcompact cardinal.*

The author recently generalized Steel’s proof of Theorem 0.3 and obtained a model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. Recall that AD is the Axiom of Determinacy which states that all two player games of perfect information on integers are determined, i.e., one of the players has a winning strategy. $AD_{\mathbb{R}}$ is the axioms which states that all two player games of perfect information on reals are determined. We let

$$\Theta = \sup\{\alpha : \text{there is a surjection of } \mathbb{R} \text{ onto } \alpha\}.$$

The axiom $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ is a natural closure point of the *Solovay hierarchy* which is a determinacy hierarchy whose base theory is AD^+ . In the subsequent sections, we will give a more precise definitions of AD^+ and the Solovay hierarchy.

Theorem 0.5 *Suppose $\neg\Box_\kappa$ holds for some singular strong limit cardinal κ . Then there is M such that $\text{Ord}, \mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$.*

Corollary 0.6 *Assume PFA. Then there is M such that $\text{Ord}, \mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$.*

In recent years, the Solovay hierarchy has been used as intermediary for establishing reversals. Using the core model induction, working under *PFA* or any other theory in question, we construct a model satisfying some theory from the Solovay hierarchy. The next step, then, is to reduce the resulting theory from the Solovay hierarchy to one in large cardinal hierarchy. Theorem 0.5 is an example of the first step of this process. Recently the author and Yizheng Zhu showed that $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ is equiconsistent with Θ -regular hypothesis, which is weaker than a Woodin limit of Woodin cardinals but much stronger than a proper class of Woodin cardinals and strong cardinals. For the details see [6] or [10]. We then get the following corollary.

Corollary 0.7 *Suppose PFA holds or just that $\neg\Box_\kappa$ holds for some singular strong limit cardinal κ . Then there is an inner model satisfying $ZFC + \Theta$ -regular hypothesis. In particular, there is an inner model with a proper class of Woodin cardinals and strong cardinals.*

The proof of Theorem 0.5 uses the core model induction at the level of $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ and basic machinery at this level hasn’t yet appeared in print. Because of this, it is unrealistic to hope to present an intelligible proof of Theorem 0.5. Instead, we will concentrate on the first step of the induction and will present the full proof of Theorem 0.5 in future publications. The main theorem of this paper is the following.

Theorem 0.8 (Main Theorem) *Suppose $\neg \square_{\kappa}$ holds for some singular strong limit cardinal κ . Then there is a non-tame mouse.*

1 Preliminaries

1.1 Premice, mice and their iteration strategies

We establish some notation and list some basic facts about mice and their iteration strategies. The reader can find more detail in [20]. Suppose \mathcal{M} is a premouse. We then let $o(\mathcal{M}) = \text{Ord} \cap \mathcal{M}$. If \mathcal{M} is a premouse and $\xi \leq o(\mathcal{M})$ then we let $\mathcal{M}||\xi$ be \mathcal{M} cutoff at ξ , i.e., we keep the predicate indexed at ξ . We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of \mathcal{M} if there is no extender E on \mathcal{M} such that $\xi \in (\text{crit}(E), \text{lh}(E)]$. We say ξ is a *strong cutpoint* if there is no E on \mathcal{M} such that $\xi \in [\text{crit}(E), \text{lh}(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in \mathcal{M} if η isn’t a strong cutpoint of \mathcal{M} . Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}_{\eta}^{\mathcal{M}} = \cup \{ \mathcal{N} \triangleleft \mathcal{M} : \rho(\mathcal{N}) = \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N} \}.$$

If \mathcal{M} is a k -sound premouse, then a (k, θ) -iteration strategy for \mathcal{M} is a winning strategy for player II in the iteration game $G_k(\mathcal{M}, \theta)$, and a k -normal iteration tree on \mathcal{M} is a play of this game in which II has not yet lost. (That is, all models are wellfounded.) k -normal trees are called “ k -maximal” in [20], but we shall use “maximal” for a completely different property of trees here.) We shall drop the reference to the fine-structural parameter k whenever it seems safe to do so, and speak simply of normal trees.

We say \mathcal{M} is θ -iterable if II has a winning strategy in $G_k(\mathcal{M}, \theta)$. We say \mathcal{M} is *countably* θ -iterable if any countable substructure of \mathcal{M} is θ -iterable. It follows from the copying construction that a θ -iterable mouse is countably θ -iterable. We say \mathcal{M} is *countably* iterable if all of its countable substructures are $\omega_1 + 1$ -iterable.

If \mathcal{T} is a normal iteration tree, then \mathcal{T} has the form

$$\mathcal{T} = \langle T, \text{deg}, D, \langle E_\alpha, \mathcal{M}_{\alpha+1} \mid \alpha + 1 < \eta \rangle \rangle.$$

Recall that D is the set of *dropping* points. Recall also that if η is limit then

$$\begin{aligned} \vec{E}(\mathcal{T}) &= \cup_{\alpha < \eta} (\dot{E}^{\mathcal{M}_\alpha \upharpoonright lh(E_\alpha)}), \\ \mathcal{M}(\mathcal{T}) &= \cup_{\alpha < \eta} \mathcal{M}_\alpha \upharpoonright lh(E_\alpha), \\ \delta(\mathcal{T}) &= \sup_{\alpha < \eta} lh(E_\alpha). \end{aligned}$$

If b is a branch of \mathcal{T} such that $D \cap b$ is finite, then $\mathcal{M}_b^\mathcal{T}$ is the direct limit of the models along b . If $\alpha \leq_T \beta$ and $(\alpha, \beta]_\mathcal{T} \cap D = \emptyset$ then

$$\pi_{\alpha, \beta}^\mathcal{T} : \mathcal{M}_\alpha^\mathcal{T} \rightarrow \mathcal{M}_\beta^\mathcal{T}$$

is the iteration map, and if $\alpha \in b$ and $(b - \alpha) \cap D = \emptyset$, then

$$\pi_{\alpha, b}^\mathcal{T} : \mathcal{M}_\alpha^* \rightarrow \mathcal{M}_b^\mathcal{T}$$

is the iteration map. If \mathcal{T} has a last model $\mathcal{M}_\alpha^\mathcal{T}$, and the branch $[0, \alpha]_\mathcal{T}$ does not drop, then we often write $\pi^\mathcal{T}$ for $\pi_{0, \alpha}^\mathcal{T}$.

Recall that the strategy for a sound mouse projecting to ω is determined by \mathcal{Q} -structures. For \mathcal{T} normal, let $\Phi(\mathcal{T})$ be the phalanx of \mathcal{T} .

Definition 1.1 *Let \mathcal{T} be a k -normal tree of limit length on a k -sound premouse, and let b be a cofinal branch of \mathcal{T} . Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment \mathcal{Q} of $\mathcal{M}_b^\mathcal{T}$, if one exists, such that \mathcal{Q} projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Theorem 1.2 *Let \mathcal{M} be a k -sound premouse such that $\rho_k(\mathcal{M}) = \omega$. Then \mathcal{M} has at most one $(k, \omega_1 + 1)$ iteration strategy. Moreover, any such strategy Σ is determined by: $\Sigma(\mathcal{T})$ is the unique cofinal b such that the phalanx $\Phi(\mathcal{T}) \frown \langle \delta(\mathcal{T}), \text{deg}^\mathcal{T}(b), \mathcal{Q}(b, \mathcal{T}) \rangle$ is $\omega_1 + 1$ -iterable.*

In some cases, however, it is enough to assume that $\mathcal{Q}(b, \mathcal{T})$ is countably iterable. This happens, for instance, when \mathcal{M} has no local Woodin cardinals with extenders overlapping it. While the mice we will consider do have local overlapped Woodin cardinals, the mice themselves will not have such Woodin cardinals. This simplifies our situation somewhat and below we describe exactly how this will be used. We say an iteration tree \mathcal{T} is above η if all the extenders used in \mathcal{T} have critical points $> \eta$.

Definition 1.3 *Suppose \mathcal{M} is a premouse and \mathcal{T} is a normal tree on \mathcal{M} . We say \mathcal{T} has a fatal drop if for some α such that $\alpha + 1 < lh(\mathcal{T})$, there is some $\eta < o(\mathcal{M}_\alpha^\mathcal{T})$ such that the rest of \mathcal{T} is an iteration tree on $\mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ above η . Suppose \mathcal{T} has a fatal drop. Let (α, η) be lexicographically least witnessing the fact that \mathcal{T} has a fatal drop. Then we say \mathcal{T} has a fatal drop at (α, η) .*

Definition 1.4 (Definition 2.1 of [15]) *Let \mathcal{T} be a normal iteration tree; then $\mathcal{Q}(\mathcal{T})$ is the unique premouse extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1 + 1$ -iterable above $\delta(\mathcal{T})$, and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$, if there is any such premouse.*

Countable iterability is enough to guarantee there is at most one premouse with the properties of $\mathcal{Q}(\mathcal{T})$. If it exists, $\mathcal{Q}(\mathcal{T})$ might identify the good branch of \mathcal{T} , the one any sufficiently powerful iteration strategy must choose. This is the content of the next lemma which can be proved by analyzing the proof of Theorem 6.12 of [20]

Lemma 1.5 *Suppose \mathcal{M} is a k -sound premouse such that no measurable cardinal of \mathcal{M} is a limit of Woodin cardinals. Suppose \mathcal{T} is a k -normal iteration tree on \mathcal{M} of limit length which doesn't have a fatal drop and suppose $\mathcal{Q}(\mathcal{T})$ exists. Then there is at most one cofinal branch b of \mathcal{T} such that either $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^\mathcal{T}$ or $\mathcal{Q}(\mathcal{T}) = \mathcal{M}_b^\mathcal{T} \upharpoonright \xi$ for some ξ in the wellfounded part of $\mathcal{M}_b^\mathcal{T}$.*

We shall need to look more closely at what is behind the uniqueness results above, at how the “fragments” of $\mathcal{Q}(b, \mathcal{T})$ (or better, its theory) determine the initial segments of b . The following is the crucial lemma which is essentially due to Martin and Steel ([4])

Lemma 1.6 *Suppose \mathcal{T} is a normal iteration tree on \mathcal{M} of limit length and s is a cofinal subset of $\delta(\mathcal{T})$; then there is at most one cofinal branch b such that there is $\alpha \in b$ with the property that $i_{\alpha, b}^\mathcal{T}$ exists and $s \subseteq \text{rng}(i_{\alpha, b}^\mathcal{T})$.*

Proof. Towards a contradiction, suppose there are two cofinal branches b and c such that for some α, β , both $i_{\alpha, b}^\mathcal{T}$ and $i_{\beta, c}^\mathcal{T}$ exist and $s \subseteq \text{ran}(i_{\alpha, b}^\mathcal{T}) \cap \text{ran}(i_{\beta, c}^\mathcal{T})$. Without loss of generality we can assume that α and β are the least ordinals with this property, $\alpha \leq \beta$ and that b and c diverge at α or earlier, i.e., if γ is the least ordinal in $b \cap c$ then $\gamma \leq \alpha$. By [4], we can assume that $b = \langle \alpha_n : n < \omega \rangle$, $c = \langle \beta_n : n < \omega \rangle$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Let then ξ be the least ordinal in $\text{ran}(i_{\alpha, b}^\mathcal{T}) \cap \text{ran}(i_{\beta, c}^\mathcal{T})$. Let n be the least such that $\text{crit}(i_{\alpha_n, b}^\mathcal{T}) > \xi$. This means that

$\text{crit}(E_{\alpha_{n+1-1}}^{\mathcal{T}}) > \xi$ and that $\text{lh}(E_{\alpha_n}^{\mathcal{T}}) < \xi$. By the proof of Theorem 2.2 of [4], this means that for some $m \geq 1$, $\xi \in [\text{crit}(E_{\beta_{m-1}}^{\mathcal{T}}), \text{lh}(E_{\beta_{m-1}}^{\mathcal{T}}))$. This then implies that $\xi \notin \text{ran}(i_{\beta_{m-1},c}^{\mathcal{T}})$, which is a contradiction. \square

$\mathcal{Q}(b, \mathcal{T})$ identifies b because it determines a canonical cofinal subset of $\text{rng}(i_{\alpha,b}^{\mathcal{T}} \cap \delta(\mathcal{T}))$, for some $\alpha \in b$, to which we can apply Lemma 1.6. But now, the proof of Lemma 1.6 gives the following refinement:

Lemma 1.7 *Suppose \mathcal{T} is an iteration tree on \mathcal{M} of limit length and b, c are two cofinal branches of \mathcal{T} such that $i_b^{\mathcal{T}}$ and $i_c^{\mathcal{T}}$ exist. Suppose that for some α ,*

$$i_b^{\mathcal{T}}(\alpha) = i_c^{\mathcal{T}}(\alpha) < \delta(\mathcal{T}).$$

Then $i_b^{\mathcal{T}} \upharpoonright \alpha = i_c^{\mathcal{T}} \upharpoonright \alpha$. Moreover, if $\xi \in b$ is the least such that $\text{crit}(i_{\xi,b}^{\mathcal{T}}) > i_b^{\mathcal{T}}(\alpha)$ then $\xi \in c$, so that $b \cap (\xi + 1) = c \cap (\xi + 1)$.

In addition to normal trees, we must consider linear stacks of normal trees. These are plays of the iteration game $G_k(\mathcal{M}, \alpha, \theta)$ in which II has not yet lost. See [20, Def. 4.4] for the formal definition. We shall generally use the vector notation $\vec{\mathcal{T}}$ for a stack of normal trees, and then \mathcal{T}_η for its η^{th} normal component. \mathcal{M}_η will denote the model at the beginning of the η th round of $\vec{\mathcal{T}}$. If Σ is a (k, α, θ) -iteration strategy for \mathcal{M} , and $\vec{\mathcal{T}}$ is a stack of trees on \mathcal{M} played according to Σ and having last model \mathcal{N} , then we let $\Sigma_{\mathcal{N}, \vec{\mathcal{T}}}$ be the (l, α, θ) strategy for \mathcal{N} induced by Σ . (l is the degree of the branch \mathcal{M} -to- \mathcal{N} of $\vec{\mathcal{T}}$. We assume here α is additively closed, so that there is such a strategy.) We say \mathcal{M} is countable (α, θ) -iterable if all of its countable submodels are (α, θ) -iterable.

Suppose \mathcal{M} is a mouse and Σ is an (α, θ) -iteration strategy. We then let

$$I(\mathcal{M}, \Sigma) = \{ \mathcal{N} : \text{there is a stack } \vec{\mathcal{T}} \text{ on } \mathcal{M} \text{ according to } \Sigma \text{ with last model } \mathcal{N} \text{ and } \pi^{\vec{\mathcal{T}}} \text{ exists} \}.$$

Given a premouse \mathcal{M} with a unique Woodin cardinal δ , we let $\mathbb{B}^{\mathcal{M}}$ be the countably generated extender algebra of \mathcal{M} at δ . In order to have a unique choice, we stipulate that the identities determining $\mathbb{B}^{\mathcal{M}}$ are precisely those coming from extenders E on the \mathcal{M} -sequence such that $\nu(E)$ is an inaccessible, but not a limit of inaccessibles, in \mathcal{M} , and $\nu(E) < \delta$. If $G \subseteq \mathbb{B}^{\mathcal{M}}$ then we let x_G be the set naturally coded by G . For basic facts about the extender algebra, we refer the reader to [1] and [20].

1.2 AD^+ and the Solovay hierarchy

We will not need the exact formulation of AD^+ . The interested readers can consult [22] or the introductory chapters of [7]. The *Solovay hierarchy* is a hierarchy of axioms extending AD^+ . To define the hierarchy, we first need to define the *Solovay sequence*. First recall the *Wadge ordering* of $\mathcal{P}(\mathbb{R})$. For $A, B \subseteq \mathbb{R}$, we say A is *Wadge reducible* to B and write $A \leq_W B$ if there is a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{-1}A = B$. Martin showed that under AD , \leq_W is a wellfounded relation. For $A \subseteq \mathbb{R}$, we let $w(A)$ be the rank of A in \leq_W . Under AD , $\Theta = \sup_{A \subseteq \mathbb{R}} w(A)$.

The Solovay sequence is defined as follows.

Definition 1.8 (The Solovay sequence) *Assume AD . The Solovay sequence is a closed increasing sequence $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ of ordinals defined by*

1. $\theta_0 = \sup\{\alpha : \text{there is an OD surjection } f : \mathbb{R} \rightarrow \alpha\}$,
2. if $\theta_\beta < \Theta$ then letting $A \subseteq \mathbb{R}$ be such that $w(A) = \theta_\beta$,

$$\theta_{\beta+1} = \sup\{\alpha : \text{there is an } OD_A \text{ surjection } f : \mathcal{P}(\theta_\beta) \rightarrow \alpha\},$$

3. if λ is a limit then $\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha$.

The Solovay hierarchy is the hierarchy we get by requiring that Ω is large. The following are the first few theories of this hierarchy. \leq_{con} is the consistency strength relation: given two theories T and S , $S \leq_{con} T$ if $Con(T) \vdash Con(S)$.

$$AD^+ + \Omega = 0 <_{con} AD^+ + \Omega = 1 < AD^+ + \Omega = 2 \cdots <_{con} AD^+ + \Omega = \omega <_{con} \cdots <_{con} AD^+ + \Omega = \omega_1 <_{con} AD^+ + \Omega = \omega_1 + 1 \cdots$$

For more on the Solovay hierarchy consult [6]. We will use the following theorem to construct a non-tame mouse from our hypothesis.

Theorem 1.9 (Woodin, [18]) *Assume there is a transitive inner model M containing \mathbb{R} such that $M \models AD^+ + \Theta = \theta_1$. Then there is a non-tame mouse.*

1.3 Suitable mice

The core model induction is a method for constructing iteration strategies for various models and often times the iteration strategies we construct are strategies for HOD's of various models of determinacy. In many situation, including our current situation, such HOD's are known to be fine structural models and the proof that they are will be used throughout this paper. Because of this we take a moment to review the background material for the HOD computation of models of determinacy. In later sections we will heavily rely on notions introduced in this section. In particular, notions such as *suitable premouse* or *quasi-iterable premouse* will be of crucial importance for us. Many of the notions introduced in this section are due to Woodin who introduced them in his computation of $\text{HOD}^{L[x][g]}$ (see [9]). We start with suitability.

Often times, the notion of a suitable premouse is needed simultaneously in models of determinacy and in ZFC context. Our current situation is an instance of this and we chose to define the notion in a most general way to avoid confusions when applying it in different contexts.

Fix then some cardinal λ and let $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$. We assume $ZF + DC_\lambda$. Notice that any function $f : H_{\lambda^+} \rightarrow H_{\lambda^+}$ can be naturally coded by a subset of $\mathcal{P}(\lambda)$. We then let $Code_\lambda : H_{\lambda^+}^{H_{\lambda^+}} \rightarrow \mathcal{P}(\mathcal{P}(\lambda))$ be one such coding. If $\lambda = \omega$ thence just write $Code$. Because any λ^+ -iteration strategy for a premouse of size $\leq \lambda$ is in $H_{\lambda^+}^{H_{\lambda^+}}$, we have that any such strategy is in the domain of $Code_\lambda$. Given a premouse \mathcal{M} , we say \mathcal{M} has an iteration strategy in Γ if $|\mathcal{M}| \leq \lambda$ and \mathcal{M} has a λ^+ -iteration strategy (or (α, λ^+) -iteration strategy for $\alpha \leq \lambda^+$) Σ such that $Code_\lambda(\Sigma) \in \Gamma$. We let $Mice^\Gamma$ be the set of mice that have an iteration strategy in Γ . Given a countable set a we let

$$\mathcal{W}^\Gamma(a) = \cup\{\mathcal{N} : \mathcal{N} \text{ is a sound mouse over } a \text{ such that } \rho(\mathcal{N}) = a \text{ and } \mathcal{N} \in Mice^\Gamma\}$$

and define $Lp_\xi^\Gamma(a)$ for $\xi < \omega_1$ by induction as follows:

1. $Lp_0^\Gamma(a) = \mathcal{W}^\Gamma(a)$,
2. for $\xi < \omega_1$, $Lp_{\xi+1}^\Gamma = \mathcal{W}^\Gamma(Lp_\xi^\Gamma(a))$,
3. for limit $\xi < \omega_1$, $Lp_\xi^\Gamma = \cup_{\alpha < \xi} Lp_\alpha^\Gamma(a)$.

Definition 1.10 (Γ -suitable premouse) *A premouse \mathcal{P} is Γ -suitable if there is a unique cardinal δ such that*

1. $\mathcal{P} \models \text{“}\delta \text{ is the unique Woodin cardinal”}$,

2. $o(\mathcal{P}) = \sup_{n < \omega} (\delta^{+n})^{\mathcal{P}}$,
3. for every $\eta \neq \delta$, $\mathcal{W}^\Gamma(\mathcal{P}|\eta) \models \text{“}\eta \text{ isn't Woodin”}$.
4. for any $\eta < o(\mathcal{P})$, $\mathcal{O}_\eta^{\mathcal{P}} = \mathcal{W}^\Gamma(\mathcal{P}|\eta)$.

Suppose \mathcal{P} is Γ -suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 1.10. Given an iteration tree \mathcal{T} on \mathcal{P} , we say \mathcal{T} is *nice* if \mathcal{T} has no fatal drops. Notice that Γ -suitable premice satisfy the hypothesis of Lemma 1.5. A nice tree \mathcal{T} is Γ -correctly guided if for every limit $\alpha < lh(\mathcal{T})$, $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha)$ exists and

$$\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \leq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T} \upharpoonright \alpha)).$$

\mathcal{T} is Γ -short if it is nice, correctly guided and $\mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T})) \models \text{“}\delta(\mathcal{T}) \text{ is not Woodin”}$. \mathcal{T} is Γ -maximal if it is nice, Γ -correctly guided yet not Γ -short. Notice that if \mathcal{T} is a maximal tree and b is a branch such that $i_b^{\mathcal{T}}(\delta^{\mathcal{P}}) = \delta(\mathcal{T})$ then \mathcal{T} doesn't have a nice normal continuation.

Definition 1.11 (Γ -correctly guided finite stack) *Suppose \mathcal{P} is Γ -suitable. We say $\langle \mathcal{T}_i, \mathcal{P}_i : i < m \rangle$ is a Γ -correctly guided finite stack on \mathcal{P} if*

1. $\mathcal{P}_0 = \mathcal{P}$,
2. \mathcal{P}_i is Γ -suitable and \mathcal{T}_i is a nice Γ -correctly guided tree on \mathcal{P}_i below $\delta^{\mathcal{P}_i}$,
3. for every i such that $i + 1 < m$ either \mathcal{T}_i has a last model and $\pi^{\mathcal{T}}$ -exists or \mathcal{T} is maximal, and
 - (a) if \mathcal{T}_i has a last model then \mathcal{P}_{i+1} is the last model of \mathcal{T}_i and if $lh(\mathcal{T}_i) = \alpha + 1$ where α is limit then if \mathcal{T}_i^- is \mathcal{T}_i without its last branch then $\mathcal{Q}(\mathcal{T}_i^-)$ -exists,
 - (b) if \mathcal{T}_i is Γ -maximal then $\mathcal{P}_{i+1} = Lp_\omega^\Gamma(\mathcal{M}(\mathcal{T}_i))$.

Notice that if $\langle \mathcal{T}_i, \mathcal{P}_i : i \leq m \rangle$ is a correctly guided finite stack on \mathcal{P} then only \mathcal{T}_m can have a dropping last branch.

Definition 1.12 (The last model of a Γ -correctly guided finite stack) *Suppose \mathcal{P} is Γ -suitable and $\vec{\mathcal{T}} = \langle \mathcal{T}_i, \mathcal{P}_i : i \leq k \rangle$ is a Γ -correctly guided finite stack on \mathcal{P} . We say \mathcal{R} is the last model of $\vec{\mathcal{T}}$ if one of the following holds:*

1. \mathcal{T}_k has a last model and \mathcal{R} is the last model of \mathcal{T}_k ,
2. \mathcal{T}_k is of limit length, \mathcal{T}_k is Γ -short and there is a cofinal well-founded branch b such that $\mathcal{Q}(b, \mathcal{T}_k)$ exists, $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T}_k))$ and $\mathcal{R} = \mathcal{M}_b^\mathcal{T}$,
3. \mathcal{T}_k is of limit length, \mathcal{T}_k is Γ -maximal, \mathcal{R} is Γ -suitable and

$$\mathcal{R} = Lp_\omega^\Gamma(\mathcal{M}(\mathcal{T}_k)).$$

We say \mathcal{R} is a correct iterate of \mathcal{P} if there is a correctly guided finite stack on \mathcal{P} with last model \mathcal{R} .

Definition 1.13 ($S(\Gamma)$ and $F(\Gamma)$) We let $S(\Gamma) = \{\mathcal{Q} : \mathcal{Q} \text{ is } \Gamma\text{-suitable}\}$. Also, we let $F(\Gamma)$ be the set of functions f such that $\text{dom}(f) = S(\Gamma)$ and for each $\mathcal{P} \in S(\Gamma)$, $f(\mathcal{P}) \subseteq \mathcal{P}$ and $f(\mathcal{P})$ is amenable to \mathcal{P} , i.e., for every $X \in \mathcal{P}$, $X \cap f(\mathcal{P}) \in \mathcal{P}$.

Given $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$ we let $f^n(\mathcal{P}) = f(\mathcal{P}) \cap \mathcal{P} \setminus ((\delta^\mathcal{P})^{+n})^\mathcal{P}$. Then $f(\mathcal{P}) = \bigcup_{n < \omega} f^n(\mathcal{P})$. We also let

$$\gamma_f^\mathcal{P} = \delta^\mathcal{P} \cap \text{Hull}_1^\mathcal{P}(\{f^n(\mathcal{P}) : n < \omega\}).$$

Notice that

$$\gamma_f^\mathcal{P} = \delta^\mathcal{P} \cap \text{Hull}_1^\mathcal{P}(\gamma_f^\mathcal{P} \cup \{f^n(\mathcal{P}) : n < \omega\}).$$

We then let

$$H_f^\mathcal{P} = \text{Hull}_1^\mathcal{P}(\gamma_f^\mathcal{P} \cup \{f^n(\mathcal{P}) : n < \omega\}).$$

If $\mathcal{P} \in S(\Gamma)$, $f \in F(\Gamma)$ and $i : \mathcal{P} \rightarrow \mathcal{Q}$ is an embedding then we let $i(f(\mathcal{P})) = \bigcup_{n < \omega} i(f^n(\mathcal{P}))$.

Definition 1.14 (f -iterability) Suppose $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. We say \mathcal{P} is f -iterable if whenever $\langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ is a finite correctly guided stack on \mathcal{P} with last model \mathcal{R} then there is a sequence $\langle b_k : k < m \rangle$ such that the following holds.

1. For $k < m - 1$,

$$b_k = \begin{cases} \emptyset & : \mathcal{T}_k \text{ has a successor length} \\ \text{cofinal well-founded branch} & \\ \text{such that } \mathcal{M}_{b_k}^\mathcal{T} = \mathcal{P}_\kappa & : \mathcal{T}_k \text{ is } \Gamma\text{-maximal} \end{cases}$$

2. The following three cases hold.

- (a) If \mathcal{T}_{m-1} has a successor length then $b_{m-1} = \emptyset$.
- (b) If \mathcal{T}_{m-1} is Γ -short then there is a cofinal well-founded branch b such that $\mathcal{Q}(b, \mathcal{T}_{m-1})$ exists, $\mathcal{Q}(b, \mathcal{T}_{m-1}) \trianglelefteq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T}_{m-1}))$ and b_{m-1} is the unique such branch.
- (c) If \mathcal{T}_{m-1} is Γ -maximal then b_{m-1} is a cofinal well-founded branch.

3. Letting

$$\pi_k = \begin{cases} \pi^{\mathcal{T}_k} & : \mathcal{T}_k \text{ has a successor length} \\ \pi_{b_k}^{\mathcal{T}_k} & : \mathcal{T}_k \text{ is } \Gamma\text{-maximal} \end{cases}$$

and $\pi = \pi_{m-1} \circ \pi_{m-2} \circ \cdots \circ \pi_0$ then

$$\pi(f(\mathcal{P})) = f(\mathcal{R}).$$

Suppose again that $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. Suppose $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ is a Γ -correctly guided finite stack on \mathcal{P} with last model \mathcal{R} . We say $\vec{b} = \langle b_k : k < m \rangle$ witness f -iterability for $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ if 2 above is satisfied. We then let

$$\pi_{\vec{\mathcal{T}}, \vec{b}, k} = \begin{cases} \pi^{\mathcal{T}_k} & : \mathcal{T}_k \text{ has a successor length} \\ \pi_{b_k}^{\mathcal{T}_k} & : \mathcal{T}_k \text{ is maximal} \\ \text{undefined} & : \text{otherwise} \end{cases}$$

and $\pi_{\vec{\mathcal{T}}, \vec{b}} = \pi_{\vec{\mathcal{T}}, \vec{b}, m-1} \circ \pi_{\vec{\mathcal{T}}, \vec{b}, m-2} \circ \cdots \circ \pi_{\vec{\mathcal{T}}, \vec{b}, 0}$. Notice that clause three isn't vacuous as it might be that \mathcal{T}_k is Γ -short and its unique branch has a drop.

Continuing with the notation of the previous paragraph, let \vec{b} and \vec{c} be two f -iterability branches for $\vec{\mathcal{T}}$. It then follows from Theorem 1.6 that

$$\pi_{\vec{\mathcal{T}}, \vec{b}} \upharpoonright H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{T}}, \vec{c}} \upharpoonright H_f^{\mathcal{P}}.$$

Lemma 1.15 (Uniqueness of f -iterability embeddings) *Suppose $\mathcal{P} \in S(\Gamma)$, $f \in F(\Gamma)$ and $\vec{\mathcal{T}}$ is a finite correctly guided stack on \mathcal{P} . Suppose \vec{b} and \vec{c} are two f -iterability branches for $\vec{\mathcal{T}}$. Then*

$$\pi_{\vec{\mathcal{T}}, \vec{b}} \upharpoonright H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{T}}, \vec{c}} \upharpoonright H_f^{\mathcal{P}}.$$

Moreover, if $\vec{\mathcal{T}}$ consists of just one normal tree \mathcal{T} , \mathcal{Q} is the last model of \mathcal{T} and b and c witness f -iterability for \mathcal{T} then if $\xi \in b$ is the least such that $\text{crit}(E_\xi^{\mathcal{T}}) > \gamma_f^{\mathcal{Q}}$ then $b \cap \xi = c \cap \xi$.

Definition 1.16 Suppose $\mathcal{P} \in S(\Gamma)$ and f -iterable. Given a Γ -correctly guided maximal \mathcal{T} on \mathcal{P} with last model \mathcal{Q} , we let $b_{\mathcal{T},f} = b \cap \xi$ where b witnesses f -iterability of \mathcal{P} for \mathcal{T} and $\xi \in b$ is the least such that $\text{crit}(E_\xi^{\mathcal{T}}) > \gamma_f^{\mathcal{Q}}$.

Notice that, if \mathcal{P} is f -iterable, $\vec{\mathcal{T}}$ is a correctly guided finite stack on \mathcal{P} , and \vec{b} witnesses f -iterability of \mathcal{P} for $\vec{\mathcal{T}}$, then even though $\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_f^{\mathcal{P}}$ is independent of \vec{b} it may very well depend on $\vec{\mathcal{T}}$. This observation motivates the following definition.

Definition 1.17 (Strong f -iterability) Suppose $\mathcal{P} \in S(\Gamma)$ and $f \in F(\Gamma)$. We say \mathcal{P} is strongly f -iterable if \mathcal{P} is f -iterable and whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_j, \mathcal{P}_j : j < u \rangle \in H_{\lambda^+}$ and $\vec{\mathcal{U}} = \langle \mathcal{U}_j, \mathcal{P}_j : j < v \rangle \in H_{\lambda^+}$ are two correctly guided finite stacks on \mathcal{P} with common last model \mathcal{R} , \vec{b} witnesses f -iterability for $\vec{\mathcal{T}}$ and \vec{c} witnesses f -iterability for $\vec{\mathcal{U}}$ then $\pi_{\vec{b}}^{\vec{\mathcal{T}}}$ is defined iff $\pi_{\vec{c}}^{\vec{\mathcal{U}}}$ is defined and

$$\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_f^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{c}} \upharpoonright H_f^{\mathcal{P}}.$$

If \mathcal{P} is strongly f -iterable and $\vec{\mathcal{T}}$ is a correctly guided finite stack on \mathcal{P} with last model \mathcal{R} then we let

$$\pi_{\mathcal{P},\mathcal{R},f} : H_f^{\mathcal{P}} \rightarrow H_f^{\mathcal{R}}$$

be the embedding given by any \vec{b} which witnesses the f -iterability of $\vec{\mathcal{T}}$, i.e., fixing \vec{b} which witnesses f -iterability for $\vec{\mathcal{T}}$,

$$\pi_{\mathcal{P},\mathcal{R},f} = \pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_f^{\mathcal{P}}.$$

Clearly, $\pi_{\mathcal{P},\mathcal{R},f}$ is independent of $\vec{\mathcal{T}}$ and \vec{b} .

Given a finite sequence of functions $\vec{f} = \langle f_i : i < n \rangle \in F(\Gamma)$, we let $\oplus_{i < n} f_i \in F(\Gamma)$ be the function given by $(\oplus_{i < n} f_i)(\mathcal{P}) = \langle f_i(\mathcal{P}) : i < n \rangle$. We set $\oplus \vec{f} = \oplus_{i < n} f_i$.

We then let

$$\mathcal{I}_{\Gamma,F} = \{(\mathcal{P}, \vec{f}) : \mathcal{P} \in S(\Gamma), \vec{f} \in (F(\Gamma))^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } \oplus \vec{f}\text{-iterable}\}.$$

Definition 1.18 Given $F \subseteq F(\Gamma)$, we say F is closed if for any $\vec{f} \subseteq F^{<\omega}$ there is \mathcal{P} such that $(\mathcal{P}, \oplus \vec{f}) \in \mathcal{I}_{\Gamma, F}$ and for any $\vec{g} \subseteq F^{<\omega}$, there is a Γ -correct iterate \mathcal{Q} of \mathcal{P} such that $(\mathcal{Q}, \vec{f} \cup \vec{g}) \in \mathcal{I}_{\Gamma, F}$.

Fix now a closed $F \subseteq F(\Gamma)$. Let

$$\mathcal{F}_{\Gamma, F} = \{H_f^{\mathcal{P}} : (\mathcal{P}, f) \in \mathcal{I}_{\Gamma, F}\}.$$

We then define $\preceq_{\Gamma, F}$ on $\mathcal{I}_{\Gamma, F}$ by letting $(\mathcal{P}, \vec{f}) \preceq_{\Gamma, F} (\mathcal{Q}, \vec{g})$ iff \mathcal{Q} is a Γ -correct iterate of \mathcal{P} and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{P}, \vec{f}) \preceq_{\Gamma, F} (\mathcal{Q}, \vec{g})$, we have that

$$\pi_{\mathcal{P}, \mathcal{Q}, \vec{f}} : H_{\oplus \vec{f}}^{\mathcal{P}} \rightarrow H_{\oplus \vec{f}}^{\mathcal{Q}}.$$

Notice that if F is closed then $\preceq_{\Gamma, F}$ is directed. Let then

$$\mathcal{M}_{\infty, \Gamma, F}$$

be the direct limit of $(\mathcal{F}_{\Gamma, F}, \preceq_{\Gamma, F})$ under $\pi_{\mathcal{P}, \mathcal{Q}, \vec{f}}$'s. Given $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{\Gamma, F}$, we let $\pi_{\mathcal{P}, \vec{f}, \infty} : H_{\oplus \vec{f}}^{\mathcal{P}} \rightarrow \mathcal{M}_{\infty, \Gamma, F}$ be the direct limit embedding.

Lemma 1.19 $\mathcal{M}_{\infty, \Gamma, F}$ is wellfounded.

Proof. If not then we can fix $\langle (\mathcal{P}_i, f_i) : i < \omega \rangle \subseteq F$ and $\langle \alpha_i : i < \omega \rangle$ such that $\alpha_i \in H_{f_i}^{\mathcal{P}_i}$, $(\mathcal{P}_i, \oplus_{j \leq i} f_j) \in \mathcal{I}_{\Gamma, F}$ and $\pi_{\mathcal{P}_{i+1}, f_{i+1}, \infty}(\alpha_{i+1}) < \pi_{\mathcal{P}_i, f_i, \infty}(\alpha_i)$. By simultaneously comparing \mathcal{P}_i 's we get a common Γ -correct iterate \mathcal{P} such that if $\beta_i = \pi_{\mathcal{P}_i, \mathcal{P}, f_i}(\alpha_i)$ then $\beta_{i+1} < \beta_i$, contradiction!
□

It turns out that V_{Θ}^{HOD} of many models of determinacy can be obtained as $\mathcal{M}_{\infty, \Gamma, F}$ for some Γ and F . We will give the details in the next few subsection.

1.4 Quasi-iterability

We are now in a position to define quasi-iterability which will later be used to construct strategies for certain HODs of models of determinacy. We continue with the set up of Section 1.3. Thus, recall that we are assuming ZF and $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$ for some cardinal λ . Fix some Γ -suitable \mathcal{P} and a closed $F \subseteq F(\Gamma)$. Let $G \subseteq F$.

Definition 1.20 We say $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G) -quasi iteration of \mathcal{P} of length ν if

1. $\mathcal{M}_0 = \mathcal{P}$ and for all $\alpha < \nu$, \mathcal{M}_α is Γ -suitable,
2. for $\alpha < \nu$ then \mathcal{T}_α is a Γ -correctly guided tree on \mathcal{M}_α and if $\alpha + 1 < \nu$ then $\mathcal{M}_{\alpha+1}$ is the last model of \mathcal{T}_α ,
3. for all $\alpha < \nu$, \mathcal{M}_α is strongly f -iterable for every $f \in G$,
4. for all limit $\alpha < \nu$, for any $g \in F$ there is $\beta < \alpha$ such that for any $\gamma \in [\beta, \alpha)$
 - (a) if $\text{lh}(\mathcal{T}_\gamma)$ is a successor then $i^{\mathcal{T}_\gamma}(g(\mathcal{M}_\gamma)) = g(\mathcal{M}_{\gamma+1})$,
 - (b) if $\text{lh}(\mathcal{T}_\gamma)$ is limit then there is a cofinal wellfounded branch b of \mathcal{T}_γ such that $\mathcal{M}_b^{\mathcal{T}_\gamma} = \mathcal{M}_{\gamma+1}$ and $i_b^{\mathcal{T}_\gamma}(g(\mathcal{M}_\gamma)) = g(\mathcal{M}_{\gamma+1})$,

Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G) -quasi iteration of \mathcal{P} of length ν . Given limit $\alpha < \nu$ and $g \in F$ we let $\beta_{g,\alpha} < \alpha$ be the least ordinal satisfying 4b of Definition 1.20.

Definition 1.21 Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G) -quasi iteration of \mathcal{P} of length ν . We say $\langle \pi_{\gamma,\xi}^{g,\alpha} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in \text{Lim} \cap \nu + 1 \rangle$ are the embeddings of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ if for all limit $\alpha \leq \nu$ and for all $g \in F$

1. $\pi_{\gamma,\xi}^{g,\alpha}$ is defined for all $\gamma, \xi \in [\beta_{g,\alpha}, \alpha]$,
2. $\pi_{\gamma,\gamma+1}^{g,\alpha} = i^{\mathcal{T}_\gamma}$ if it exists and otherwise $\pi_{\gamma,\gamma+1}^{g,\alpha} = i_b^{\mathcal{T}_\gamma} \upharpoonright H_g^{\mathcal{M}_\gamma}$ where b witnesses clause 4b for α and g ,
3. fixing $\gamma \in [\beta_{g,\alpha}, \alpha]$, $\pi_{\gamma,\xi}^{g,\alpha}$ is defined by induction on $\xi \in (\gamma, \alpha]$ where the first step of the induction is clause 2 above and the rest is given by the following scheme:
 - (a) if $\xi \leq \alpha$ is limit then if $H_g^{\mathcal{M}_\xi}$ is a direct limit of $H_\xi^{\mathcal{M}_\lambda}$'s under $\pi_{\gamma,\lambda}^{g,\alpha}$'s for $\gamma < \lambda < \xi$ then $\pi_{\gamma,\xi}^{g,\alpha}$ is the direct limit embedding and otherwise it is undefined,
 - (b) if $\xi = \lambda + 1$ then $\pi_{\gamma,\xi}^{g,\alpha} = \pi_{\lambda,\lambda+1}^{g,\alpha} \circ \pi_{\gamma,\lambda}^{g,\alpha}$.

Next we define (F, G) -quasi iterations.

Definition 1.22 Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a semi (F, G) -quasi iteration of \mathcal{P} of length ν . We say $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a (F, G) -quasi iteration of \mathcal{P} of length ν if letting $\langle \pi_{\gamma,\xi}^{g,\alpha} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in \text{Lim} \cap \nu + 1 \rangle$ be the embeddings of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ then

1. for every α , $\mathcal{M}_\alpha = \cup_{g \in F} H_g^{\mathcal{M}_\alpha}$, and
2. for every limit $\alpha < \nu$ and for every $g \in F$, $H_g^{\mathcal{M}_\alpha}$ is the direct of $H_g^{\mathcal{M}_\gamma}$ for $\gamma \in [b_{g,\alpha}, \alpha)$ under the maps $\pi_{\gamma,\xi}^{g,\alpha}$'s for $\gamma < \xi \in [b_{g,\alpha}, \alpha)$.

Finally we define the last model of (F, G) -quasi iterations.

Definition 1.23 Suppose $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ is a (F, G) -quasi iteration of \mathcal{P} of length ν with embeddings $\langle \pi_{\gamma,\xi}^{g,\alpha} : \gamma < \xi < \alpha \leq \nu \wedge \alpha \in \text{Lim} \cap \nu + 1 \rangle$. Then we say \mathcal{Q} is the last model of $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu < \lambda^+ \rangle$ if \mathcal{Q} is Γ -suitable, \mathcal{Q} is strongly f -iterable for all $f \in G$ and one of the following holds:

1. $\nu = \alpha + 1$, \mathcal{Q} is the last model of \mathcal{T}_α as defined in Definition 1.12 and $\mathcal{Q} = \cup_{f \in F} H_f^{\mathcal{Q}}$ or
2. ν is limit, $\mathcal{Q} = \cup_{g \in F} H_g^{\mathcal{Q}}$ and for $g \in F$, $H_g^{\mathcal{Q}}$ is the direct limit of $H_g^{\mathcal{M}_\gamma}$ under $\pi_{\gamma,\xi}^{g,\nu}$'s for $\gamma < \xi \in [b_{g,\nu}, \nu)$.

Notice that it may be the case that quasi iterations don't have last models but when they do it is uniquely determined.

Definition 1.24 We say \mathcal{P} is (F, G) -quasi iterable if all of its (F, G) -quasi iterations have a last model.

Before moving on we fix some notation. When $G = \{f\}$ then we use (F, f) instead of (F, G) . When $G = \emptyset$ then we write F instead of (F, \emptyset) . Suppose $F \subseteq F(\Gamma)$ is closed. We say \mathcal{P} is (strongly) F -iterable if \mathcal{P} is (strongly) f -iterable for all $f \in F$. Suppose now that \mathcal{P} is (strongly) F -iterable. Let \mathcal{Q} be a correctly guided iterate of \mathcal{P} . Then we let $\pi_{\mathcal{P}, \mathcal{Q}, F} = \cup_{f \in F} \pi_{\mathcal{P}, \mathcal{Q}, f}$. Suppose further there is some $F^* \subseteq F(\Gamma)$ such that \mathcal{P} is (F^*, F) -quasi iterable and suppose that \mathcal{Q} is a (F^*, F) -quasi iterate of \mathcal{P} . We then let $\pi_{\mathcal{P}, \mathcal{Q}, F} : \cup_{f \in F} H_f^{\mathcal{P}} \rightarrow \cup_{f \in F} H_f^{\mathcal{Q}}$ be the iteration embedding coming from the composition of the quasi iterability embeddings. We also define $\pi_{\mathcal{Q}, F, \infty}$ similarly.

1.5 \vec{f} -guided strategies

We continue with the set up of Section 1.3. Thus, recall that we are assuming $ZF + DC_\lambda$. In this subsection, our goal is to develop tools for constructing strategies by putting together pieces

of various f -iterability branches. In later sections, we will need to construct iteration strategies that have the so-called branch condensation and one method for producing such strategies is via producing strongly \vec{f} -guided strategies. Below we let ZFC^- stand for ZFC without Replacement or without Powerset Axiom. We start by introducing *branch condensation*, a notion that played an important role in [7].

Definition 1.25 *Suppose Σ is an (α, β) -iteration strategy for some structure M such that $M \models ZFC^-$ (M need not be a fine structural model). We say Σ has branch condensation if whenever $(\mathcal{R}, \vec{\mathcal{T}}, \mathcal{Q}, \vec{\mathcal{U}}, c, \pi)$ is such that*

1. $\vec{\mathcal{T}}$ is a stack on M according to Σ with last model \mathcal{R} and $i^{\vec{\mathcal{T}}}$ exists,
2. $\vec{\mathcal{U}}$ is a stack on M according to Σ such that its last normal component is of limit length and c is some branch on $\vec{\mathcal{U}}$ such that $\pi_c^{\vec{\mathcal{U}}}$ exists,
3. $\mathcal{Q} = \mathcal{M}_c^{\vec{\mathcal{U}}}$ and $\pi : \mathcal{Q} \rightarrow_{\Sigma_1} \mathcal{R}$ is such that $i^{\vec{\mathcal{T}}} = \pi \circ i_c^{\vec{\mathcal{U}}}$

then $c = \Sigma(\vec{\mathcal{U}})$.

Below we introduce one of the main methods for producing strategies with branch condensation. The basic idea is that if the strategy moves many operators correctly then it must have branch condensation. For example, let $\mathcal{M} = \mathcal{M}_1 | (\delta^+)^{\mathcal{M}_1}$ where \mathcal{M}_1 is the minimal mouse with a Woodin cardinal and δ is the Woodin cardinal of \mathcal{M}_1 . Then \mathcal{M} may have many iteration strategies but it has one strategy that moves the theory of sharps correctly. Let Σ be this strategy. It can be shown, using the fact that the theory of sharps *condenses*, that Σ has branch condensation. Below we will explain how exactly the aforementioned condensation of the theory of sharps works in a more general context.

Definition 1.26 *Suppose $\mathcal{P} \in S(\Gamma)$ and Σ is a (λ^+, λ^+) -iteration strategy for \mathcal{P} . We say Σ is Γ -fullness preserving if whenever $i : \mathcal{P} \rightarrow \mathcal{Q}$ comes from an iteration produced by Σ , $\mathcal{Q} \in S(\Gamma)$. Given $f \in F(\Gamma)$, we say Σ respects f if whenever $i : \mathcal{P} \rightarrow \mathcal{Q}$ and $j : \mathcal{Q} \rightarrow \mathcal{R}$ are iterations produced via Σ then $j(f(\mathcal{Q})) = f(\mathcal{R})$. We say Σ strongly respects f if whenever $i, j, \mathcal{P}, \mathcal{Q}, \mathcal{R}$ are as above and \mathcal{S} is such that there are $\sigma : \mathcal{Q} \rightarrow \mathcal{S}$ and $\tau : \mathcal{S} \rightarrow \mathcal{R}$ such that $j = \tau \circ \sigma$ then $\mathcal{S} \in S(\Gamma)$ and $\sigma(f(\mathcal{Q})) = f(\mathcal{S})$.*

If Σ strongly respects many f 's then it is possible to show that Σ has branch condensation. Fix some $F \subseteq F(\Gamma)$.

Definition 1.27 We say F is a quasi-self-justifying-system (qsjs) if there is $\mathcal{P} \in S(\Gamma)$ such that

1. F is closed,
2. for any $\vec{f} \in F^{<\omega}$, $\oplus \vec{f} \in F$,
3. for every $f \in F$, \mathcal{P} is (F, f) -quasi iterable,
4. whenever \mathcal{Q} is a (F, f) -quasi iterate of \mathcal{P} , $\sup_{f \in F} \gamma_f^{\mathcal{Q}} = \mathcal{Q}$ (this condition follows from clause 3 above),
5. whenever \mathcal{Q} is a F -quasi iterate of \mathcal{P} and $\sigma : \mathcal{R} \rightarrow_{\Sigma_1} \mathcal{Q}$ is such that $f(\mathcal{Q}) \in \text{rng}(\sigma)$ for all $f \in F$ then $\mathcal{R} \in S(\Gamma)$.

The following lemmas are essentially due to Woodin and are based on Lemma 1.6. They collectively imply that if F is a qsjs then it produces a Γ -fullness preserving iteration strategy for some $\mathcal{P} \in S(\Gamma)$.

Lemma 1.28 (ZF) Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then there is a λ^+ -iteration strategy Σ^F for \mathcal{P} such that Λ^F is Γ -fullness preserving.

Proof. We define Λ^F for normal trees. First we describe Λ^F on Γ -correctly guided trees. Given such a tree \mathcal{T} on \mathcal{P} we have two cases. If \mathcal{T} is Γ -short then we let $\Lambda^F(\mathcal{T}) = b$ be the unique branch of \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T})$ -exists. If \mathcal{T} is a maximal then we do the following. Let \mathcal{Q} be the last model of \mathcal{T} . Recall the definition of b_f for $f \in F$ (see Definition 1.16). For each $f \in F$, let b_f be a branch of \mathcal{T} witnessing f -iterability of \mathcal{P} for \mathcal{T} . Because $\sup_{f \in F} \gamma_f^{\mathcal{Q}} = \delta^{\mathcal{Q}}$, we have that $b = \cup_{f \in F} b_f$ is a cofinal branch of F . Notice now that if $\mathcal{S} = \cup_{f \in F} \pi_b^{\mathcal{T}}(H_f^{\mathcal{P}})$ then there is $\sigma : \mathcal{S} \rightarrow_{\Sigma_1} \mathcal{Q}$. By clause 5 of Definition 1.27, we have that $\mathcal{S} \in S(\Gamma)$ and by clause 4 of Definition 1.27, we have that $\sigma \upharpoonright \delta(\mathcal{T}) = \text{id}$. Hence, $\mathcal{S} = \mathcal{Q}$. We then let $\Lambda^F(\mathcal{T}) = b$.

If \mathcal{T} has a fatal drop then let (α, η) be such that \mathcal{T} has a fatal drop at (α, η) . Then we let $\Lambda^F(\mathcal{T})$ be the $\Lambda(\mathcal{T}^*)$ where Λ is the strategy of \mathcal{O}_η^M and \mathcal{T}^* is the \mathcal{T} from α on. \square

Notice that if F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it then any F -quasi iterate of \mathcal{P} also witnesses that F is a qsjs. Given then any $\mathcal{Q} \in S(\Gamma)$ witnessing that F is a qsjs we let $\Lambda_{\mathcal{Q}}^F$ be the λ^+ -iteration strategy given by Lemma 1.28. Let then $\mathcal{P} \in S(\Gamma)$ witness that F is qsjs.

To get a (λ^+, λ^+) -strategy for \mathcal{P} , we just need to dovetail all of $\Lambda_{\mathcal{Q}}^F$'s together for \mathcal{Q} 's that are F -quasi iterate of \mathcal{P} . The following notion will be used in the proof of the next lemma.

Given an iteration tree \mathcal{T} such that $lh(\mathcal{T}) = \gamma + 1$ where γ is limit, we let \mathcal{T}^- be \mathcal{T} without its last branch. Suppose $\vec{\mathcal{T}}$ is a stack on a Γ -suitable premouse \mathcal{P} . We say $\vec{\mathcal{T}}$ is *good* if letting $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \eta \rangle$ be the normal components of $\vec{\mathcal{T}}$, for each $\alpha < \eta$, \mathcal{M}_α is Γ -suitable and \mathcal{T}_α is a Γ -correctly guided tree on \mathcal{M}_α . We let $\vec{\mathcal{T}}^*$ be defined as follows: $\vec{\mathcal{T}}^* = \langle \mathcal{U}_\alpha, \mathcal{M}_\alpha : \alpha < \eta \rangle$ where $\mathcal{U}_\alpha = \mathcal{T}_\alpha$ if either \mathcal{T}_α^- is undefined or it is defined but it is Γ -short and otherwise, $\mathcal{U}_\alpha = \mathcal{T}_\alpha^-$. We say $\vec{\mathcal{T}}^*$ is the quasi-rearrangement of $\vec{\mathcal{T}}$.

Lemma 1.29 (ZF) *Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then there is a (λ^+, λ^+) -iteration strategy Σ^F for \mathcal{P} such that Σ^F is Γ -fullness preserving.*

Proof. We handle trees with fatal drops exactly the same way as in Lemma 1.28. We leave the details to the reader. By induction we define $\langle \Sigma_\alpha^F : \alpha < \lambda^+ \rangle$ such that

1. Σ_α^F is a Γ -fullness preserving (α, λ^+) -iteration strategy for \mathcal{P} ,
2. for $\alpha < \beta \leq \lambda^+$, $\Sigma_\alpha^F \subseteq \Sigma_\beta^F$,
3. for all $\alpha < \lambda^+$, whenever $\vec{\mathcal{T}}$ is according to Σ_α^F , $\vec{\mathcal{T}}^*$ is an F -quasi iteration of \mathcal{P} , and
4. if β is limit and $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ is a stack on \mathcal{P} such that for each $\alpha < \beta$, $\bigoplus_{\xi < \alpha} \mathcal{T}_\xi$ is according to Σ_α^F then letting \mathcal{M} be the direct limit along the main branch of $\vec{\mathcal{T}}$, \mathcal{M} is suitable and $(\vec{\mathcal{T}}^*) \frown \mathcal{M}$ is a F -quasi iteration of \mathcal{P} .

For $\alpha = 1$ we let $\Sigma_\alpha^F = \Lambda_{\mathcal{P}}^F$. Clearly, Σ_1^F satisfies clauses 1-4 above. Suppose now we have defined $\langle \Sigma_\alpha^F : \alpha < \beta \rangle$. For $\alpha < \beta$, let I_α be the set of pairs $(\vec{\mathcal{T}}, \mathcal{M})$ such that $\vec{\mathcal{T}}$ is according to Σ_α^F , it has α many rounds, \mathcal{M} is the last model of $\vec{\mathcal{T}}$, \mathcal{M} is Γ -suitable and if α is limit then \mathcal{M} is the direct limit of the models along the main branch. Suppose now $\beta = \nu + 1$ for some ν . Then $\Sigma_\beta^F(\vec{\mathcal{T}}) = b$ iff either

1. $\vec{\mathcal{T}}$ is a run of $\mathcal{G}(\mathcal{P}, \nu, \lambda^+)$, $\vec{\mathcal{T}}$ is according to Σ_ν^F and $\Sigma_\nu^F(\vec{\mathcal{T}}) = b$ or
2. $\vec{\mathcal{T}}$ is of the form $\vec{\mathcal{U}} \frown \mathcal{M} \frown \mathcal{U}$ such that $(\vec{\mathcal{U}}, \mathcal{M}) \in I_\nu$, \mathcal{U} is according to $\Lambda_{\mathcal{M}}^F$ and $\Lambda_{\mathcal{M}}^F(\mathcal{U}) = b$.

Again, clearly Σ_β^F has the desired properties.

Lastly, suppose β is limit. We have that $\langle \Sigma_\alpha^F : \alpha < \beta \rangle$ satisfies 1-4. It is then enough to show that clause 4 holds. To see this, fix $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ as in clause 4. We have that for each $\alpha < \beta$, $(\vec{\mathcal{T}} \upharpoonright \alpha)^*$ is an F -quasi iteration of \mathcal{P} . What we need to show is that the direct limit along the main branch of $\vec{\mathcal{T}}$ is the same as the direct limit as defined by the two clauses of Definition 1.22. Let then, for $\alpha < \gamma < \beta$, $\pi_{\alpha,\gamma} : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\gamma$ be the iteration embeddings given by $\vec{\mathcal{T}}$. We have that for each $f \in F$ and $\alpha < \gamma < \beta$, $\pi_{\alpha,\gamma}(f(\mathcal{M}_\alpha)) = f(\mathcal{M}_\gamma)$ and that $\mathcal{M}_\alpha = \cup_{f \in F} H_f^{\mathcal{M}_\alpha}$. Let then \mathcal{M} be the direct limit of \mathcal{M}_α 's under the maps $\pi_{\alpha,\gamma}$ and let $\pi_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{M}$ be the direct limit embedding. We then have that $\mathcal{M} = \cup_{\alpha < \beta, f \in F} \pi_0(H_f^{\mathcal{P}})$ and for every $f \in F$, $\pi_0(H_f^{\mathcal{P}})$ is the direct limit under $\pi_{\alpha,\gamma}^{f,\beta}$ where $\langle \pi_{\alpha,\gamma}^{f,\xi} : \alpha < \gamma \leq \xi \leq \beta \rangle$ are the embeddings of $\vec{\mathcal{T}}^*$. It then follows that \mathcal{M} is indeed the model constructed by the two clauses of Definition 1.22. \square

Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. We then let $\Sigma_{\mathcal{P}}^F$ be the (λ^+, λ^+) -iteration strategy given by Lemma 1.29. Next we show that $\Sigma_{\mathcal{P}}^F$ has branch condensation.

Lemma 1.30 (ZF) *Suppose F is a qsjs and $\mathcal{P} \in S(\Gamma)$ witnesses it. Then $\Sigma_{\mathcal{P}}^F$ is a (λ^+, λ^+) -iteration strategy which is Γ -fullness preserving and has branch condensation.*

Proof. We only need to show that $\Sigma_{\mathcal{P}}^F$ has branch condensation as the rest follows from Lemma 1.29. Let then $(\mathcal{R}, \vec{\mathcal{T}}, \mathcal{Q}, \vec{\mathcal{U}}, c, \pi)$ be as in Definition 1.25 where we let $M = \mathcal{P}$ and $\Sigma = \Sigma_{\mathcal{P}}^F$. We need to see that $c = \Sigma_{\mathcal{P}}^F(\vec{\mathcal{U}})$. Because we have that $\pi : \mathcal{Q} \rightarrow \mathcal{R}$ and $i^{\vec{\mathcal{T}}} = \pi \circ i_c^{\vec{\mathcal{U}}}$ we get that, using clause 5 of Definition 1.27, that \mathcal{Q} is Γ -suitable and $i_c^{\vec{\mathcal{U}}}(f(\mathcal{P})) = f(\mathcal{Q})$ for all $f \in F$. Let then $\langle \mathcal{U}_\alpha, \mathcal{M}_\alpha : \alpha \leq \eta \rangle$ be the normal components of $\vec{\mathcal{U}}$. Because $\vec{\mathcal{U}} \upharpoonright \eta$ is via $\Sigma_{\mathcal{P}}^F$, we have that $i_{0,\eta}^{\vec{\mathcal{U}}}(f(\mathcal{P})) = f(\mathcal{M}_\eta)$ for every $f \in F$. It then follows that $i_c^{\mathcal{U}_\eta}(f(\mathcal{M}_\eta)) = f(\mathcal{Q})$ for all $f \in F$. Hence, \mathcal{U}_η is according to $\Lambda_{\mathcal{M}_\eta}^F$, implying that $\Sigma_{\mathcal{P}}^F(\vec{\mathcal{U}}) = c$. \square

We finish with the following lemma whose proof we leave to the reader as it is very close to the proof of Lemma 1.29.

Lemma 1.31 *Suppose F and G are two qsjs such that $F \subseteq G$. Then for any $\mathcal{P} \in S(\Gamma)$ witnessing that F is qsjs, \mathcal{P} witnesses that G is also qsjs and $\Sigma_{\mathcal{P}}^F = \Sigma_{\mathcal{P}}^G$.*

1.6 (ω, Γ) -suitable preimage

In this paper, our primary tool for constructing iteration strategies with branch condensation will be Lemma 1.29 which heavily relies on clause 4 and 5 of Definition 1.27. Here we develop

some notions that we will later use to show that various F 's are qsj's and in particular, satisfy clause 4 and 5 of Definition 1.27.

We continue with the set up of the previous sections. Recall that we have fixed a cardinal λ and $\Gamma \subseteq \mathcal{P}(\mathcal{P}(\lambda))$. The basic notion we will need is that of (ω, Γ) -suitable premeice. These are formed by stacking ω many Γ -suitable premeice and hence, they all have ω -Woodin cardinals.

Definition 1.32 *A premeice \mathcal{P} is (ω, Γ) -suitable if there is an increasing sequence of \mathcal{P} -cardinals $\langle \delta_i : i < \omega \rangle$ such that letting $\delta_\omega = \sup_{i < \omega} \delta_i$,*

1. $\mathcal{P} \models$ “ δ_ω is the largest cardinal”,
2. for each i , δ_i is a Woodin cardinal in \mathcal{P} ,
3. if $\mathcal{P}_i = \mathcal{P} | (\delta_i^{+\omega})^{\mathcal{P}}$ then \mathcal{P}_0 is a Γ -suitable premeice and \mathcal{P}_{i+1} is a Γ -suitable premeice over \mathcal{P}_i .

If \mathcal{P} is (ω, Γ) -suitable then we let $\delta_i^{\mathcal{P}} = \delta_i$. We say \mathcal{P} is an *anomalous* (ω, Γ) -suitable premeice if $\rho_\omega(\mathcal{P}) < \delta_\omega^{\mathcal{P}}$.

Suppose now \mathcal{P} is any premeice with exactly ω -Woodin cardinals. Let $\langle \delta_i : i \in [-1, \omega) \rangle$ be such that $\delta_{-1} = \emptyset$ and $\langle \delta_i : i < \omega \rangle$ enumerates the Woodin cardinals of \mathcal{P} in increasing order. Let $\delta_\omega = \sup_{i < \omega} \delta_i$. Suppose $\mathcal{P} \models$ “ δ_ω is the largest cardinal”. Let \mathcal{T} be a normal tree on \mathcal{T} constructed via $\mathcal{G}(\mathcal{P}, \lambda^+)$. Notice that there is a natural way of rearranging \mathcal{T} so that it is constructed via a run of $\mathcal{G}(\mathcal{P}, \omega + 1, \lambda^+)$. In this rearrangement of \mathcal{T} , if \mathcal{M}_n is the model of the beginning of the n th round and $n \leq \omega$ then the iteration embedding $i : \mathcal{P} \rightarrow \mathcal{M}_n$ exists. We have that $\mathcal{M}_0 = \mathcal{P}$. Furthermore, if $n < \omega$ and \mathcal{T}_n is the tree played in the n th round then \mathcal{T}_n is based on the window $(i(\delta_{n-1}), i(\delta_n))$. The tree played in the ω th round is played on \mathcal{M}_ω and is above $i(\delta_\omega)$. Notice that not every normal \mathcal{T} will be constructed in exactly $\omega + 1$ non-trivial rounds (where we say that n th round is trivial if I doesn't play any extender from the window $(i(\delta_{n-1}), i(\delta_n))$).

In what follows, whenever we have a premeice like \mathcal{P} above (and these include all (ω, Γ) -suitable premeice), we will think of normal iteration trees on \mathcal{P} as stacks produced via a run of $\mathcal{G}(\mathcal{P}, \omega + 1, \lambda^+)$. Given such a \mathcal{P} and a normal tree \mathcal{T} on \mathcal{P} , we say $\langle \mathcal{T}_n, \mathcal{M}_n : n < k \rangle$ are the normal components of \mathcal{T} where $k \leq \omega + 1$ is the least such that for all $m \geq k$, \mathcal{T}_m is undefined. We say \mathcal{M} is *the last model* of \mathcal{T} if

$$\mathcal{M} = \begin{cases} \mathcal{M}_\alpha^{\mathcal{T}_n} & : k = n + 1, \alpha + 1 = lh(\mathcal{T}_n) \\ \mathcal{M}_\omega & : k = \omega \text{ and } \mathcal{M}_\omega \text{ is the direct limit of } \mathcal{M}_n \text{'s} \\ \text{undefined} & : \text{otherwise.} \end{cases}$$

Definition 1.33 Suppose \mathcal{P} is (ω, Γ) -suitable. We say Σ is an (ω, Γ) -fullness preserving strategy for \mathcal{P} if Σ is an $(\omega^2, \lambda^+ + 1)$ -iteration strategy such that whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \beta \rangle$ is a stack on \mathcal{P} according to Σ the following holds:

1. $\beta \leq \omega^2$,
2. $\mathcal{M}_0 = \mathcal{P}$ and for $\alpha < \beta$, \mathcal{M}_α is (ω, Γ) -suitable,
3. for $\alpha < \beta$, letting $\langle \mathcal{T}_i^\alpha, \mathcal{M}_i^\alpha : i < k_\alpha \rangle$ be the normal components of \mathcal{T}_α we have that

- (a) $k_\alpha \leq \omega$,
- (b) if $\alpha + 1 < \beta$ then $\mathcal{M}_{\alpha+1}$ is the last model of \mathcal{T}_β ,
- (c) for each $i < k_\alpha$ and limit $\xi < lh(\mathcal{T}_i^\alpha)$, $\mathcal{Q}(\mathcal{T}_i^\alpha \upharpoonright \xi)$ exists and

$$\mathcal{Q}(\mathcal{T}_i^\alpha \upharpoonright \xi) \trianglelefteq \mathcal{W}^\Gamma(\mathcal{M}(\mathcal{T}_i^\alpha \upharpoonright \xi)),$$

4. if $\beta = \omega^2$ and \mathcal{M}_{ω^2} is the direct limit of \mathcal{M}_α 's under the iteration embeddings given by $\vec{\mathcal{T}}$ then \mathcal{M}_{ω^2} is (ω, Γ) -suitable.

Next we need to introduce *simultaneous genericity iterations* which we will use to show that the strategies we construct have branch condensation. Simultaneous genericity iteration were used for this purpose in [7] as well. First, however, we need *generic genericity iterations*.

Suppose $N \in H_{\lambda^+}$ and $\mathcal{M} \in H_{\lambda^+}$ is a premouse with a Woodin cardinal δ . We say \mathcal{T} is the *generic N -genericity tree* on \mathcal{M} if \mathcal{T} is a run of $\mathcal{G}(\mathcal{M}, \lambda^+)$ in which I plays as follows: at stage α , $E_\alpha^{\mathcal{T}}$ is the least extender such that for some $p \in \text{Coll}(\omega, N)$, p forces that if x is the generic code of N then x violates some axiom generated by $E_\alpha^{\mathcal{T}}$. If Σ is a $\lambda^+ + 1$ -strategy for \mathcal{M} then it can be shown that generic N -genericity iterations terminate and produce an iteration $i : \mathcal{M} \rightarrow \mathcal{Q}$ such that whenever $g \subseteq \text{Coll}(\omega, N)$ is generic and x is the generic code of N then x is generic over \mathcal{Q} for the extender algebra at $i(\delta)$. The proof of this fact is just like the proof of the same fact for the usual genericity iterations (see [1] or [20]).

Definition 1.34 We say $\langle \mathcal{R}_i, \mathcal{Q}_i, m_i, \sigma_i, \nu_i, \Sigma : i < \omega \rangle$ is a Γ -sequence of triangles with direct limit \mathcal{R}_ω if

1. for $i \leq \omega$, \mathcal{R}_i and \mathcal{Q}_i are (ω, Γ) -suitable premice,
2. $m_i : \mathcal{R}_i \rightarrow_{\Sigma_1} \mathcal{R}_{i+1}$, $\sigma_i : \mathcal{Q}_i \rightarrow_{\Sigma_1} \mathcal{R}_{i+1}$ and $\nu_i : \mathcal{R}_i \rightarrow_{\Sigma_1} \mathcal{Q}_i$ are such that $m_i = \sigma_i \circ \nu_i$,
3. \mathcal{R}_ω is the direct limit of \mathcal{R}_i 's under the embeddings $m_{i,j} =_{\text{def}} m_{j-1} \circ m_{j-2} \circ \cdots \circ m_i$,
4. m_ω , σ_ω , and ν_ω are undefined, and
5. Σ is an (ω, Γ) -fullness preserving strategy for \mathcal{R}_ω .

Suppose $\langle \mathcal{R}_i, \mathcal{Q}_i, m_i, \sigma_i, \nu_i, \Sigma : i \leq \omega \rangle$ is a Γ -sequence of triangles. Then we let $m_{i,\omega} : \mathcal{R}_i \rightarrow \mathcal{R}_\omega$ be the direct limit embedding and $\sigma_{i,\omega} = m_{i+1,\omega} \circ \sigma_i$. We then let Σ^i be $m_{i,\omega}$ -pullback of Σ and Λ^i be $\sigma_{i,\omega}$ -pullback of Σ .

Definition 1.35 (Simultaneous genericity iterations) *Suppose $\langle \mathcal{R}_k, \mathcal{Q}_k, m_k, \sigma_k, \nu_k, \Sigma : k < \omega \rangle$ is a Γ -sequence of triangles with direct limit \mathcal{R}_ω and $\vec{N} = \langle N_k : k < \omega \rangle \subseteq H_{\lambda^+}$. Suppose further that either $\lambda = \omega$ and Σ is an $(\omega^2, \omega_1 + 1)$ -strategy or that $\lambda > \omega$ and for each i , $N_k \in H_\lambda$. We say $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k \leq \omega \rangle$ is the simultaneous \vec{N} -genericity iteration of $\langle \mathcal{R}_k, \mathcal{Q}_k : i < \omega \rangle$ via Σ if the following holds:*

1. $\langle \mathcal{R}_k^0, \mathcal{Q}_k^0, m_k^0, \sigma_k^0, \nu_k^0 : k < \omega \rangle = \langle \mathcal{R}_k, \mathcal{Q}_k, m_k, \sigma_k, \nu_k : k < \omega \rangle$,
2. for all $j, k < \omega$, $\vec{\mathcal{S}}_k^j$ is a non-dropping stack of finite length on \mathcal{R}_k^j based on the window $(\delta_j^{\mathcal{R}_k^j}, \delta_{j+1}^{\mathcal{R}_k^j})$, $\vec{\mathcal{S}}_k^j$ is according to $\Sigma_{\mathcal{R}_k^j, \oplus_{l < j} \vec{\mathcal{S}}_k^l}$, and \mathcal{R}_k^{j+1} is the last model of $\vec{\mathcal{S}}_k^j$,
3. for all $j, k < \omega$, $\vec{\mathcal{W}}_k^j$ is a non-dropping stack of finite length on \mathcal{Q}_k^j based on the window $(\delta_j^{\mathcal{Q}_k^j}, \delta_{j+1}^{\mathcal{Q}_k^j})$, $\vec{\mathcal{W}}_k^j$ is according to $\Lambda_{\mathcal{Q}_k^j, \oplus_{l < j} \vec{\mathcal{W}}_k^l}$, and \mathcal{Q}_k^{j+1} is the last model of $\vec{\mathcal{W}}_k^j$,
4. for $j, k < \omega$, $\sigma_k^j : \mathcal{Q}_k^j \rightarrow \mathcal{R}_{k+1}^j$, $m_k^j : \mathcal{R}_k^j \rightarrow \mathcal{R}_{k+1}^j$, $\nu_k^j : \mathcal{R}_k^j \rightarrow \mathcal{Q}_k^j$, and

$$m_k^j = \sigma_k^j \circ \nu_k^j,$$

5. for each j , $\vec{\mathcal{S}}_0^j$ is the tree of generic N_j -genericity iteration of \mathcal{R}_0^j in which II plays according to $\Sigma_{\mathcal{R}_0^j, \oplus_{l < j} \vec{\mathcal{S}}_0^l}$,
6. for each $k, j < \omega$, letting $\vec{\mathcal{W}}^* = \nu_k^j \vec{\mathcal{S}}_k^j$, $\vec{\mathcal{W}}_k^j = \vec{\mathcal{W}}^* \frown \mathcal{W}$ where, letting \mathcal{M} be the last model of $\vec{\mathcal{W}}^*$, we have that \mathcal{W} is the tree of generic N_j -genericity iteration of \mathcal{M} which is based on the window $(\delta_j^{\mathcal{M}}, \delta_{j+1}^{\mathcal{M}})$ and is according to $\Lambda_{\mathcal{M}, (\oplus_{l < j} \vec{\mathcal{W}}_k^l) \frown \vec{\mathcal{W}}^*}$,

7. for each $k, j < \omega$, letting $\vec{\mathcal{S}}^* = \sigma_k^j \vec{\mathcal{W}}_k^j$, $\vec{\mathcal{S}}_k^j = \vec{\mathcal{S}}^* \frown \mathcal{S}$ where, letting \mathcal{N} be the last model of $\vec{\mathcal{S}}^*$, we have that \mathcal{S} is the tree of generic N_j -genericity iteration of \mathcal{N} which is based on the window $(\delta_j^{\mathcal{N}}, \delta_{j+1}^{\mathcal{N}})$ and is according to $\Sigma_{\mathcal{N}, (\oplus_{l < j} \vec{\mathcal{S}}_k^l)}^{k+1} \frown \vec{\mathcal{S}}^*$,
8. keeping the notation of clause 6 and 7, for each $k, j < \omega$, letting $s_k^j : \mathcal{R}_k^{j+1} \rightarrow \mathcal{M}$ and $w_k^j : \mathcal{Q}_k^{j+1} \rightarrow \mathcal{N}$ be the maps coming from the copying constructions, we have that $\nu_k^{j+1} = i^{\mathcal{W}} \circ s_k^j$, $\sigma_k^{j+1} = i^{\mathcal{S}} \circ w_k^j$ and $m_k^{j+1} = \sigma_k^{j+1} \circ \nu_k^{j+1}$.

Suppose $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k < \omega \rangle$ is as in Definition 1.35. Then we say $\langle \mathcal{R}_k^\omega, \mathcal{Q}_k^\omega, \vec{\mathcal{S}}_k^\omega, \vec{\mathcal{W}}_k^\omega, m_k^\omega, \sigma_k^\omega, \nu_k^\omega : k < \omega \rangle$ is the direct limit of $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k < \omega \rangle$ if

1. for each $k \leq \omega$, \mathcal{R}_k^ω is the direct limit of \mathcal{R}_k^j 's under $i^{\vec{\mathcal{S}}_k^j}$'s,
2. for each $k < \omega$, \mathcal{Q}_k^ω is the direct limit of \mathcal{Q}_k^j 's under $i^{\vec{\mathcal{W}}_k^j}$'s,
3. for each $k < \omega$, $\sigma_k^\omega, \nu_k^\omega$ and m_k^ω come from direct limit constructions.

Notice that we have that for each $k < \omega$, $\sigma_k^\omega : \mathcal{Q}_k^\omega \rightarrow \mathcal{R}_{k+1}^\omega$, $m_k^\omega : \mathcal{R}_k^\omega \rightarrow \mathcal{R}_{k+1}^\omega$, $\nu_k^\omega : \mathcal{R}_k^\omega \rightarrow \mathcal{Q}_k^\omega$, and

$$m_k^\omega = \sigma_k^\omega \circ \nu_k^\omega,$$

We also let $\mathcal{R}_\omega^\omega$ be the direct limit of \mathcal{R}_k^ω 's under m_k^ω 's. We say $\mathcal{R}_\omega^\omega$ is the direct limit of $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{\mathcal{S}}_k^j, \vec{\mathcal{W}}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k \leq \omega \rangle$. Notice that, by the copying construction, $\mathcal{R}_\omega^\omega$ is a Σ -iterate of \mathcal{R}_ω and the length of the stack producing the iteration is ω^2 .

1.7 Review of HOD analysis

In this section, we review HOD analysis of models satisfying $AD^+ + V = L(\mathcal{P}(\mathbb{R})) + MC + \Theta = \theta_0$. Until the end of this subsection, we assume V satisfies the above theory. We let $\Gamma = \mathcal{P}(\mathbb{R})$. For the duration of this subsection, we will drop Γ -from our notation. Thus, a suitable premouse is a Γ -suitable premouse and etc.

Suppose \mathcal{P} is suitable and $A \subseteq \mathbb{R}$ is OD. We say \mathcal{P} weakly term captures A if letting $\delta = \delta^{\mathcal{P}}$, for each $n < \omega$ there is a term relation $\tau \in \mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$ such that for comeager many \mathcal{P} -generics, $g \subseteq Coll(\omega, (\delta^{+n})^{\mathcal{P}})$, $\tau_g = \mathcal{P}[g] \cap A$. We say \mathcal{P} term captures A if the equality holds for all generics. The following lemma is essentially due to Woodin and the proof can be found in [14].

Lemma 1.36 *Suppose \mathcal{P} is suitable and $A \subseteq \mathbb{R}$ is OD. Then \mathcal{P} weakly term captures A . Moreover, there is a suitable \mathcal{Q} which term captures A .*

Given a suitable \mathcal{P} and an OD set of reals A , we let $\tau_{A,n}^{\mathcal{P}}$ be the standard name for a set of reals in $\mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$ witnessing the fact that \mathcal{P} weakly captures A . We then define $f_A \in F(\Gamma)$ by letting

$$f_A(\mathcal{P}) = \langle \tau_{A,n}^{\mathcal{P}} : n < \omega \rangle.$$

Let $F_{od} = \{f_A : A \subseteq \mathbb{R} \wedge A \in OD\}$.

All the notions we have defined in Section 1.10, Section 1.4 and Section 1.5 can be redefined for ordinal definable sets $A \subseteq \mathbb{R}$ using f_A as the relevant function. To save some ink, in what follows, we will say A -iterable instead of f_A -iterable and similarly for other notions. Also, we will use A in our subscripts instead of f_A .

The following lemma is one of the most fundamental lemmas used to compute HOD and it is originally due to Woodin. Again, the proof can be found in [14].

Theorem 1.37 *For each $f \in F_{od}$, there is $\mathcal{P} \in S(\Gamma)$ which is (F_{od}, f) -quasi iterable.*

Let $\mathcal{M}_\infty = \mathcal{M}_{\infty, F_{od}}$.

Theorem 1.38 (Woodin, [14]) $\delta^{\mathcal{M}_\infty} = \Theta$, $\mathcal{M}_\infty \in \text{HOD}$ and $\mathcal{M}_\infty|_\Theta = (V_\Theta^{\text{HOD}}, \vec{E}^{\mathcal{M}_\infty|_\Theta}, \in)$.

Finally, if $a \in H_{\omega_1}$, then we could define $\mathcal{M}_\infty(a)$ by working with suitable premice over a . Everything we have said about suitable premice can also be said about suitable premice over a and in particular, the equivalent of Theorem 1.38 can be proven using $\text{HOD}_{a \cup \{a\}}$ instead of HOD and $\mathcal{M}_\infty(a)$ instead of \mathcal{M}_∞ .

2 The proof of Theorem 0.8

The rest of this paper is devoted to the proof of Theorem 0.8. From now on κ is as in the hypothesis of Theorem 0.8. Given $a \subseteq V_\kappa$, let

$$Lp(a) = \cup \{ \mathcal{N} : \mathcal{N} \text{ is a sound countably iterable mouse over } a \text{ such that } \rho_\omega(\mathcal{N}) = a \}.$$

We define $\langle Lp_\xi(a) : \xi < \kappa^+ \rangle$ by the following recursion:

1. for $\alpha < \kappa^+$, $Lp_{\alpha+1} = Lp_1(Lp_\alpha(a))$,
2. for $\lambda < \kappa^+$ limit, $Lp_\lambda(a) = \bigcup_{\alpha < \lambda} Lp_\alpha(a)$.

We will use the following fundamental result throughout this paper.

Theorem 2.1 (Schimmerling-Zeman, [12]) *For all $A \subseteq V_\kappa$, $Lp(A) \models \square_\kappa$.*

From now on we fix $A \subseteq \kappa$ such that A codes V_κ . Because $\neg \square_\kappa$, we must have that $o(Lp(A)) < \kappa^+$. Let $\xi = o(Lp(A))$. Because κ is singular, we have that $\text{cf}(\xi) < \kappa$.

Definition 2.2 *We say $\mu < \kappa$ is a good point if $\text{cf}(\kappa), \text{cf}(\xi) < \mu$, μ is regular and $\mu^\omega = \mu$.*

Clearly there are good points. Suppose then that μ is a good point and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Working in $V[g]$, we let

$$\Gamma_{\mu,g} = \{B \subseteq \mathbb{R} : L(B, \mathbb{R}) \models AD^+\}.$$

The following theorem shows that $\Gamma_{\mu,g}$ is not empty. Its proof is essentially Steel's proof that $L(\mathbb{R}) \models AD$ (see [17]). See [7] for the definition of $L^\Sigma(\mathbb{R})$. It is the minimal Σ -mouse over \mathbb{R} which contains all the ordinals and has no extenders on its sequence¹.

Theorem 2.3 *Suppose μ is a good point and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Suppose in $V[g]$, \mathcal{M} is a κ^+ -iterable countable mouse over some set X and that $\rho_\omega(\mathcal{M}) = X$. Let Σ be the κ^+ -iteration strategy of \mathcal{M} . Then $L^\Sigma(\mathbb{R}) \models AD^+$. Hence, in $V[g]$, letting $\Lambda = \Sigma \upharpoonright H_{\omega_1}$,*

$$\text{Code}(\Lambda) \in \Gamma_{\mu,g}.$$

Our goal is to show that for some good μ and generic $g \subseteq \text{Coll}(\omega, \mu)$, there is $B \in \Gamma_{\mu,g}$ such that

$$L(B, \mathbb{R}) \models AD^+ + \Theta = \theta_1.$$

We can then use Theorem 1.9 and the homogeneity of the forcing to prove Theorem 0.8. Let then (*) be the following statement: For any good μ and a generic $g \subseteq \text{Coll}(\omega, \mu)$, in $V[g]$, there is no inner model M such that $\text{Ord}, \mathbb{R} \subseteq M$ and

¹Below we stop feeding Σ after stage κ^+ .

$$M \models AD^+ + \Theta = \theta_1 \quad (*).$$

Towards a contradiction we assume that (*) holds.

Borrowing some lemmas from the next subsection we can characterize sets in $\Gamma_{\mu,g}$ in terms of \mathbb{R} -mice.

Theorem 2.4 *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Let $B \in \Gamma_{\mu,g}$. Then, in $V[g]$, there is a sound \mathbb{R} -mouse \mathcal{N} such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$, $B \in \mathcal{N}$ and countable submodels of \mathcal{N} are κ^+ -iterable.*

Proof. Fix g and B as in the hypothesis. We work in $V[g]$. We have that $L(B, \mathbb{R}) \models MC + \Theta = \theta_0$. It follows from the main theorem of [8] that in $L(B, \mathbb{R})$ there is a countably iterable sound \mathbb{R} -mouse \mathcal{N} such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$ and $B \in \mathcal{N}$. Fix such a mouse \mathcal{N} . It is then enough to show that, in $V[g]$, countable submodels of \mathcal{N} are κ^+ -iterable. To see this, let $\pi : \mathcal{S} \rightarrow \mathcal{N}$ be a countable submodel of \mathcal{M} . We have that $L(B, \mathbb{R}) \models \text{“}\mathcal{S} \text{ is } \omega_1\text{-iterable”}$. It then follows from clause 2 of Lemma 2.10 (in particular, see clause 2c and 2d) that \mathcal{S} is κ^+ -iterable. \square

Definition 2.5 *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Working in $V[g]$, we let $\mathcal{S}_{\mu,g}^-$ be the union of those sound \mathbb{R} -mice \mathcal{N} such that $\rho_\omega(\mathcal{N}) = \mathbb{R}$, countable submodels of \mathcal{N} are κ^+ -iterable and there is a set of reals $B \in \Gamma_{\mu,g}$ such that B codes \mathcal{N} . We let $\mathcal{S}_{\mu,g} = L(\mathcal{S}_{\mu,g}^-)$.*

Theorem 2.6 *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Then in $V[g]$, $\mathcal{S}_{\mu,g} \models AD^+ + \theta_0 = \Theta$.*

Proof. It follows from the previous theorem that $\mathcal{P}(\mathbb{R})^{\mathcal{S}_{\mu,g}^-} = \Gamma_{\mu,g}$. Hence, $\mathcal{S}_{\mu,g}^- \models AD^+$. Also, notice that $\mathcal{S}_{\mu,g}^- \models \theta_0 = \Theta$. This is because otherwise there is $B \in \Gamma_{\mu,g}$ such that $w(B) = \theta_0^{\mathcal{S}_{\mu,g}^-}$. It then follows that $L(B, \mathbb{R}) \models \theta_0 < \Theta$ which contradicts (*). But now, using the scales analysis of [16] and [19] and the core model induction of [17], we get that $\mathcal{S}_{\mu,g} \models AD^+$. It then follows from (*) that $\mathcal{S}_{\mu,g} \models AD^+ + \theta_0 = \Theta$. \square

Suppose μ is good. We let $\Theta^\mu = \Theta^{\mathcal{S}_{\mu,g}}$ and if $a \in H_{\mu^+}$ then we let $\mathcal{P}_{\mu,a} = (\mathcal{M}_\infty(a))^{\mathcal{S}_{\mu,g}}$ where $g \subseteq \text{Coll}(\omega, \mu)$ is some generic. Notice that $\mathcal{P}_{\mu,a}$ is independent of g and $\mathcal{P}_{\mu,a} \in V$. We let $\mathcal{P}_\mu = \mathcal{P}_{\mu,\emptyset}$. Given a good hull (M, π) at μ such that a is in $M[g]$ and is countable there, we let $\mathcal{P}_{\mu,a}^M = \pi^{-1}(\mathcal{P}_{\mu,a}^M)$ and $\mathcal{S}_{\mu,g}^M = (L(\pi^{-1}(\mathcal{S}_{\mu,g})))^{M[g]}$.

2.1 Good points and good hulls

Clearly there are good points. Let μ be a good point. Recall that $\xi = o(Lp(A))$. Let $\nu = \text{cf}(\xi)$ and let $f : \nu \rightarrow \xi$ be an increasing cofinal function. Let $\zeta = \kappa^{+\omega}$.

Definition 2.7 *We say (M, π) is a good hull at μ if $\pi : M \prec V_\zeta$ is such that $\mu + 1 \subseteq M$, $|M| = \mu$, $M^\omega \subseteq M$ and $\{A, f\} \in \text{ran}(\pi)$.*

An easy Skolem hull argument shows that there are good hulls at μ . If (M, π) is a good hull at μ then we let $\kappa_{M,\pi} = \pi^{-1}(\kappa)$ and $A_{M,\pi} = \pi^{-1}(A)$. Often times, when it is clear what π is, we will omit it from subscripts. The fact that M is countably closed implies that M is *full* with respect to countably iterable mice. The proof of this lemma is essentially the covering argument.

Lemma 2.8 *Suppose μ is good and (M, π) is a good hull at μ . Then $Lp(A_M) \in M$.*

Proof. We only outline the proof of this well-known fact. Suppose not. Let $\xi_M = \pi^{-1}(\xi)$. Let $\mathcal{M} \trianglelefteq Lp(A_M)$ be the least such that $\rho_\omega(\mathcal{M}) = A_M$ and $\mathcal{M} \notin M$. Let E be the (κ_M, κ) -extender from π and let $\mathcal{M}^* = \text{Ult}(\mathcal{M}, E)$. Then \mathcal{M}^* is countably iterable (because E is countably closed) and is sound A -mouse such that $\rho_\omega(\mathcal{M}^*) = A$. Hence, $\mathcal{M}^* \trianglelefteq Lp(A)$. But because $\pi \upharpoonright \xi_M$ is cofinal in ξ , we get that $Lp(A) \triangleleft \mathcal{M}^*$, contradiction. \square

The following lemma is our main tool for extending iteration strategies to κ^+ -iteration strategies. It is essentially due to Steel (see Lemma 1.25 of [17]) and we leave the proof to the readers.

Lemma 2.9 *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Then the following holds.*

1. *Suppose \mathcal{M} is a sound premouse over some set $X \in V_\kappa[g]$ such that $\rho_\omega(\mathcal{M}) = X$ and in $V[g]$, \mathcal{M} is κ -iterable. Then \mathcal{M} is κ^+ -iterable.*
2. *Suppose $B \in \Gamma_{\mu,g}$ and $\Gamma \subset (\Delta_1^2)^{L(B, \mathbb{R})}$ is a good pointclass. Suppose $(\mathcal{P}, \Lambda) \in V[g]$ is such that in $V[g]$, \mathcal{P} is countable and Γ -suitable, and Λ is a Γ -fullness preserving (ω_1, ω_1) -iteration strategy for \mathcal{P} with branch condensation. Then Λ can be extended to a (κ^+, κ^+) -iteration strategy which acts on non-dropping trees and has the branch condensation.*

Let Σ be the extension of Λ mentioned in clause 2 of Lemma 2.9. Then notice that it follows from branch condensation that if $\vec{\mathcal{T}}$ is a stack according to Σ and $\pi : N \rightarrow V_\zeta[g]$ is such that N is countable in $V[g]$ and $\vec{\mathcal{T}} \in \text{rng}(\pi)$ then $\pi^{-1}(\vec{\mathcal{T}})$ is according to Λ . Moreover, whenever $\mathcal{Q} \in I(\mathcal{P}, \Sigma)$, η is a cutpoint of \mathcal{Q} and \mathcal{M} is a mouse over $\mathcal{Q}|_\eta$ such that, in $V[g]$, its countable hulls have iteration strategy in Γ , then $\mathcal{M} \trianglelefteq \mathcal{Q}$.

The following lemma will be used to show that good hulls are correct about κ^+ -iterable mice.

Lemma 2.10 *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Working in $V[g]$, suppose $B \in \Gamma_{\mu, g}$ and let $\Gamma^* = \mathcal{P}(\mathbb{R}) \cap L(B, \mathbb{R})$. Suppose that for some countable set X , $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma^*}(X)$ is such that $\rho_\omega(\mathcal{M}) = \omega$. Then, in $V[g]$, there are*

1. a good pointclass² $\Gamma \subset (\underline{\Delta}_1^2)^{L(B, \mathbb{R})}$,
2. a Γ -suitable \mathcal{P} over X which has an (ω_1, ω_1) -iteration strategy $\Lambda \in L(B, \mathbb{R})$ with branch condensation and a (κ^+, κ^+) -iteration strategy Σ with branch condensation which acts on non-dropping trees such that
 - (a) $L(B, \mathbb{R}) \models$ “ Λ is Γ -fulness preserving”,
 - (b) Λ and Σ agree on $\text{dom}(\Sigma) \cap H_{\omega_1}$,
 - (c) there is some α such that if \mathcal{N}_α is the α th model of $(L[\vec{E}][X])^\mathcal{P}$ -construction then $\text{Core}(\mathcal{N}_\alpha) = \mathcal{M}$,
 - (d) for any $\nu < \kappa$ and for any $h \subseteq \text{Coll}(\omega, \nu)$, Σ can be extended to a (κ^+, κ^+) -iteration strategy in $V[g * h]$ and
 - (e) if $X \in V$ then $\mathcal{P} \in V$ and $\Sigma \upharpoonright V \in V$.

Proof. Because \mathcal{M} is iterable in $L(B, \mathbb{R})$, we can find a good pointclass $\Gamma \subset (\underline{\Delta}_1^2)^{L(B, \mathbb{R})}$ such that \mathcal{M} has an iteration strategy coded into Γ . Let $\vec{C} \in L(B, \mathbb{R})$ be a Γ -sjs³ and let $\Gamma_1 \subset (\underline{\Delta}_1^2)^{L(B, \mathbb{R})}$ be a good pointclass such that $\vec{C} \in \underline{\Delta}_{\Gamma_1}$. Let $F : \mathbb{R} \rightarrow L(B, \mathbb{R})$ be a function such that for cone of x , $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures Γ_1 (as in Theorem 1.2.9 of [7]).

²Recall a good pointclass is a pointclass which is closed under existential real quantification, is ω -parametrized and has the scale property.

³Recall that a Γ -sjs is a countable subset S of Γ such each $A \in S$ has a Γ -scale all of whose norms are in S , see [14] or [17] for more on sjs.

Let $x \in \mathbb{R}$ be such that X is coded by x , $F(x)$ is defined and \vec{C} is Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$. It follows that $\mathcal{W}^\Gamma(\mathcal{N}_x^*|\delta_x) \in \mathcal{N}_x^*$. Because sjs's condense to themselves, by a Skolem hull argument we get that there are club many η such that $\mathcal{W}^\Gamma(\mathcal{N}_x^*|\eta) \models$ “ η is Woodin”. Let then η be the least such that $\mathcal{W}^\Gamma(\mathcal{N}_x^*|\eta) \models$ “ η is Woodin”. It then follows that if $\mathcal{P}^- = (L[\vec{E}][X])^{\mathcal{N}_x^*|\eta}$ and $\mathcal{P} = Lp_\omega^\Gamma(\mathcal{P}^-)$ then \mathcal{P} is Γ -suitable over X .

Appealing to universality (see Lemma 1.1.26 of [7]), we get that $\mathcal{M} \triangleleft \mathcal{P}$ and that it is reached by $(L[\vec{E}][X])^\mathcal{P}$ -construction of \mathcal{P} . Working in $V[g]$, let $\Lambda \in L(B, \mathbb{R})$ be the (ω_1, ω_1) -iteration strategy of \mathcal{P} induced by Σ_x (recall that background constructions inherit a strategy from the background as in [5]). It then follows that $L(B, \mathbb{R}) \models$ “ Λ is Γ -fullness preserving” (one can use an argument like the one in Lemma 3.2.3 of [7]). Using Steel’s lifting techniques from [17] which was summarized in clause 2 of Lemma 2.9, we can lift Λ to a (κ^+, κ^+) -iteration strategy Σ such that Σ acts on trees with no drops. We now have that $(\mathcal{P}, \Gamma, \Lambda, \Sigma)$ satisfies 1 and 2a-2c.

To get 2d, we can use generic comparisons. First, we show that Σ can be extended to a (κ, κ) -iteration strategy in $V[g * h]$ where for some $\nu < \kappa$, $h \subseteq \text{Coll}(\omega, \nu)$ is $V[g]$ -generic. Working in $V[g]$, let $\pi : M \rightarrow V_\zeta[g]$ be a countable hull such that $(\mathcal{P}, \Sigma) \in \text{rng}(\pi)$. Let $\kappa_M = \pi^{-1}(\kappa)$ and let $\Sigma^M = \pi^{-1}(\Sigma)$. It is enough to show that 2d holds in M for κ_M and Σ^M . We have that $\Sigma^M = \Sigma \upharpoonright V_{\kappa_M}^M$. Let $\nu < \kappa_M$ and let $k \subseteq \text{Coll}(\omega, \nu)$ be M -generic. Let

$$D_{\nu, k} = \{(a, \mathcal{W}^\Gamma(a)) : a \in V_{\kappa_M}^{M[k]}\}.$$

Let $C = \{(x, y) : x, y \in \mathbb{R} \wedge \text{“}y \text{ codes an } \mathcal{M} \triangleleft \mathcal{W}^\Gamma(x) \text{ such that } \rho_\omega(\mathcal{M}) = x\text{”}\}$. Without loss of generality, we can assume that Λ respects C . Notice that if $i : \mathcal{P} \rightarrow \mathcal{Q}$ comes from an iteration according to Λ and $x \in \mathbb{R}$ is generic over \mathcal{Q} for $\mathbb{B}^\mathcal{Q}$ then

$$\mathcal{W}^\Gamma(x) = \cup\{\mathcal{M} : \mathcal{M} \in \mathcal{Q}[x] \text{ is coded by } y \in \mathbb{R}^{\mathcal{Q}[x]} \text{ and } \mathcal{Q}[x] \models y \in \tau_C^\mathcal{Q}\} \quad (1).$$

For each $\eta < \kappa_M$ let $\mathcal{P}_\eta \in M$ be the Σ^M -iterate of \mathcal{P} which is obtained via the H_η^M -generic genericity iteration. Let $i_\eta : \mathcal{P} \rightarrow \mathcal{P}_\eta$ be the iteration embedding according to Σ^M .

It then follows from (1) that $D_{\nu, k} \in M[k]$. This is because $(a, \mathcal{N}) \in D_{\nu, k}$ iff whenever $\eta \in (\nu, \kappa_M)$, $l \subseteq \text{Coll}(\omega, \eta)$ is M -generic, x codes a and

$$\mathcal{S} = \cup\{\mathcal{M} : \mathcal{M} \in \mathcal{P}_\eta[x] \text{ is coded by } y \in \mathbb{R}^{\mathcal{P}_\eta[x]} \text{ and } \mathcal{P}_\eta[x] \models y \in i_\eta(\tau_C^\mathcal{P})\}$$

then \mathcal{N} is obtained from \mathcal{S} via S -constructions (see [7] or [13]).

Because $D_{\nu,k} \in M[k]$ for all ν, k , we can do generic comparisons in M . Suppose $\nu < \kappa_M$ is an M -cardinal and $k \subseteq \text{Coll}(\omega, \nu)$ is M -generic. Then we let $\mathcal{S}_\eta \in M$ be the Σ^M -iterate of \mathcal{P} which is obtained by generically comparing all Γ -suitable \mathcal{Q} 's which are in $H_{\omega_1}^{M[k]}$ (see [14] or [17]). We have that $\mathcal{S}_\eta \in M$ and whenever $l \subseteq \text{Coll}(\omega, \nu)$ is M -generic then \mathcal{S}_η is the result of generically comparing all Γ -suitable \mathcal{Q} 's which are in $H_{\omega_1}^{M[l]}$. Let $j_\eta : \mathcal{P} \rightarrow \mathcal{S}_\eta$ be the iteration map.

Fix now some $\nu < \kappa_M$ and let $h \subseteq \text{Coll}(\omega, \nu)$ be M -generic. Let $\vec{\mathcal{T}} \in H_{\omega_1}^{M[h]}$ be a stack on \mathcal{P} which is according to Σ such that the last component of $\vec{\mathcal{T}}$ is of limit length. We let \mathcal{Q} be the last model of $\vec{\mathcal{T}}$. Then $\mathcal{Q} \in M[h]$ is Γ -suitable. We then have that $\Sigma(\vec{\mathcal{T}}) = b$ iff b is the unique branch of $\vec{\mathcal{T}}$ such that $\mathcal{M}_b^\vec{\mathcal{T}} = \mathcal{Q}$ and there is $\sigma : \mathcal{Q} \rightarrow \mathcal{S}_\nu$ such that

$$j_\eta = \sigma \circ i_b^\vec{\mathcal{T}}.$$

Let $\phi[\vec{\mathcal{T}}, \mathcal{Q}, b, \mathcal{S}_\nu]$ be the formula on the right side of the equivalence. It follows from absoluteness that $\Sigma(\vec{\mathcal{T}}) = b$ iff $M \models \phi[\vec{\mathcal{T}}, \mathcal{Q}, b, \mathcal{S}_\nu]$. Hence, $\Sigma \upharpoonright V_{\kappa_M}^{M[g]} \in M[g]$. Using the elementarity of π we get that Σ can be extended to a (κ, κ) -strategy in $V[g * h]$ where for some $\nu < \kappa$, $h \subseteq \text{Coll}(\omega, \nu)$ is $V[g]$ -generic. Σ can then be extended to a (κ^+, κ^+) -strategy by using clause 2 of Lemma 2.9. We then get that $(\mathcal{P}, \Gamma, \Lambda, \Sigma)$ also satisfies 2d. It remains to show that 2e can also be satisfied.

Notice that if $X \in V$ then $\mathcal{M} \in V$. Let then $\mathcal{R} = \mathcal{M}_\infty(\mathcal{P}, \Lambda)$. By homogeneity of the collapse, we have that $\mathcal{R} \in V$. Let now (M, π) be a good hull at μ such that $(\mathcal{R}, \Sigma_{\mathcal{R}}) \in \text{rng}(\pi)^4$ and \mathcal{P} is countable in $M[g]$. Let $\mathcal{S} = \pi^{-1}(\mathcal{R})$. Then \mathcal{S} is a Λ -iterate of \mathcal{P} and $\mathcal{S} \in V$. It follows that $(\mathcal{S}, \Gamma, \Lambda_{\mathcal{S}}, \Sigma_{\mathcal{S}})$ satisfies 1 and 2a-2d. We claim that $(\mathcal{S}, \Sigma_{\mathcal{S}})$ satisfies 2e. The proof is just like the proof of 2d above using the fact that, by the homogeneity of the collapse, if G is the function $a \rightarrow \mathcal{W}^\Gamma(a)$ then $G \upharpoonright V$ is definable in V . We leave the details to the reader. \square

Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Working in $V[g]$, for $a \in V_\kappa[g]$ we let

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}^{(\mathcal{P}(\mathcal{P}(\kappa)))^{V[g]}}(a).$$

Lemma 2.11 *Suppose μ is a good point and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Then the following holds.*

1. For any $a \in H_{\omega_1}^{V[g]}$, $\mathcal{W}_{\mu,g}(a) = \mathcal{W}^{\Gamma_{\mu,g}}(a)$.

⁴It is shown in [7] that if \mathcal{P} is Γ -suitable, Σ is a Γ -fullness preserving iteration strategy for \mathcal{P} with branch condensation and $\mathcal{R} \in I(\mathcal{P}, \Sigma)$ is an iterate of \mathcal{P} via some $\vec{\mathcal{T}}$ then $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. This is the reason for omitting the stack from the subscript in $\Sigma_{\mathcal{R}}$. We will do this throughout the paper.

2. Suppose (M, π) is a good hull at μ and $a \in V_{\kappa_M}^{M[g]}$. Then

$$\mathcal{W}_{\mu,g}(a) \in M.$$

3. Suppose $\nu > \mu$ is also good and let $h \subseteq \text{Coll}(\omega, \nu)$ be $V[g]$ -generic. Suppose a is countable in $V[g]$. Then

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}_{\nu,h}(a).$$

Proof.

1. It follows from Theorem 2.3 that $\mathcal{W}_{\mu,g}(a) \trianglelefteq \mathcal{W}^{\Gamma_{\mu,g}}(a)$. Suppose then $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. Let $B \in \Gamma_{\mu,g}$ be such that $L(B, \mathbb{R}) \models$ “ \mathcal{M} is ω_1 -iterable”. It then follows from 2c of Lemma 2.10 that \mathcal{M} is κ^+ -iterable. Hence, $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$.

2. We work in $V[g]$. First it follows from Lemma 2.8 that $\mathcal{W}_{\mu,g}(A_M) \in M$. It follows from clause 1 that for every $a \in V_{\kappa_M}^M$, $\mathcal{W}_{\mu,g}(a) \in \mathcal{W}_{\mu,g}(A_M)$. Hence, clause 2 follows.

3. It follows from part 1 that it is enough to show that $\mathcal{W}^{\Gamma_{\mu,g}}(a) = \mathcal{W}^{\Gamma_{\nu,h}}(a)$. Suppose then that $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. It follows from 2d of Lemma 2.10 that \mathcal{M} is κ^+ -iterable in $V[h]$. Hence, $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\nu,h}}(a)$.

Conversely, suppose $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\nu,h}}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$. It then follows from 2e of Lemma 2.10 that \mathcal{M} is κ^+ -iterable in $V[g]$ implying that $\mathcal{M} \triangleleft \mathcal{W}^{\Gamma_{\mu,g}}(a)$.

□

Next we would like to show that good hulls capture κ^+ -iterable mice, i.e., if (M, π) is good and $a \in H_{\kappa_M}^M$ then $\mathcal{W}_{\mu,g}(a) = (\mathcal{W}_{\mu,g}(a))^{M[g]}$. For this we will need the following lemma which will come handy later on as well.

Lemma 2.12 *Suppose μ is good and let $g \subseteq \text{Coll}(\omega, \mu)$ be generic. Then in $V[g]$, \mathcal{P}_μ is $\mathcal{P}(\mathcal{P}(\kappa))$ -short tree iterable.*

Proof. Working in $\mathcal{S}_{\mu,g}$, let \mathcal{Q} be a $\Gamma_{\mu,g}$ -suitable premouse which is F_{od} -quasi-iterable. Let (M, π) be a good hull at μ such that $\mathcal{Q} \in M$ and $M[g] \models$ “ \mathcal{Q} is countable”. By elementarity of π , it is enough to show that

$M[g] \models \text{“}\mathcal{P}_\mu^M \text{ is } \mathcal{P}(\mathcal{P}(\kappa_M))\text{-short tree iterable”}$.

We verify this only for trees. The proof for stacks is very similar and only notationally more involved. To show this for trees then it is enough to show that whenever $\mathcal{T} \in (H_{(\kappa_M^+)^M})^{M[g]}$ is a tree on \mathcal{P}_μ such that $M[g] \models \text{“}\mathcal{T} \text{ is } \mathcal{P}(\mathcal{P}(\kappa_M))\text{-short”}$ then there is $b \in M[g]$ such that b is a cofinal well-founded branch of \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq (\mathcal{W}_{\mu,g}(\mathcal{M}(\mathcal{T})))^M$. Fix then such a tree \mathcal{T} and let $\mathcal{N} = \mathcal{Q}(\mathcal{T})$. Notice that M is countable in $\mathcal{S}_{\mu,g}$ and in $\mathcal{S}_{\mu,g}$, \mathcal{P}_μ^M is an F_{od} -quasi-iterate of \mathcal{Q} . It then follows that

$$\mathcal{S}_{\mu,g} \models \text{“}\mathcal{P}_\mu^M \text{ is } \Gamma_{\mu,g}\text{-short tree iterable”} \quad (1).$$

It follows from elementarity of π that $\pi(\mathcal{N})$ is κ^+ -iterable in $V[g]$ and hence, \mathcal{N} is κ^+ -iterable in $V[g]$. Therefore, using clause 1 of Lemma 2.11, we get that

$$\mathcal{S}_{\mu,g} \models \text{“}\mathcal{N} \text{ is } \omega_1\text{-iterable”} \quad (2).$$

Combining (1) and (2), we get a cofinal branch b of \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T})$ exists and $\mathcal{N} = \mathcal{Q}(b, \mathcal{T})$. By absoluteness $b \in M[g]$. \square

Theorem 2.13 (Correctness of good hulls) *Suppose μ is good and $g \subseteq \text{Coll}(\omega, \mu)$ is generic. Suppose $a \in V_{\kappa_M}^{M[g]}$. Then*

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}_{\mu,g}(a)^{M[g]}$$

Proof. Notice that because of π , $(\mathcal{W}_{\mu,g}(a))^M \trianglelefteq \mathcal{W}_{\mu,g}(a)$. It is then enough to show that if $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$ is such that $\rho_\omega(\mathcal{M}) = a$ then $M \models \text{“}\mathcal{M} \text{ is } \kappa_M^+\text{-iterable”}$. Because of Lemma 2.9, it is enough to show that $M \models \text{“}\mathcal{M} \text{ is } \kappa_M\text{-iterable”}$. It follows from clause 2 of Lemma 2.11 that $\mathcal{M} \in M$.

Next, we have that the set

$$D = \{(x, \mathcal{W}_{\mu,g}(x)) : x \in V_{\kappa_M}^M\}$$

is OD_{A_M} and hence, by MC^5 in $\mathcal{S}_{\mu,g}$ and by clause 1 of Lemma 2.11, $D \in \mathcal{W}_{\mu,g}(A_M)$. Since $\mathcal{W}_{\mu,g}(A_M) \in M$, $D \in M$. Fix then some $\eta < \kappa_M$ such that $a \in H_\eta^M$. Let \mathcal{Q}_η be the result of a generic genericity iteration of \mathcal{P}_μ^M making H_η generically generic. Notice that because $D \in M$

⁵ MC stands for mouse capturing. It was shown in [7] that it holds in the minimal model of $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. For more on MC see [7] or [18].

and \mathcal{P}_μ^M is $\Gamma_{\mu,g}$ -short tree iterable we have that the entire iteration can be done in M . This iteration produces a $\Gamma_{\mu,g}$ -correctly guided tree \mathcal{T} with last model \mathcal{Q}_η (where “last model” is in the sense of Definition 1.12).

Working in $S_{\mu,g}$, let Λ be the ω_1 -iteration strategy of \mathcal{M} . We have that whenever \mathcal{Q} is $\Gamma_{\mu,g}$ -suitable and \mathcal{M} is generic over \mathcal{Q} then $\Lambda \upharpoonright H_{\omega_2}^{\mathcal{Q}[\mathcal{M}]} \in \mathcal{Q}[\mathcal{M}]$. It then follows that $\mathcal{Q}[\mathcal{M}] \models$ “ \mathcal{M} is ω_2 -iterable”. We let $\Lambda^{\mathcal{Q}} = \Lambda \upharpoonright H_{\omega_2}^{\mathcal{Q}[\mathcal{M}]}$. Notice that $\mathcal{Q}[\mathcal{M}]$ satisfies that “ $\Lambda^{\mathcal{Q}}$ is the unique ω_2 -iteration strategy of \mathcal{M} ”.

We can now show that $\Lambda \upharpoonright V_{\kappa_M}^{M[g]} \in M[g]$ as follows. Given an iteration tree $\mathcal{T} \in V_{\kappa_M}^{M[g]}$ on \mathcal{M} which is according to Λ and has a limit length, we have that $\Lambda(\mathcal{T}) = b$ iff the following holds in $M[g]$: for all $\eta < \kappa_M$ such that $\mathcal{T} \in H_\eta^{M[g]}$, \mathcal{T} is according to $\Lambda^{\mathcal{Q}_\eta}$ and $b = \Lambda^{\mathcal{Q}_\eta}(\mathcal{T})$. Hence, $\Lambda \upharpoonright V_{\kappa_M}^{M[g]} \in M[g]$. \square

The following are important corollaries.

Corollary 2.14 *Suppose μ is good and let $g \subseteq \text{Coll}(\omega, \mu)$ be generic. Let $\eta < o(\mathcal{P}_\mu)$. Then*

$$\mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta) = \mathcal{O}_\eta^{\mathcal{P}_\mu}.$$

Proof. Suppose first that $\mathcal{M} \trianglelefteq \mathcal{O}_\eta^{\mathcal{P}_\mu}$ is such that $\rho_\omega(\mathcal{M}) = \eta$. To show that $\mathcal{M} \trianglelefteq \mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta)$ it is enough to show that \mathcal{M} is κ^+ -iterable. To show this, it is enough to show the following claim.

Claim. Suppose μ is good and let $g \subseteq \text{Coll}(\omega, \mu)$ be generic. Let $\eta < o(\mathcal{P}_\mu)$ and let $\mathcal{M} \triangleleft \mathcal{O}_\eta^{\mathcal{P}_\mu}$ be such that $\rho_\omega(\mathcal{M}) = \eta$. Then, in $V[g]$, \mathcal{M} is κ^+ -iterable.

Proof. Let (M, π) be a good hull. It is enough to show the claim in M . Let then $\eta < o(\mathcal{P}_\mu^M)$ and let $\mathcal{M} \triangleleft \mathcal{O}_\eta^{\mathcal{P}_\mu^M}$ be such that $\rho_\omega(\mathcal{M}) = \eta$. We need to see that \mathcal{M} is κ_M^+ -iterable in M . Notice that since \mathcal{P}_μ^M is $\Gamma_{\mu,g}$ -suitable, it follows from clause 1 of Lemma 2.11 that $\mathcal{M} \trianglelefteq \mathcal{W}^{\Gamma_{\mu,g}}(\mathcal{P}_\mu^M|\eta)$. It then follows from Lemma 2.13 that \mathcal{M} is κ_M^+ -iterable in M . \square

Let now $\mathcal{M} \trianglelefteq \mathcal{W}_{\mu,g}(\mathcal{P}_\mu|\eta)$ be such that $\rho(\mathcal{M}) = \eta$. We want to show that $\mathcal{M} \trianglelefteq \mathcal{P}_\mu$. Let (M, π) be a good hull such that $\mathcal{M} \in \text{rng}(\pi)$. Let $\mathcal{N} = \pi^{-1}(\mathcal{M})$. Notice that \mathcal{N} is κ^+ -iterable and hence, by Lemma 2.3, $\mathcal{S}_{\mu,g} \models$ “ \mathcal{N} is ω_1 -iterable”. Because \mathcal{P}_μ^M is $\Gamma_{\mu,g}$ -suitable, we have that $\mathcal{N} \trianglelefteq \mathcal{P}_\mu^M$. Using elementarity of π we get that $\mathcal{M} \trianglelefteq \mathcal{P}_\mu$. \square

Corollary 2.15 *Suppose $\mu < \nu$ are two good points. Suppose $g \subseteq \text{Coll}(\omega, \mu)$ is V -generic and $h \subseteq \text{Coll}(\omega, \nu)$ is $V[g]$ -generic. Then for $a \in V_\kappa[g]$*

$$\mathcal{W}_{\mu,g}(a) = \mathcal{W}_{\nu,g*h}(a).$$

Proof. Using clause 2e of Lemma 2.10 applied in $V[g]$, we have that $\mathcal{W}_{\nu,g*h}(a) \trianglelefteq \mathcal{W}_{\mu,g}(a)$. Fix then $\mathcal{M} \triangleleft \mathcal{W}_{\mu,g}(a)$ such that $\rho_\omega(\mathcal{M}) = a$ and let (M, π) be a good hull at μ such that $\{\eta, \mathcal{M}\} \in \text{rng}(\pi)$. Let $\lambda = \pi^{-1}(\nu)$ and $\mathcal{N} = \pi^{-1}(\mathcal{M})$. It is enough to show that whenever $k \subseteq \text{Coll}(\omega, \lambda)$ is $M[g]$ -generic, $\mathcal{N} \trianglelefteq (\mathcal{W}_{\lambda,g*k}(\pi^{-1}(a)))^{M[g][k]}$. Moreover, by genericity, it is enough to show this fact for $k \in V[g]$. Fix then such a k . The rest of the proof is then just a word-by-word generalization of the proof of Theorem 2.13 with $V = M[g][k]$. We leave it to the reader. \square

Corollary 2.16 *Suppose $\mu < \nu$ are two good points. Suppose $g \subseteq \text{Coll}(\omega, \mu)$ is V -generic and $h \subseteq \text{Coll}(\omega, \nu)$ is $V[g]$ -generic. Then for $a \in V_\kappa[g]$*

$$\mathcal{W}_{\mu,g}^\Gamma(a) = \mathcal{W}_{\nu,g*h}^\Gamma(a).$$

Proof. This follows immediately from clause 1 of Lemma 2.11 and Corollary 2.15. \square

Let ν_0 be the least good point and let $h_0 \subseteq \text{Coll}(\omega, \nu_0)$. We let $W = V[h_0]$ and for $a \in H_{\kappa^+}^{V[h_0]}$ we let

$$\mathcal{W}(a) = \mathcal{W}^{(\mathcal{P}(\mathcal{P}(\kappa^+)))^W}(a).$$

In what follows, we will often deal with $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$ -suitable premice. To make the notation convenient, we drop $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$ from our notation. Thus, in what follows, a suitable premouse is $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$ -suitable, short tree is $(\mathcal{P}(\mathcal{P}(\kappa^+)))^W$ -short and etc.

2.2 Excellent points and excellent hulls

Definition 2.17 *We say $\mu > \nu_0$ is excellent if μ is good and $\mu^{\nu_0} = \mu$.*

Definition 2.18 *Suppose μ is excellent and (M, π) is a good hull at μ . We say M is an excellent hull if $M^{\nu_0} \subseteq M$.*

Clearly if μ is excellent then there is an excellent hull at μ . Also, it is clear that there are at least ω many excellent points. The next two lemmas go back to Ketchersid's work done in [3].

Lemma 2.19 *Suppose μ is excellent. Then $\text{cf}^V(\Theta^\mu) \leq \nu_0$.*

Proof. Suppose not and let $\pi : M \rightarrow V_\zeta$ be good at ν_0 such that $\mu \in \text{ran}(\pi)$. Let $\lambda = \pi^{-1}(\Theta^\mu)$. Because we are assuming $\text{cf}^V(\Theta^\mu) > \nu_0$ and because $|\lambda| = \nu_0$, we have that

$$\nu =_{\text{def}} \sup(\pi \upharpoonright \lambda) < \Theta^\mu.$$

Let $\mathcal{Q} = \pi^{-1}(\mathcal{P}_\mu)$. Let E be the $(\text{crit}(\pi), \nu)$ -extender derived from $\pi \upharpoonright \mathcal{Q} : \mathcal{Q} \rightarrow \mathcal{P}_\mu$. We let $\mathcal{N} = \text{Ult}(\mathcal{Q}, E_\pi \upharpoonright \lambda)$ and let $j : \mathcal{Q} \rightarrow \mathcal{N}$ be the ultrapower embedding. Because $\delta^\mathcal{Q}$ is regular in \mathcal{Q} , we have that $j \upharpoonright \delta^\mathcal{Q}$ is cofinal in $j(\delta^\mathcal{Q})$ and hence, $j(\delta^\mathcal{Q}) = \nu$. Also, we have that $\mathcal{P}_\mu \upharpoonright \nu = \mathcal{N} \upharpoonright \nu$.

Claim. $\mathcal{N} \trianglelefteq \mathcal{P}_\mu$.

Proof. Notice that if $(\mathcal{M}, k) \in V$ is such that $k : \mathcal{M} \rightarrow \mathcal{N}$ and \mathcal{M} is countable in V then in W there is $\sigma : \mathcal{M} \rightarrow \mathcal{Q}$. Hence, in W , \mathcal{M} is κ^+ -iterable above $\sigma^{-1}(\delta^\mathcal{Q})$. It then follows that in W , \mathcal{N} is countably iterable above ν . On the other hand, since $\nu < \Theta^\mu$, we have that $\mathcal{P}_\mu \models \text{“}\nu \text{ isn't Woodin”}$. Let then $\mathcal{N}^* \trianglelefteq \mathcal{P}_\mu$ be the least initial segment such that $\mathcal{J}_1(\mathcal{N}^*) \models \text{“}\nu \text{ isn't Woodin”}$. Then by Corollary 2.14, it follows that \mathcal{N}^* is countably iterable. Therefore, because $\mathcal{N} \models \text{“}\nu \text{ is Woodin”}$, we have that $\mathcal{N} \trianglelefteq \mathcal{N}^* \trianglelefteq \mathcal{P}_\mu$. \square

Let \mathcal{S} be the largest initial segment of \mathcal{P}_μ such that $\mathcal{S} \models \text{“}\nu \text{ is Woodin”}$. Because $\mathcal{S} \in \text{HOD}^{\mathcal{S}_\mu, h_0 * h}$ for any W -generic $h \subseteq \text{Coll}(\omega, \mu)$, it follows from the fact that MC holds in $\mathcal{S}_\mu, h_0 * h$ that

$$\mathcal{P}(\delta^\mathcal{Q}) \cap L[\mathcal{S}, \mathcal{Q}] = (\mathcal{P}(\delta^\mathcal{Q}))^\mathcal{Q}.$$

It then also follows that we can lift j to

$$j^+ : L[\mathcal{S}, \mathcal{Q}] \rightarrow L[j^+(\mathcal{S}), \mathcal{N}].$$

Because E is countably closed, $j^+(\mathcal{S})$ is countably iterable in V . This means that $\mathcal{S} \in L[j^+(\mathcal{S}), \mathcal{N}]$ as it can be identified in $L[j^+(\mathcal{S}), \mathcal{N}]$ as the unique sound \mathcal{N} -premouse \mathcal{M} such that $\rho_\omega(\mathcal{M}) = \nu$, $\mathcal{J}_1(\mathcal{M}) \models \text{“}\nu \text{ is not Woodin”}$ and in $L[j^+(\mathcal{S}), \mathcal{N}]^{\text{Coll}(\omega, \mathcal{N})}$, there is $\sigma : \mathcal{M} \rightarrow j^+(\mathcal{S})$. However, $L[j^+(\mathcal{S}), \mathcal{N}] \models \text{“}j(\delta^\mathcal{Q}) = \nu \text{ is Woodin”}$, contradiction! \square

The proof of the lemma also gives the following the proof of which we leave to our readers.

Lemma 2.20 *Suppose μ is good. Then $\text{cf}^V(\Theta^\mu) \leq \mu$.*

Let $T_\mu \in V$ be the tree on $\omega \times (\delta_1^2)^{\mathcal{S}_{\mu,g}}$ such that whenever $g \subseteq \text{Coll}(\omega, \mu)$ is generic,

$$S_{\mu,g} \models "p[T_\mu] \text{ is the universal } \Sigma_1^2\text{-set}"$$

where $p[T]$ is the projection of T . Next we show that realizable premice are suitable.

Lemma 2.21 *Suppose μ is an excellent point, (M, π) is an excellent hull at μ and $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic. Let $H \in (V_{\kappa_M})^{M[h_0][g]}$ be such that $\mathcal{W}(H) \models "H = H_\nu \text{ for some cardinal } \nu"$. Suppose further that $N \in V_{\kappa_M}^{M[h_0][g]}$ is such that there are embeddings $\sigma : \mathcal{W}(H) \rightarrow N$ and $\tau : N \rightarrow \pi(\mathcal{W}(H))$ such that $\pi \upharpoonright \mathcal{W}(H) = \tau \circ \sigma$. Then*

$$N = \mathcal{W}(\sigma(H)).$$

Proof. Let $\eta > \mu$ be a good point such that for some λ , $\pi(\lambda) = \eta$ and $H, N \in V_\lambda^{M[h_0][g]}$. Let $T \in M$ be such that $\pi(T) = T_\eta$. Using Corollary 2.16, we have that $L[T, \mathcal{W}(H)] \models "H = H_\nu \text{ for some cardinal } \nu"$. This implies that we can extend σ to act on $L[T, \mathcal{W}(H)]$. We let $\sigma^+ : L[T, \mathcal{W}(H)] \rightarrow L[\sigma^+(T), N]$ be this extension of σ . It then follows that we can find $\tau^+ : L[\sigma^+(T), N] \rightarrow L[\mathcal{T}_\eta, \pi(\mathcal{W}(H))]$ extending τ and such that

$$\pi \upharpoonright L[T, \mathcal{W}(H)] = \tau^+ \circ \sigma^+.$$

Fix now an $n \in \omega$ such that whenever $h \subseteq \text{Coll}(\omega, \eta)$ is W -generic then in $\mathcal{S}_{\eta,h}$, $(T_\eta)_n$ projects to the set $\{(x, y) \in \mathbb{R}^2 : x \text{ codes a set } a \text{ and an } a\text{-mouse } \mathcal{M}_x \text{ and } y \text{ codes an } a\text{-mouse } \mathcal{M}_y \text{ such that } \rho_\omega(\mathcal{M}_y) = a \text{ and } \mathcal{M}_x \triangleleft \mathcal{M}_y\}$. Notice that the following holds in $L[T, \mathcal{W}(H)]$: for any generic $k \subseteq \text{Coll}(\omega, \mathcal{W}(H))$, in $L[T, \mathcal{W}(H)][k]$, for any $x \in \mathbb{R}$ coding $(H, \mathcal{W}(H))$ and for any real y , $(x, y) \notin p[T_n]$. We let this sentence be ϕ (in the language of $L[T, \mathcal{W}(H)]$).

We have that ϕ holds in $L[\sigma^+(T), N]$. Since $\tau : N \rightarrow \pi(\mathcal{W}(H))$, we also have that $N \trianglelefteq \mathcal{W}(\sigma(H))$. Moreover, since $\mathcal{P}(\mathcal{W}(H)) \cap L[T, \mathcal{W}(H)] \subseteq \mathcal{W}(\mathcal{W}(H))$, we get that, by the same argument as above, that $\mathcal{P}(N) \cap L[\sigma^+(T), N] \subseteq \mathcal{W}(N)$. It then follows that $\mathcal{P}(N) \cap L[\sigma^+(T), N]$ is countable in $W[g]$.

Suppose now that $N \neq \mathcal{W}(\sigma(H))$. Because we already know that $N \trianglelefteq \mathcal{W}(\sigma(H))$, it is enough to show that $\mathcal{W}(\sigma(H)) \trianglelefteq N$. Suppose not. There is then a κ^+ -iterable sound $\mathcal{M} \trianglelefteq \mathcal{W}(\sigma(H))$ such that $\rho_\omega(\mathcal{M}) = \sigma(H)$ and $N \triangleleft \mathcal{M}$. Fix $k \subseteq \text{Coll}(\omega, \sigma(H))$ such that $k \in W[g]$ and k is both $M[h_0][g]$ and $L[\sigma^+(T), N]$ generic. Let $x \in (\mathbb{R})^{L[\sigma^+(T), N][k]}$ code $(\sigma(H), N)$ in a canonical fashion. We have that $x \in M[h_0][g][k]$. Let $y \in \mathbb{R}^{M[g][k]}$ code \mathcal{M} . We

have that $(x, y) \in p[T_n]$. It then follows that $(x, y) \in p[\sigma^+(T_n)]$. Thus, by absoluteness, we can find $w \in L[\sigma^+(T), N][k]$ such that $(x, w) \in p[\sigma^+(T_n)]$. This, however, contradicts ϕ . \square

The following is an immediate corollary of Lemma 2.21. We will use it to produce strategies which are fullness preserving.

Corollary 2.22 *Suppose μ is an excellent point, (M, π) is a good hull at μ and $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic. Suppose $a \in H_{\omega_1}^{M[h_0]}$ and suppose further that $\mathcal{Q} \in H_{\kappa_M}^{M[h_0][g]}$ is an a -premouse such that there are Σ_1 -elementary embeddings $\sigma : \mathcal{P}_{\mu, a}^M \rightarrow \mathcal{Q}$ and $\tau : \mathcal{Q} \rightarrow \mathcal{P}_{\mu, a}^M$ such that*

$$\pi \upharpoonright \mathcal{P}_{\mu, a}^M = \tau \circ \sigma.$$

Then \mathcal{Q} is suitable.

2.3 A quasi-sjs

Our ultimate goal is to isolate a quasi-self justifying system. Excellent hulls give possible candidates of such quasi-self justifying systems. Suppose μ is excellent and $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic. Given an excellent μ , we let $F^\mu = F_{od}^{S_{\mu, h_0^*g}}$ (we drop the generics to save space). Let (M, π) be an excellent hull at μ . We then let $F^{M, \mu} = \pi^{-1}(F^\mu)$ and $F^{M, \pi} = \pi'' F^{M, \mu}$. We will eventually show that $F^{M, \pi}$ is a quasi-sjs. The proof will be spread over several lemmas. Our first lemma is the following.

Lemma 2.23 *Suppose μ is excellent, $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then*

$$\mathcal{P}_{\mu, a}^M = \cup_{f \in F^{M, \pi}} H_f^{\mathcal{P}_{\mu, a}^M}.$$

Proof. This is an immediate consequence of the definition of \mathcal{M}_∞ . \square

Lemma 2.24 *Suppose μ is excellent, $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in $\mathcal{S}_{\mu, h_0^*g}$, $\mathcal{P}_{\mu, a}^M$ is $(F^\mu, F^{M, \pi})$ -quasi iterable.*

Proof. Using Theorem 1.37 and elementarity of π , we get that for every $f \in F^{M, \pi}$ there is $\mathcal{Q} \in M[h_0][g]$ such that $\mathcal{S}_{\mu, h_0^*g} \models$ “ \mathcal{Q} is (F^μ, f) -quasi iterable”. It then follows that $\mathcal{P}_{\mu, a}^M$ is (F^μ, f) -quasi iterate of \mathcal{Q} . Since f was arbitrary, we get that in $\mathcal{S}_{\mu, h_0^*g}$, $\mathcal{P}_{\mu, a}^M$ is (F^μ, f) -quasi

iterable for every $f \in F^{M,\pi}$. It then follows that $\mathcal{S}_{\mu,h_0*g} \models \text{“}\mathcal{P}_{\mu,a} \text{ is } (F^\mu, F^{M,\pi})\text{-quasi iterable”}$.
 \square

If we restrict ourselves to stacks that are in $V_{\kappa_M}^M$ then we actually get that $F^{M,\pi}$ is a qsjs as witnessed by $\mathcal{P}_{\mu,a}$.

Lemma 2.25 ($F^{M,\pi}$ is a qsjs) *Suppose μ is excellent, $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in \mathcal{S}_{μ,h_0*g} , $F^{M,\pi}$ is a qsjs as witnessed by $\mathcal{P}_{\mu,a}^M$ for stacks in $V_{\kappa_M}^{M[h_0][g]}$.*

Proof. We do the proof for $a = \emptyset$. Clearly $F^{M,\pi}$ and $\mathcal{P} = \mathcal{P}_\mu^M$ satisfy the first two clauses of Definition 1.27. We check clause 3 for quasi iterations that are in $V_{\kappa_M}^{M[h_0][g]}$. Let $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha : \alpha < \nu \rangle \in V_{\kappa_M}^{M[h_0][g]}$ be an $(F^{M,\pi}, F^{M,\pi})$ -quasi iteration of \mathcal{P} . We need to show that it has a last model. Let $\langle \pi_{\gamma,\xi}^{\alpha,\alpha} : \gamma < \xi < \alpha \leq \nu \rangle$ be the embeddings of $\vec{\mathcal{T}}$. We need to show the $\vec{\mathcal{T}}$ has a last model and for this we need to consider two cases. Suppose first $\nu = \alpha + 1$. Let \mathcal{Q} be the last model of \mathcal{T}_α . We need to show that $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H_f^\mathcal{Q}$.

Claim. $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H_f^\mathcal{Q}$.

Proof. Since \mathcal{Q} is a $(F^{M,\pi}, F^{M,\pi})$ -quasi iterate of \mathcal{P} , it is also $(F^\mu, F^{M,\pi})$ -quasi iterate of \mathcal{P} . Hence, in \mathcal{S}_{μ,h_0*g} , \mathcal{Q} is $F^{M,\pi}$ -iterable. Let \mathcal{S} be the transitive collapse of $\cup_{f \in F^{M,\pi}} H_f^\mathcal{Q}$ and let $\sigma : \mathcal{S} \rightarrow \mathcal{Q}$ be the inverse of the transitive collapse. Let $l = \pi_{\mathcal{Q},\infty,F^{M,\pi}}$. Then $l \circ \sigma : \mathcal{S} \rightarrow \pi(\mathcal{P})$. Because $\mathcal{P} = \cup_{f \in F^{M,\pi}} H_f^\mathcal{P}$, we have $k : \mathcal{P} \rightarrow \mathcal{S}$ such that

$$\pi \upharpoonright \mathcal{P} = l \circ \sigma \circ k.$$

Because $\mathcal{S} \in M$ (this follows from the fact that $\mathcal{S} \trianglelefteq \mathcal{Q}$), it follows from Lemma 2.21 that \mathcal{S} is suitable and hence, $\mathcal{S} = \mathcal{Q}$. It then follows that $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H_f^\mathcal{Q}$. \square

Next we assume ν is limit. In this case we have to verify that if for $f \in F^{M,\pi}$, H_f is the direct limit of $H_f^{M_\gamma}$'s under $\pi_{\gamma,\xi}^{f,\nu}$'s then $\mathcal{Q} = \cup_{f \in F^{M,\pi}} H_f$ is suitable. The proof is similar to the proof of the claim. We define $l : \mathcal{Q} \rightarrow \pi(\mathcal{P})$ as follows. Given $x \in \mathcal{Q}$ let α, f be such that $\alpha \in [\beta_{f,\nu}, \nu)$ and there is $\bar{x} \in H_f^{M_\alpha}$ such that $\pi_{\alpha,\nu}^{f,\nu}(\bar{x}) = x$. Then let $l(x) = \pi_{M_\alpha,\infty,f}(\bar{x})$. We then have $k : \mathcal{P} \rightarrow \mathcal{Q}$ coming from the direct limit construction and such that $\pi \upharpoonright \mathcal{P} = l \circ k$. Therefore, using Lemma 2.21, we get that \mathcal{Q} is suitable.

Actually, a little argument is needed to show that $\mathcal{Q} \in M$. To see this, first let $\vec{B} = \langle B_i : i < \omega \rangle \subseteq F^{M,\pi}$ be such that $\sup_{i < \omega} w(B_i) = \Theta^\mu$. Let $D_i = \{(x, y) : x \in \mathbb{R}^{M[h_0 * g]}, x \text{ codes a continuous function } f \text{ and } y \in f^{-1} B_i\}$. Then for any $\eta < \kappa_M$ and any $M[h_0 * g]$ -generic $k \subseteq \text{Coll}(\omega, \eta)$ such that $k \in W[g]$, $\langle D_i \cap M[h_0 * g * k] : i < \omega \rangle \in M[h_0 * g * k]$. It is then easy to see that this is enough to compute \mathcal{Q} in $M[h_0 * g]$.

As part of the proof of clause 3 we have also shown clause 4. We leave the details to the reader. To show clause 5, let $\mathcal{Q} \in V_{\kappa_M}^{M[h_0][g]}$ be a $(F^{M,\pi}, F^{M,\pi})$ -quasi iterate of \mathcal{P} and $\sigma : \mathcal{R} \rightarrow_{\Sigma_1} \mathcal{Q}$ be such that $f(\mathcal{Q}) \in \text{rng}(\sigma)$ for all $f \in F^{M,\pi}$ and $\mathcal{R} \in M$. We need to see that \mathcal{R} is suitable. Again, we have $\text{rng}(\pi_{\mathcal{P}, \mathcal{Q}, F^{M,\pi}}) \subseteq \text{rng}(\sigma)$ and hence, there is an embedding $k : \mathcal{P} \rightarrow_{\Sigma_1} \mathcal{R}$ such that $\pi_{\mathcal{P}, \mathcal{Q}, F^{M,\pi}} = \sigma \circ k$. Let then $l = \pi_{\mathcal{Q}, \infty, F^{M,\pi}} \circ \sigma$. It follows that $\pi \upharpoonright \mathcal{P} = l \circ k$ and hence, \mathcal{R} is suitable. \square

Next, we show that we can substitute $F^{M,\pi}$ by a subset of it which is countable in W . We make this move as we believe it makes the arguments that follow more transparent. We could just as well carry on with $F^{M,\pi}$.

Definition 2.26 (Pre-sjs) *Suppose μ is excellent and $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic. We say $\vec{B} = \langle B_i : i < \omega \rangle$ is a pre-sjs at μ if for every $a \in H_{\omega_1}^{W[g]}$.*

1. for every $i < \omega$, $B_i \in F^\mu$,
2. for every $n < \omega$ there is $k < \omega$ such that $B_k = \bigoplus_{i < n} B_i$,
3. $\mathcal{P}_{\mu,a} = \bigcup_{i < \omega} H_{B_i}^{\mathcal{P}_{\mu,a}}$.

Lemma 2.27 *Suppose μ is excellent and $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic. Then there is a pre-sjs at μ .*

Proof. It is enough to prove that for some excellent (M, π) , $M \models$ “there is a pre-sjs at μ ”. Fix then some excellent (M, π) . We have that $M[h_0] \models \text{cf}(\delta^{\mathcal{P}_\mu^M}) = \omega$. Fix then some increasing sequence $\langle \gamma_i : i < \omega \rangle \in M$ cofinal in $\delta^{\mathcal{P}_\mu^M}$. Let $\delta = \delta^{\mathcal{P}_\mu^M}$. We can then find $\vec{B}^* = \langle B_i^* : i < \omega \rangle \subseteq F^{M,\pi}$ such that for every $a \in H_{\omega_1}^{M[h_0 * g]}$, $\sup_{i < \omega} \gamma_{B_i^*}^{\mathcal{P}_{\mu,a}^M} = \delta^{\mathcal{P}_{\mu,a}^M}$. Next, close \vec{B}^* under \oplus operator to obtain $\vec{B} = \langle B_i : i < \omega \rangle$. We claim that $M \models$ “ $\pi^{-1}(\vec{B})$ is a pre-sjs”.

To see this, it is enough to show that for every a , $\mathcal{P}_{\mu,a}^M = \bigcup_{i < \omega} H_{B_i}^{\mathcal{P}_{\mu,a}^M}$. Let then \mathcal{T} on $\mathcal{P}_{\mu,a}^M$ be the correctly guided maximal tree for making $\mathcal{P}_{\mu,a}^M | (\delta^{+n-1})^{\mathcal{P}_{\mu,a}^M}$ generic where we let $\delta^{+n-1} = \delta$.

We have $\mathcal{T} \in \mathcal{P}_{\mu,a}^M$ and $\delta(\mathcal{T}) = (\delta^{+n})^{\mathcal{P}_{\mu,a}^M}$. Also, because $\mathcal{P}_{\mu,a}^M$ is $(F^{M,\pi}, F^{M,\pi})$ -quasi iterable, we have a branch b of \mathcal{T} such that \mathcal{M}_b^T is suitable and $i_b^T(\delta) = \delta(\mathcal{T})$. Moreover, letting $\mathcal{Q} = \mathcal{M}_b^T$, for every $k, n < \omega$, $i_b^T(\tau_{B_k,n}^{\mathcal{P}_{\mu,a}^M}) = \tau_{B_k,n}^{\mathcal{Q}}$. Since for each $i < \omega$ there is $k < \omega$ such that $\pi_{\mathcal{P}_{\mu,a}^M, \mathcal{Q}, B_i} \in H_{B_k}^{\mathcal{P}_{\mu,a}^M}$, we have that if \mathcal{S} is the transitive collapse of $\cup_{i < \omega} H_{B_i}^{\mathcal{P}_{\mu,a}^M}$ and $\sigma : \mathcal{S} \rightarrow \mathcal{P}_{\mu,a}^M$ is the inverse of the collapse then $\sigma \upharpoonright (\delta^{+n})^{\mathcal{S}}$ is cofinal implying that $\sigma = id$. \square

Suppose μ is excellent, (M, π) is an excellent hull at μ , $g \subseteq \text{Coll}(\omega, \mu)$ is V -generic and $\vec{B} \in V[g]$ is a pre-sjs at μ . We let $B_i^M =_{\text{def}} B_i \cap M[g]$ and $\vec{B}^M =_{\text{def}} \langle B_i^M : i < \omega \rangle$. We say (M, π) captures \vec{B} if $\vec{B}^M \in M$ and $\pi(\vec{B}^M) = \vec{B}$. We let $F_{\vec{B}} = \{f_{B_i} : i < \omega\}$. It follows from Lemma 2.25 and the proof of Lemma 2.27 that

Lemma 2.28 *Suppose μ is excellent, $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic, (M, π) is an excellent hull at μ and $a \in (H_{\omega_1})^{M[g]}$. Then in $\mathcal{S}_{\mu, h_0 * g}$, $F_{\vec{B}}$ is a qsjs as witnessed by $\mathcal{P}_{\mu,a}^M$ for stacks in $V_{\kappa_M}^{M[h_0][g]}$.*

We continue with the set up of the lemma. Given any $B \in \mathcal{P}(\mathbb{R}^{W[g]}) \cap OD^{\mathcal{S}_{\mu, h_0 * g}}$, we let

$$f_B^{M,g} = \{(\mathcal{Q}, f_B(\mathcal{Q})) : \mathcal{Q} \in V_{\kappa_M}^{M[g]} \text{ and } \mathcal{Q} \text{ is } \Gamma_{\mu,g}\text{-suitable}\}.$$

We define $f_{\vec{B}}^M$ similarly. Given a pre-sjs \vec{B} at μ such that M captures \vec{B} , we let

$$f_{\vec{B}}^{M,g} = \langle f_{B_i}^{M,g} : i < \omega \rangle \text{ and } f_{\vec{B}}^M = \langle f_{B_i}^M : i < \omega \rangle.$$

Lemma 2.29 *Suppose μ is excellent, (M, π) is an excellent hull at μ , $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic and $\vec{B} \in V[g]$ is a pre-sjs at μ captured by (M, π) . Then $f_{\vec{B}}^{M,g} \in M[h_0 * g]$.*

Proof. We have that in $\mathcal{S}_{\mu, h_0 * g}$, for each i , $f_{B_i}^{M,g}$ is $OD_{V_{\kappa_M}^{M[h_0][g]}}$ and hence, $f_{B_i}^{M,g} \in M[h_0][g]$ (recall that $\mathcal{W}(A_M) \in M$). Let then for each $i < \omega$, $\dot{f}_i \in M[h_0]^{\text{Coll}(\omega, \mu)}$ be the term that is always realized as $f_{B_i}^{M,g}$. Because $M[h_0]$ is ω -closed in W , we have that $\langle \dot{f}_i : i < \omega \rangle \in M[h_0]$. Hence, $f_{\vec{B}}^{M,g} \in M$. \square

Suppose now μ is excellent, $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic, \vec{B} is a pre-sjs at μ , and (M, π) is an excellent hull at μ capturing \vec{B} . Suppose further that $a \in (H_{\omega_1})^{M[g]}$. Lemma 2.25 and Lemma 1.29 imply that we can define $\Sigma^{F_{\vec{B}}}$ for $\mathcal{P}_{\mu,a}$. In our case, however, we can only restrict to iterations that are in $V_{\kappa_M}^{M[h_0][g]}$. We then let $\Sigma_{\mu, \vec{B}, a}^{M,g}$ be this iteration strategy.

Lemma 2.30 *$\Sigma_{\mu, \vec{B}, a}^{M,g}$ is a (κ_M, κ_M) -iteration strategy which acts on iterations that are in $M[h_0][g]$ and is fullness preserving. Moreover, if $\vec{T} \in V_{\kappa_M}^{M[h_0][g]}$ is via $\Sigma_{\mu, \vec{B}, a}^{M,g}$ and the last normal component of \vec{T} has limit length then $\Sigma_{\mu, \vec{B}, a}^{M,g}(\vec{T}) \in M[h_0][g]$.*

Proof. The first part is an immediate consequence of Lemma 1.29. The second part follows from the fact that $f_{\vec{B}}^{M,g} \in M[h_0][g]$. \square

Notice that $\Sigma_{\mu, \vec{B}, a}^{M,g} \upharpoonright V_{\kappa_M}^{M[h_0]} \in M[h_0]$ given that $a \in M[h_0]$. We let $\Sigma_{\mu, \vec{B}, a}^M = \Sigma_{\mu, \vec{B}, a}^{M,g} \upharpoonright V_{\kappa_M}^{M[h_0]}$. When $a = \emptyset$, we omit it from subscripts. In the next subsection we will show that for each $i < \omega$ there is some $\Sigma_{\mu, \vec{B}, a}^{M,g}$ -iterate \mathcal{Q} of $\mathcal{P}_{\mu, a}^M$ such that if Λ is the strategy of \mathcal{Q} induced by $\Sigma_{\mu, \vec{B}, a}^{M,g}$ then Λ strongly respect B_i . It then easily follows from Lemma 1.30 that Λ has branch condensation.

2.4 An ω -suitable \mathcal{P}

Recall that we have that $\text{cf}^W(\kappa) = \omega$. Let then $\langle \mu_i : i < \omega \rangle$ be a sequence of excellent points such that for each i , $\mu_{i+1}^+ = \mu_{i+1}$ and $\sup_{i < \omega} \mu_i = \kappa$. For the rest of the paper, we fix a W -generic $g_0 \subseteq \text{Coll}(\omega, \mu_0)$ and a pre-sjs \vec{B} at μ_0 .

Fix now an excellent (M^*, π^*) at μ_0 capturing \vec{B} . For each $X \in V_{\kappa_{M^*}}^{M^*[h_0][g_0]}$ we let \mathcal{T}_X be the tree on $\mathcal{P}_\mu^{M^*}$ according to $\Sigma_{\mu_0, \vec{B}}^{M^*}$ that makes X generic. Let \mathcal{Q}_X be the last model of \mathcal{T}_X and let $\pi_X : \mathcal{P}_\mu^{M^*} \rightarrow \mathcal{Q}_X$ be the iteration embedding according to $\Sigma_{\mu_0, \vec{B}}^{M^*}$. We then let $\tau_{B_i, X} = \pi_X(\tau_{B_i, 0}^{\mathcal{P}_\mu^{M^*}})$. We let

$$\tau_i = \{(X, \mathcal{Q}_X, \tau_{B_i, X}) : X \in V_{\kappa_{M^*}}^{M^*[h_0][g]}\}$$

and let $\vec{\tau} = \langle \tau_i : i < \omega \rangle$. The proof of Lemma 2.29 gives that $\vec{\tau} \in M^*[h_0][g]$.

To continue, we introduce the following notation. Suppose $\mu \leq \nu$ are two excellent points and (M, π) and (N, σ) are two excellent hulls at μ and ν respectively. We write $(M, \pi) < (N, \sigma)$ if $(M, \pi) \in \text{rng}(\sigma)$. In this case, we let $\pi_{M, N} = \sigma^{-1}(\pi) : M \rightarrow N$. We have that $\pi_{M, N} \in N$.

Our first goal here is to show that we can extend \vec{B} to a pre-sjs at any other excellent point $> \mu_0$. Given an excellent $\mu > \mu_0$, we let $B_{\mu, i} \in W[g_0]^{\text{Coll}(\omega, \mu)}$ be the name for the set of reals given by $\pi^*(\tau_i)$ as follows: given a standard name for a real $\dot{x} \in W[g_0]^{\text{Coll}(\omega, \mu)}$ and $p \in \text{Coll}(\omega, \mu)$,

$$(p, \dot{x}) \in B_{\mu, i} \leftrightarrow W[g_0] \models \text{“if } \eta = \mu^{++} \text{ and } (\mathcal{Q}, \tau) \text{ is such that } (V_\eta, \mathcal{Q}, \tau) \in \pi^*(\tau_i) \text{ then } p \Vdash_{\text{Coll}(\omega, \mu)}^{\mathcal{Q}[V_\eta]} \dot{x} \in \tau\text{”}.$$

If $g \subseteq \text{Coll}(\omega, \mu)$ is $W[g_0]$ generic then we let $B_{\mu, i, g}$ be the realization of $B_{\mu, i}$. We let $\vec{B}_\mu = \langle B_{\mu, i} : i < \omega \rangle$ and $\vec{B}_{\mu, g} = \langle B_{\mu, i, g} : i < \omega \rangle$.

Lemma 2.31 *Suppose $\mu > \mu_0$ and $g \subseteq \text{Coll}(\omega, \mu)$ is $W[g_0]$ -generic. Then $B_{\mu,i,g} \in \mathcal{S}_{\mu,h_0*g_0*g}$.*

Proof. It is enough to show that $L(B_{\mu,i,g}, \mathbb{R}^{W[g_0*g]}) \models AD^+$. Let (M, π) be a good hull at μ_0 such that $(M^*, \pi^*) < (M, \pi)$ and $\mu \in \text{rng}(\pi)$. Let $B = \pi^{-1}(B_{\mu,i})$. It is enough to show that if $\lambda = \pi^{-1}(\mu)$ and $h \subseteq \text{Coll}(\omega, \lambda)$ is $M[h_0 * g_0]$ -generic such that $h \in W[g_0]$ then $M[h_0 * g_0 * h] \models L(B_h, \mathbb{R}) \models AD^+$ where B_h is the realization of B in $M[h_0 * g_0 * h]$. Fix then such an h . Notice that we have that

$$B_h = B_i \cap \mathbb{R}^{M[h_0*g_0*h]}.$$

Since B_i is *OD* in $\mathcal{S}_{\mu_0,h_0*g_0}$, we have that there is a sound $\mathbb{R}^{M[h_0*g_0*h]}$ -mouse \mathcal{M} such that $B_h \in \mathcal{M}$, $\rho_\omega(\mathcal{M}) = \mathbb{R}^{M[h_0*g_0*h]}$ and $\mathcal{S}_{\mu_0,h_0*g_0} \models \text{“}\mathcal{M} \text{ is } \omega_1\text{-iterable”}$. It then follows from Lemma 2.11 and Lemma 2.13 that $\mathcal{M} \in M[h_0 * g_0 * h]$ and $M[h_0 * g_0 * h] \models \text{“}\mathcal{M} \text{ is } \kappa_M^+\text{-iterable”}$. Therefore, $\mathcal{M} \trianglelefteq (\mathcal{S}_{\lambda,h_0*g_0*h}^-)^{M[h_0*g_0*h]}$ implying that $M[h_0 * g_0 * h] \models L(B_h, \mathbb{R}) \models AD^+$. \square

Lemma 2.32 *Suppose $\mu > \mu_0$ and $g \subseteq \text{Coll}(\omega, \mu)$ is $W[g_0]$ -generic. Then $\vec{B}_{\mu,i,g}$ is a pre-sjs at μ .*

Proof. The proof is like the proof of Lemma 2.31. It follows from the proof of Lemma 2.27 that it is enough to show that for any $a \in H_{\omega_1}^{W[g_0*g]}$, $\sup_{i < \omega} \gamma_{B_{\mu,i,g}}^{\mathcal{P}_{\mu,a}} = \Theta^\mu$. We first show this for $a = \emptyset$. Let (M, π) be an excellent hull at μ_0 such that $(M^*, \pi^*) < (M, \pi)$ and $\mu \in \text{rng}(\pi)$. Let $\lambda = \pi^{-1}(\mu)$ and let $h \subseteq \text{Coll}(\omega, \lambda)$ be $M[h_0 * g_0]$ -generic such that $h \in W[g_0]$. It is enough to show that in $M[h_0 * g_0 * h]$,

$$\sup_{i < \omega} \gamma_{B_{\mu,i,g}}^{\mathcal{P}} = \delta^{\mathcal{P}}$$

where $\mathcal{P} = \pi^{-1}(\mathcal{P}_\mu)$. This follows from the fact that $(B_{\mu,i,g})^{M[h_0*g_0*h]} = B_i \cap M[h_0 * g_0 * h]$ and in M , \mathcal{P} is a $F_{\vec{B}}$ -quasi-iterate of $\mathcal{P}_{\mu_0}^M$.

To show the claim for every a use Lemma 2.25 and Lemma 2.27. \square

We define $\langle \mathcal{P}_i : i < \omega \rangle$ as follows:

1. $\mathcal{P}_0 = \mathcal{P}_{\mu_0}$,
2. $\mathcal{P}_{i+1} = \mathcal{P}_{\mu_{i+1}, \mathcal{P}_i}$.

Let $\delta_i = \delta^{\mathcal{P}_i}$.

Lemma 2.33 $\mathcal{P}_{i+1} \models \text{“}\delta_i \text{ is Woodin”}$ and no level of \mathcal{P}_{i+1} projects across δ_i .

Proof. It is enough to show that no level of \mathcal{P}_{i+1} projects across δ_i . Towards a contradiction, suppose $\mathcal{M} \trianglelefteq \mathcal{P}_{i+1}$ is such that $\rho(\mathcal{M}) \leq \delta_i$. It follows from Corollary 2.14 that \mathcal{M} , regarded as a mouse over \mathcal{P}_i , is κ^+ -iterable in W and hence, $\mathcal{M} \trianglelefteq \mathcal{P}_i$, contradiction. \square

Let $\delta_\omega = \sup_{i < \omega} \delta_i$ and $\mathcal{P}^- = \cup_{i < \omega} \mathcal{P}_i$ and let

$$\mathcal{P} = \begin{cases} \mathcal{W}(\mathcal{P}^-) & : \text{ if no level of } \mathcal{W}(\mathcal{P}^-) \text{ projects across } \delta_\omega \\ \mathcal{N} & : \text{ where } \mathcal{N} \triangleleft \mathcal{W}(\mathcal{P}^-) \text{ is the least such that } \rho(\mathcal{N}) < \delta_\omega \end{cases}$$

Notice that $\kappa = \delta_\omega$. In subsequent sections we will show that $\mathcal{P} = \mathcal{W}(\mathcal{P}^-)$. Before we move on, we fix some notation.

Suppose (M, π) is an excellent hull at μ_k for some $k \leq \omega$. Let $\mu_{-1} = \nu_0$. We say (M, π) is *perfect* if $(M^*, \pi^*) < (M, \pi)$, $M^{\mu_{k-1}^+} \subseteq M$ and $\langle \mu_i : i < \omega \rangle \in \text{rng}(\pi)$. Notice that we have that $\mathcal{P} \in \text{rng}(\pi)$. We then let $\langle \mu_i^M : i \in \omega \rangle = \pi^{-1}(\langle \mu_i : i \in \omega \rangle)$, $\mathcal{P}_i^M = \pi^{-1}(\mathcal{P}_i)$ and $\mathcal{P}^M = \pi^{-1}(\mathcal{P})$.

Notice that if for some $k < \omega$, $g \subseteq \text{Coll}(\omega, \mu_k)$ is $W[g_0]$ generic and (M, π) is a perfect hull at μ_k then $\Sigma_{\mu_k, a, \vec{B}_{\mu_k, g}}^{M, g_0 * g}$ is defined for every $a \in (H_{\omega_1})^{M[h_0 * g_0]}$. We then let $\Sigma_{\mu_k, a}^{M, g} = \Sigma_{\mu_k, a, \vec{B}_{\mu_k, g}}^{M, g_0 * g}$ where we let $\mathcal{P}_{-1} = \emptyset$. Notice that we have that $\Sigma_{\mu_k, a}^{M, g} \upharpoonright M[h_0 * g_0] \in M[h_0 * g_0]$. We then let $\Sigma_{\mu_k, a}^M = \Sigma_{\mu_k, a}^{M, g} \upharpoonright M[h_0 * g_0]$. It is again true that $\Sigma_{\mu_k, a}^M \in M[h_0 * g_0]$.

2.5 Iteration strategy for \mathcal{P}

Suppose $(R, \tau) > (M^*, \pi^*)$ is a perfect hull at some μ_p for $p \leq \omega$. Here we will describe a (κ_R, κ_R) -strategy for \mathcal{P}^R which acts on stacks all of whose normal components are in $V_{\kappa_R}^R$ and are above $\delta_{p-1}^{\mathcal{P}^R}$ (recall that $\delta_{-1} = 0$). To do this, we introduce the auxiliary game $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ in which player *II*, after a notification that *I* is about to start a new round, will play an embedding $\pi_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{P}$ such that $\tau \upharpoonright \mathcal{P} = \pi_\alpha \circ i_{0, \alpha}$ where $i_{0, \alpha}$ is the iteration embedding. More precisely, $\langle \mathcal{T}_\alpha, \mathcal{M}_\alpha, \pi_\alpha : \alpha < \eta \rangle \subseteq V_{\kappa_R}^R$ is a run of $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ in which neither player has lost if

1. for $\alpha < \eta$, \mathcal{M}_α is the model at the beginning of the α th round and $\mathcal{M}_0 = \mathcal{P}^R$,
2. for $\alpha < \eta$, \mathcal{M}_α is (anomalous) ω -suitable premouse,
3. for $\alpha < \eta$, \mathcal{T}_α is a tree with no fatal drops based on some window $(\delta_k^{M_\alpha}, \delta_{k+1}^{M_\alpha})$ for $k \geq p-1$,

4. for $\alpha < \eta$, \mathcal{T}_α is correctly guided and if \mathcal{T}_α has a last branch and \mathcal{T}_α^- is short then $\mathcal{Q}(\mathcal{T}_\alpha^-) \sqsubseteq \mathcal{M}_{lh(\mathcal{T}_\alpha)-1}^{\mathcal{T}_\alpha}$,
5. $\pi_0 = \pi \upharpoonright \mathcal{P}^R$ and for $\alpha < \beta < \eta$, $\pi_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{P}$,
6. if for $\alpha < \beta < \eta$, $i_{\alpha,\beta} : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta$ is the iteration embedding then for $\alpha < \beta < \eta$, $\pi_\alpha = \pi_\beta \circ i_{\alpha,\beta}$.

The game has at most ω^3 many rounds. As usual player I plays the extenders and starts new rounds. II plays branches. I cannot start a new round if the current model isn't suitable. If I decides to start a new round with \mathcal{M}_α as its starting model then II has to play the embedding π_α that satisfies clauses 5 and 6. The game stops if one of the models produced is ill founded or if II cannot play π_α . II wins if all the models appearing in the iteration are wellfounded, she can always play the embeddings π_α , the game runs ω^3 many rounds or if one of the rounds runs κ_R -steps.

Lemma 2.34 *Player II has a winning strategy in $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$.*

Proof. We define the strategy Λ for II by induction. First fix some bijection $f : \kappa_R \rightarrow V_{\kappa_R}^R$ such that $f \in R$. In the first round, I plays a tree on $\mathcal{P}^R = \mathcal{M}_0$. Suppose I starts playing the first round on $(\delta_k^{\mathcal{M}_0}, \delta_{k+1}^{\mathcal{M}_0})$ for $k \geq p-1$ and suppose \mathcal{T} is a tree constructed according to Λ and \mathcal{T} has limit length. Below we describe how II should play her move.

1. If \mathcal{T} is short then II plays the unique cofinal branch $b \in R$ of \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T}) \sqsubseteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$ exists,
2. if \mathcal{T} is maximal then II plays b such that
 - (a) $b \in R$,
 - (b) $\mathcal{M}_b^\mathcal{T}$ is ω -suitable,
 - (c) there is $\sigma : \mathcal{M}_b^\mathcal{T} \rightarrow \mathcal{P}$ such that $\tau \upharpoonright \mathcal{P}^R = \sigma \circ i_b^\mathcal{T}$.
 - (d) b is the f -least branch satisfying 2a-2c.

We need to see that II can satisfy these conditions along with satisfying the rules of the iteration game. It is enough to show that there is some $b \in R$ satisfying all the clauses above except

2d. We can then choose f -least such b .

Claim. There is $b \in R$ satisfying 1 and 2a-2c.

Proof. Fix some perfect hull (M, π) at μ_{k+1} such that $(R, \tau) < (M, \pi)$.

Subclaim. \mathcal{T} is according to $\pi_{R,M}$ -pullback of $\Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M$.

Proof. Fix limit $\alpha < lh(\mathcal{T})$. By induction, we assume that $\mathcal{T} \upharpoonright \alpha$ is according to $\pi_{R,M}$ -pullback of Σ^M . Notice that $\pi_{R,M}\mathcal{T} \upharpoonright \alpha \in M$. To prove the subclaim, we need to see that if b is the branch of $\mathcal{T} \upharpoonright \alpha$ then b is also the branch of $\pi_{R,M}\mathcal{T} \upharpoonright \alpha$. Notice that we must have that $\mathcal{T} \upharpoonright \alpha$ isn't maximal and hence, $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha)$ exists and $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \trianglelefteq \mathcal{M}_\alpha^{\mathcal{T}}$. Let $c = \Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M(\pi_{R,M}\mathcal{T} \upharpoonright \alpha)$. It follows that

$$\mathcal{Q}(c, \pi_{R,M}\mathcal{T} \upharpoonright \alpha) \text{ exists iff } \mathcal{Q}(c, \mathcal{T} \upharpoonright \alpha) \text{ exists}$$

implying that if $b \neq c$ then $\mathcal{Q}(c, \mathcal{T} \upharpoonright \alpha)$ doesn't exist. We then have that $\pi_{R,M}\mathcal{T} \upharpoonright \alpha$ is a maximal tree. Let $\mathcal{U} = \pi_{R,M}\mathcal{T} \upharpoonright \alpha$. Because $\mathcal{U} \in M$, it follows from the construction of $\Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M$ that there is a map $\sigma : \mathcal{M}_c^{\mathcal{U}} \rightarrow \mathcal{P}$ such that $\pi \upharpoonright \mathcal{P}^M = \sigma \circ i_c^{\mathcal{U}}$. Let $k : \mathcal{M}_c^{\mathcal{T} \upharpoonright \alpha} \rightarrow \mathcal{M}_c^{\mathcal{U}}$ be the copy map. It then follows that $\tau \upharpoonright \mathcal{P}^R = \sigma \circ k \circ i_c^{\mathcal{T} \upharpoonright \alpha}$ which implies, using Lemma 2.22, that $\mathcal{M}_c^{\mathcal{T} \upharpoonright \alpha}$ is ω -suitable. Hence, $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha)$ cannot exist, contradiction! \square

We then let $b = \Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M(\pi_{R,M}\mathcal{T})$. It follows from the proof of the subclaim that b satisfies clause 1. Suppose then \mathcal{T} is maximal. It follows from the copying construction that $\pi_{R,M}\mathcal{T}$ is also maximal. It also follows from the construction of $\Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M$ that there is $\sigma : \mathcal{M}_b^{\pi_{R,M}\mathcal{T}} \rightarrow \mathcal{P}$ such that $\pi \upharpoonright \mathcal{P}^M = \sigma \circ i_b^{\pi_{R,M}\mathcal{T}}$. Letting $l : \mathcal{M}_b^{\mathcal{T}} \rightarrow \mathcal{M}_b^{\pi_{R,M}\mathcal{T}}$ come from the copying construction, we have that $\tau \upharpoonright \mathcal{P}^R = \sigma \circ l \circ i_b^{\mathcal{T}}$. Hence, $\mathcal{M}_b^{\mathcal{T}}$ is ω -suitable. Moreover, b satisfies clause 2b-2c. To finish, we need to show that $b \in R[h_0 * g_0]$. Recall that $R[h_0 * g_0]$ is ω -closed in $W[g_0]$ and that in $W[g_0]$, $cf(\delta_{k+1}^{\mathcal{P}^R}) = \omega$. Let then $\langle \gamma_i : i < \omega \rangle \subseteq rng(i_b^{\mathcal{T}})$ be cofinal in $\delta(\mathcal{T})$. Then we have that $\langle \gamma_i : i < \omega \rangle \in R$. We also have that b is the unique branch c of \mathcal{T} such that $\langle \gamma_i : i < \omega \rangle \subseteq i_c^{\mathcal{T}}$. It then follows from absoluteness that $b \in R$. \square

We then define $\Lambda(\mathcal{T}) = b$ which satisfies clause 1 and 2 above. Suppose now that we have defined Λ for stacks that have $< \alpha$ -many rounds. We want to extend Λ to act on stacks that have α -many rounds. The first step is to specify II 's move at the beginning of round α . To do this, we examine three cases. Suppose first α is limit. Let $\langle \mathcal{T}_\xi, \mathcal{M}_\xi, \pi_{\xi, \gamma} : \xi < \gamma < \alpha \rangle$ be a run of

$\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ which is according to Λ . Then I must start the α th round on \mathcal{M}_α which is the direct limit of \mathcal{M}_ξ 's under the iteration embeddings $i_{\xi, \gamma} : \mathcal{M}_\xi \rightarrow \mathcal{M}_\gamma$. Let $i_{\xi, \alpha} : \mathcal{M}_\xi \rightarrow \mathcal{M}_\alpha$ be the direct limit embedding. We then let $\pi_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{P}$ come from the direct limit construction: given $x \in \mathcal{M}_\alpha$, let ξ be such that for some $\bar{x} \in \mathcal{M}_\xi$, $x = i_{\xi, \alpha}(\bar{x})$ and let

$$\pi_\alpha(x) =_{def} \pi_\xi(\bar{x}).$$

We then let II play π_α . It follows that π_α is as desired and that \mathcal{M}_α is ω -suitable.

Next suppose $\alpha = \beta + 1$ and let $\langle \mathcal{T}_\xi, \mathcal{M}_\xi, \pi_{\xi, \gamma} : \xi < \gamma \leq \beta \rangle$ be a run of $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ which is according to Λ and suppose I wants to start the α th round of the game. Then I has to start the round on the last model \mathcal{M}_α of \mathcal{T}_β . It follows from our inductive hypothesis that \mathcal{M}_α is ω -suitable. It remains to choose π_α . Suppose first that \mathcal{T}_β^- is defined and is maximal. Let b be the last branch of \mathcal{T} . As part of our inductive hypothesis we have that there is $\pi : \mathcal{M}_b^{\mathcal{T}_\beta^-} \rightarrow \mathcal{P}$ such that $\pi_\beta = \pi \circ i_b^{\mathcal{T}_\beta^-}$. In this case, we let $\pi_\alpha = \pi$. Next suppose that either \mathcal{T}_β^- isn't defined or that it is defined but it is short. Let $k < \omega$ be such that \mathcal{T}_β is a tree played on the window $(\delta_k^{\mathcal{M}_\beta}, \delta_{k+1}^{\mathcal{M}_\beta})$. Also, let $(M, \pi) > (R, \tau)$ be a perfect hull at μ_{k+1} . It follows from the proof of the subclaim that \mathcal{T}_β is according to $\pi_{R, M}$ -pullback of $\Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M$. Let then $l : \mathcal{M}_\alpha \rightarrow \mathcal{N}$ come from the copying construction where \mathcal{N} is the last model of $\pi_{R, M} \mathcal{T}_\beta$. By the construction of $\Sigma_{\mu_{k+1}, \mathcal{P}_k^M}^M$ we have $\sigma : \mathcal{N} \rightarrow \mathcal{P}$ such that $\pi \upharpoonright \mathcal{P}^M = \sigma \circ i^{\pi_{R, M} \mathcal{T}_\beta}$. It then follows that $\pi_\beta = \sigma \circ l \circ i^{\mathcal{T}_\beta}$. Let then $\pi_\alpha = \sigma \circ l$.

Next, to describe II 's moves in the α th round we just follow the steps describing II 's move in the first round. More precisely, given a tree $\mathcal{T} \in R$ on \mathcal{M}_α which is according to Λ and has limit length, we let II play as follows:

1. If \mathcal{T} is short then II plays the unique cofinal branch $b \in R$ of \mathcal{T} such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$ exists,
2. if \mathcal{T} is maximal then II plays b such that
 - (a) $b \in R$,
 - (b) $\mathcal{M}_b^{\mathcal{T}}$ is ω -suitable,
 - (c) there is $\sigma : \mathcal{M}_b^{\mathcal{T}} \rightarrow \mathcal{P}$ such that $\pi_\alpha = \sigma \circ i_b^{\mathcal{T}}$.
 - (d) b is the f -least branch satisfying 2a-2c.

That there is such a branch $b \in R$ follows from the proof of the claim above. This finishes our description of Λ . \square

We let Λ^R be the winning strategy of \mathcal{P}^R in the iteration game $\mathcal{G}^a(\mathcal{P}^R, \omega^3, \kappa_R)$ constructed above. Next, we show that perfect hulls of \mathcal{P} are iterable. More precisely, suppose $(M^*, \pi^*) < (R, \tau)$ is a perfect hull at μ_p for some $p < \omega$. Let $\mathcal{S} \in R$ be an ω -suitable Λ^R iterate of \mathcal{P}^R and let $k : \mathcal{S} \rightarrow \mathcal{P}$ be the realization according to Λ . Let $(M, \pi) > (R, \tau)$ be a perfect hull at μ_{p+1} such that $k \in \text{rng}(\pi)$. Notice that $H_{\mu_p^{++}} \in M$. We let $\Sigma^{R, M, \mathcal{S}}$ be the k^* -pullback of Λ^M where $k^* = \pi^{-1}(k)$. Because $H_{\mu_p^{++}} \in M$, we have that $\Lambda^{R, M, \mathcal{S}}$ is a $(0, \omega^3, \mu_p^{++})$ -strategy for \mathcal{S} acting on stacks above $\delta_p^{\mathcal{S}}$.

Lemma 2.35 $\mathcal{P} = \mathcal{W}(\mathcal{P}_\omega^-)$.

Proof. Suppose not and let k be least such that $\rho_\omega(\mathcal{P}) < \delta_k$. Suppose (R, τ) is a perfect hull at μ_{k+1} . Let $\mathcal{Q} = \mathcal{P}^R$ and let p be the standard parameter of \mathcal{Q} . Let $\mathcal{N} = \text{Hull}_m^{\mathcal{Q}}(\delta_{k+1}, \{p\})$ where m is the least such that $\rho_m(\mathcal{Q}) < \delta_\omega^{\mathcal{Q}}$. We claim that $\Sigma_{\mu_{k+1}, \mathcal{P}_k^R}^R$ has branch condensation. To see this, it is enough to show that if $\vec{\mathcal{T}}$ is a stack on \mathcal{N} according to Σ^R such that $i^{\vec{\mathcal{T}}}$ -exists and \mathcal{S} is the last model of $\vec{\mathcal{T}}$ then \mathcal{S} is (m, μ_{k+1}^{++}) -iterable above $\delta_k^{\mathcal{S}}$. But choosing M as in the paragraph proceeding the lemma, we see that $\Lambda^{R, M, \mathcal{S}}$ is such a strategy for \mathcal{S} .

It then follows that $\Sigma =_{\text{def}} \tau(\Sigma_{\mu_{k+1}, \mathcal{P}_k^R}^R)$ is a fullness preserving (κ, κ) -strategy for \mathcal{P}_{k+1} acting on trees above δ_k . Hence, Σ can be extended to a (κ^+, κ^+) -iteration strategy. Let then $h \subseteq \text{Coll}(\omega, \mu_{k+2})$ be W -generic. It then follows from Theorem 2.3 that in $W[g_0 * h]$, $L^\Sigma(\mathbb{R}) \models AD^+$ implying that letting $\Sigma^* =_{\text{def}} \Sigma \upharpoonright H_{\omega_1}^{W[g_0 * h]}$, $\Sigma^* \in S_{\mu_{k+2}, h_0 * g_0 * h}$. Hence, in $W[g_0 * h]$, $L(\Sigma^*, \mathbb{R}) \models AD^+ + \theta_0 < \Theta$, contradiction! \square

2.6 A derived model at κ

Our goal now is to show that if $G \subseteq \text{Coll}(\omega, \kappa)$ is $W[g_0]$ -generic and $g_i = G \cap \text{Coll}(\omega, \mu_i)$ then $S_{\mu_i, h_0 * g_0 * g_i}$'s can be dovetailed into one model of determinacy. In the next subsection, we will use this fact to construct a strategy for some \mathcal{P}_μ which has branch condensation. Our first lemma is a strengthening of Lemma 2.8 and Lemma 2.13.

Lemma 2.36 *Suppose (M, π) is an excellent hull at some excellent μ . Let $g \subseteq \text{Coll}(\omega, \mu)$ be W -generic and let $B \subseteq \kappa_M$ be a set in $M[h_0 * g]$. Then $(\mathcal{W}(B))^{M[h_0 * g]} = \mathcal{W}(B)$.*

Proof. We clearly have that $(\mathcal{W}(B))^{M[h_0 * g]} \leq \mathcal{W}(B)$. We need to show that $\mathcal{W}(B) \leq (\mathcal{W}(B))^{M[h_0 * g]}$. Suppose not. Let then $\mathcal{M} \leq \mathcal{W}(B)$ be the least such that $\rho_\omega(\mathcal{M}) = B$ and $(\mathcal{W}(B))^{M[h_0 * g]} \triangleleft \mathcal{M}$. To derive contradiction, we need to show that $\mathcal{M} \in M[h_0 * g]$ and $M[h_0 * g] \models \text{“}\mathcal{M} \text{ is } \kappa^+ \text{-iterable”}$. Let $\mathcal{Q} = \mathcal{P}_\mu^M$. Recall that \mathcal{Q} is short tree iterable in $S_{\mu, h_0 * g}$.

Let $C \in M$ be the name for B . We claim that if \mathcal{T} is the correctly guided tree on \mathcal{Q} for making C generically generic then $\mathcal{T} \in M$. To see this, it is enough to show that if $\alpha < lh(\mathcal{T})$ is limit then $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \in M$. It will then follow that $M \models \text{“}\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \text{ is countably iterable”}$ and hence, M can correctly identify the branch of $\mathcal{T} \upharpoonright \alpha$. Suppose then $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \notin M$. Let $\mathcal{S} = (Lp(\mathcal{T} \upharpoonright \alpha))^M$.

Claim. There is a cofinal wellfounded branch c of $\mathcal{T} \upharpoonright \alpha$ such that $c \in M$.

Proof. Let $\mathcal{T}^* = \pi(\mathcal{T} \upharpoonright \alpha)$ and $\mathcal{S}^* = \pi(\mathcal{S})$. Notice that $\text{cf}^V(\delta(\mathcal{T}^*)) < \kappa$. Let $\lambda = \text{cf}^V(\delta(\mathcal{T}^*))$. Fix an excellent $\eta > \mu$ such that $\eta^\lambda = \eta$. Let $(N, \sigma) > (M, \pi)$ be an excellent hull at η such that $N^\lambda \subseteq N$. Let $\mathcal{U} = \sigma^{-1}(\mathcal{T}^*)$. We have that \mathcal{U} is correctly guided and hence, \mathcal{U} is according to $\pi(\Sigma_{\mu, \bar{B}}^M)$. Let then $b = \pi(\Sigma_{\mu, \bar{B}}^M)(\mathcal{U})$. Let $\xi \in b$ be least such that $i_{\xi, b}^{\mathcal{U}}$ is defined. Because $\text{cf}^V(\delta(\mathcal{U})) = \lambda$, we can fix a cofinal $Y \subseteq \text{rng}(i_{\xi, b}^{\mathcal{U}})$ of order type λ . We have that $Y \in N$ and b is the unique branch c such that for some $\gamma \in c$, $Y \subseteq \text{rng}(i_c^{\mathcal{U}})$. It then follows that $b \in N$. By elementarity, there is a wellfounded branch c of $\mathcal{T} \upharpoonright \alpha$ such that $c \in M$. \square

Let then c be as in the claim. If $c = \pi(\Sigma_{\mu, \bar{B}}^M)(\mathcal{T} \upharpoonright \alpha)$ then we are done as then $\mathcal{Q}(c, \mathcal{T} \upharpoonright \alpha)$ exists. Suppose then $c \neq \pi(\Sigma_{\mu, \bar{B}}^M)(\mathcal{T} \upharpoonright \alpha)$. It then follows that $\mathcal{T} \upharpoonright \alpha$ has two well-founded branches implying that $\text{cf}(\delta(\mathcal{T} \upharpoonright \alpha)) = \omega$. Let $b = \pi(\Sigma_{\mu, \bar{B}}^M)(\mathcal{T} \upharpoonright \alpha)$ and let $\xi \in b$ be the least such that $i_{\xi, b}^{\mathcal{T} \upharpoonright \alpha}$ -exists. Again, by choosing an ω -sequence cofinal in $\delta(\mathcal{T} \upharpoonright \alpha)$ and repeating the proof of the claim using the fact that M is ω -closed, we get that $b \in M$. It then follows that, after all, $\mathcal{Q}(\mathcal{T} \upharpoonright \alpha) \in M$, contradiction!

Let now $\mathcal{T} \in M$ be the correctly guided tree on \mathcal{Q} for making C -generically generic. Let now $b = \pi(\Sigma_{\mu, \bar{B}}^M)(\mathcal{T})$. Repeating the above arguments we can show that $b \in M$. Let then $\mathcal{S} = \mathcal{M}_b^{\mathcal{T}}$. We have that B is generic over \mathcal{S} for the extender algebra of \mathcal{S} . Moreover, because \mathcal{S} is full, we have that $\mathcal{W}(B) \in \mathcal{S}[B]$. Hence, $\mathcal{M} \in M$.

To show that \mathcal{M} is κ^+ -iterable in $M[h_0 * g]$ we repeat the above argument and the proof of Lemma 2.13. Let Λ be the iteration strategy of \mathcal{M} . Given a tree $\mathcal{U} \in M$ on \mathcal{M} according to Λ and of limit length, by the above proof we can find a $\pi(\Sigma_{\mu, \bar{B}}^M)$ -iterate \mathcal{S} of \mathcal{Q} such that $(\mathcal{M}, \mathcal{U})$ are generic over \mathcal{S} . It then follows that $\Lambda(\mathcal{U}) \in \mathcal{S}[\mathcal{M}, \mathcal{U}]$ implying that if $b = \Lambda(\mathcal{U})$

then $b \in M[h_0 * g]$. b can then be identified in $M[h_0 * g]$ as the unique branch of \mathcal{U} such that the phalanx $\Phi(\mathcal{U} \cap \mathcal{M}_b^{\mathcal{U}})$ is countably iterable. \square

Lemma 2.37 *Suppose (M, π) is a perfect hull at μ_m for some m and let $\mathcal{Q} = \mathcal{P}^M$. Suppose $\sigma : \mathcal{Q} \rightarrow_{\Sigma_1} \mathcal{S}$ is such that $\mathcal{S} \in (H_{\kappa_M^+})^M$ and there is $k : \mathcal{S} \rightarrow \mathcal{P}$ such that $\pi \upharpoonright \mathcal{Q} = k \circ \sigma$. Then $\mathcal{S} = \mathcal{W}(\mathcal{S} | \delta_\omega^{\mathcal{S}})$.*

Proof. Let $T = T_{\mu_{m+1}}$. Let $B \in \mathcal{P}(\kappa_M) \cap M$ code \mathcal{Q}, \mathcal{S} . Let $\eta = \kappa_M$. We then have that $L[T, A_M, B] \models H_{\eta^{++}} = \mathcal{W}(\mathcal{W}(A_M, B))$. Let $\xi = \kappa^+$ and let $k : N \rightarrow L_\xi[T, A_M, B]$ be an elementary such that $L[T, A_M, B] \models “|N| = \eta^+ \wedge \text{crit}(k) > \eta^+”$ and let $S \in N$ be such that $k(S) = T$. Let $\nu = k^{-1}(\kappa^+)$. Notice that we have that $\mathcal{P}(\delta_\omega^{\mathcal{Q}}) \cap L_\nu[S, \mathcal{Q}] = (\mathcal{P}(\delta_\omega^{\mathcal{Q}}))^{\mathcal{Q}}$. It follows from Lemma 2.36 that $N \in M$

Suppose now that $\mathcal{S} \neq \mathcal{W}(\mathcal{S} | \delta_\omega^{\mathcal{S}})$. Let $\mathcal{M} \trianglelefteq \mathcal{W}(\mathcal{S} | \delta_\omega^{\mathcal{S}})$ be least such that $\mathcal{S} \triangleleft \mathcal{M}$ and $\rho_\omega(\mathcal{M}) = \delta_\omega^{\mathcal{S}}$. Notice that if $g \subseteq \text{Coll}(\omega, \mathcal{S})$ is W -generic and $x \in L_\nu[S, \mathcal{S}][g]$ codes \mathcal{S} then there is y coding \mathcal{M} such that $(x, y) \in p[S_n]$ where n is as in Lemma 2.21. This is because there is such a $y \in N[x]$. We can then repeat the proof of Lemma 2.21 to conclude that in fact $\mathcal{M} \trianglelefteq \mathcal{S}$. \square

Lemma 2.38 *Suppose $G \subseteq \text{Coll}(\omega, \kappa)$ is $W[g_0]$ -generic and let $g_i = G \cap \text{Coll}(\omega, \mu_i)$. Let $\mathbb{R}^* = \cup_{i < \omega} (\mathbb{R}^{W[g_0 * g_i]})$ and let $C_k = \cup_{i < \omega} B_{\mu_i, k, g_i}$. In $W(\mathbb{R}^*)$, let $\Phi = \{D \subseteq \mathbb{R}^* : D \text{ is Wadge reducible to some } C_k\}$. Then $L(\Phi, \mathbb{R}^*) \models AD^+$.*

Proof. It is enough to proof that the claim holds in some perfect (M, π) . Fix then such a perfect (M, π) at some μ_m . Let $(N, \sigma) > (M, \pi)$ be a perfect hull at μ_{m+1} and let $\Lambda = \Lambda^{M, N, \mathcal{P}^M}$. Let $N_i = (H_{(\mu_i^M)^+})^M$. Let \mathcal{Q} be the Λ -iterate of \mathcal{P}^M which is obtained via $\langle N_i : i < \omega \rangle$ -generic genericity iteration of \mathcal{P}^M . We have that if $i : \mathcal{P}^M \rightarrow \mathcal{Q}$ is the iteration embedding and $l : \mathcal{Q} \rightarrow \mathcal{P}$ is the last move of II then $\pi \upharpoonright \mathcal{P}^M = l \circ i$ implying that $\mathcal{Q} = \mathcal{W}(\mathcal{Q} | \delta_\omega^{\mathcal{Q}})$ (this follows from Lemma 2.37). Fix now some $G \subseteq \text{Coll}(\omega, \kappa_M)$ generic over $M[h_0 * g_0]$ and let g_i 's be defined similarly as above. We define C_k^M and Φ^M similarly working in $M[h_0 * g_0 * G]$. Let $\tau_k = \oplus_{i < \omega} \tau_{B_{\mu_m, k, G}^{\mathcal{Q}((\delta_i^{\mathcal{Q}})^+ \omega)^{\mathcal{Q}}}}$. We have that $\tau_k \in \mathcal{Q}$ and therefore, letting $\mathbb{R}^* = \cup_{i < \omega} (\mathbb{R}^{M[g_i]})$, we have that $B_{\mu_m, k, G} \cap \mathcal{Q}(\mathbb{R}^*) = C_k^M$. It then follows from the derived model theorem that $L(\Phi^M, \mathbb{R}^*) \models AD^+$. \square

Given $G \subseteq \text{Coll}(\omega, \kappa)$, we let $\mathcal{S}_G = L(\Phi, \mathbb{R}^*)$. Similarly, if (M, π) is a perfect hull at some μ_p and $G \subseteq \text{Coll}(\omega, \kappa_M)$ is M -generic then we let $\mathcal{S}_G^M = (L(\Phi, \mathbb{R}^*))^{M[G]}$.

2.7 Strongly \vec{B} -guided strategy

Lemma 2.39 *Suppose (R, τ) is an excellent hull at μ_0 and let $\mathcal{R} = \mathcal{P}^R$. Suppose $g \subseteq \text{Coll}(\omega, \mu_0)$ is generic and $p < \omega$. Then some tail of $\Sigma_{\mu_0, \vec{B}}^R$ strongly respects B_p .*

Proof. Suppose not. Let $\Psi = \Sigma_{\mu_0, \vec{B}}^R$. Fix $\langle \mathcal{R}_j, \mathcal{Q}_j, \sigma_j, \nu_j, \vec{T}_j : j < \omega \rangle \in M$ such that

1. $\mathcal{R}_0 = \mathcal{R}$,
2. \vec{T}_j is a stack according to Ψ on \mathcal{R}_j , \mathcal{R}_{j+1} is its last model and $i^{\vec{T}_j}$ -exists,
3. $\sigma_j : \mathcal{Q}_j \rightarrow \mathcal{R}_{j+1}$ and $\nu_j : \mathcal{R}_j \rightarrow \mathcal{Q}_j$ are elementary
4. $i^{\vec{T}_j} = \sigma_j \circ \nu_j$ and
5. $\sigma_j^{-1}(\tau_{B_p}^{\mathcal{R}_{j+1}}) \neq \tau_{B_p}^{\mathcal{Q}_j}$.

We let \mathcal{R}_ω be the direct limit of \mathcal{R}_j 's under $\pi^{\vec{T}_j}$'s. We have that there is $l : \mathcal{R}_\omega \rightarrow \mathcal{P}_\omega$. Fix some perfect hull $(M, \pi) > (R, \tau)$ at μ_1 and let $\Sigma = \Lambda^{R, M, \mathcal{R}_\omega}$. Let $\pi_{i, \omega} : \mathcal{R}_j \rightarrow \mathcal{R}_\omega$ be the direct limit embedding and let $\sigma_{j, \omega} = \pi_{j+1, \omega} \circ \sigma_j$.

Next let $N_i = (H_{(\mu_i^R)_+})^R$ and let $h \subseteq \text{Coll}(\omega, \kappa_R)$ be R -generic such that $h \in W[g]$. Let $\sigma = \cup_{i < \omega} (\mathbb{R}^{N_i[h \cap \text{Coll}(\omega, \mu_i^R)]})$. Let $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{S}_k^j, \vec{W}_k^j, \sigma_k^j, \nu_k^j, m_k^j : j, k < \omega \rangle$ be the iteration coming from simultaneous $\langle N_i : i < \omega \rangle$ -generic genericity iteration of $\langle \mathcal{R}_i, \mathcal{Q}_i : i < \omega \rangle$ via Σ . Also, we let $\langle \mathcal{R}_k^\omega, \mathcal{Q}_k^\omega, \vec{S}_k^\omega, \vec{W}_k^\omega, m_k^\omega, \sigma_k^\omega, \nu_k^\omega : k < \omega \rangle$ be the direct limit of $\langle \mathcal{R}_k^j, \mathcal{Q}_k^j, \vec{S}_k^j, \vec{W}_k^j, m_k^j, \sigma_k^j, \nu_k^j : j, k < \omega \rangle$.

It is not difficult to see that σ is indeed the set of reals of some symmetric generic extension of each \mathcal{R}_k^ω and \mathcal{Q}_k^ω . Moreover, letting D_k and C_k be the derived models of respectively $\mathcal{R}_k^\omega(\sigma)$ and $\mathcal{Q}_k^\omega(\sigma)$ we have that for every k ,

$$\mathcal{S}_h^R \subseteq D_k \cap C_k.$$

Notice that $B_p \cap \sigma \in \mathcal{S}_h^R$. Let then $s \in \text{Ord}^{< \omega}$ be such that $B_p \cap \sigma$ is definable from s in \mathcal{S}_h^R . Let $\xi = \Theta^{S_h^R}$. Notice then for each k , in D_k , $B_p \cap \sigma$ is the set definable from s in $L(P_\xi(\mathbb{R}))$. Let then k be such that for all $n > k$,

$$m_n^\omega(s) = s \text{ and } m_n^\omega(\xi) = \xi.$$

We then have that for every $n < \omega$,

$$\sigma_n^\omega(s, \xi) = \nu_n^\omega(s, \xi) = (s, \xi).$$

But this implies that for $n > k$, $\sigma_n^\omega(\tau_{B_p}^{\mathcal{Q}_n^\omega}) = \tau_{B_p}^{\mathcal{R}_{n+1}^\omega}$. This then implies that for every $n > k$, $\sigma_n(\tau_{B_p}^{\mathcal{Q}_n}) = \tau_{B_p}^{\mathcal{R}_{n+1}}$, contradiction. \square

Applying Lemma 2.39 repetitively and using the proof of Lemma 1.30, we get that

Corollary 2.40 *Suppose (R, τ) is an excellent hull at μ_0 and let $\mathcal{R} = \mathcal{P}_{\mu_0}^R$. Suppose $g \subseteq \text{Coll}(\omega, \mu_0)$ is generic. Then some tail of $\Sigma_{\mu_0, \vec{B}}^R$ has branch condensation.*

2.8 Finishing the proof of Theorem 0.8

Fix then (R, τ) as in the corollary above. It then follows that some tail of $\Sigma =_{\text{def}} \tau(\Sigma_{\mu_0, \vec{B}}^R)$ has branch condensation. Let \mathcal{Q} be the Σ -iterate of \mathcal{P}_{μ_0} such that letting Λ be the corresponding tail of Σ , Λ has a branch condensation. We then have that Λ is fullness preserving, it has branch condensation and it is a (κ, κ) -iteration strategy. Appealing to Lemma 2.9, we can assume Λ is a (κ^+, κ^+) -strategy. Let μ be an excellent point such that $\mathcal{Q} \in H_\mu$. It then follows from Lemma 2.3 that if $g \subseteq \text{Coll}(\omega, \mu)$ is W -generic then in $W[g]$, $L^\Lambda(\mathbb{R}) \models AD^+$. It then also follows that in $W[g]$, $L^\Lambda(\mathbb{R}) \models AD^+ + \theta_0 < \Theta$, contradiction!

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