On the indestructibility aspects of identity crisis. *†

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Abstract

We investigate the indestructibility properties of strongly compact cardinals in universes where strong compactness suffers from identity crisis. We construct an iterative poset that can be used to establish Kimchi-Magidor theorem from [22], i.e., that the first n strongly compact cardinals can be the first n measurable cardinals. As an application, we show that the first n strongly compact cardinals can be the first n measurable cardinals while the strong compactness of each strongly compact cardinal is indestructible under Levy collapses (our theorem is actually more general, see section 3). A further application is that the class of strong cardinals can be nonempty yet coincide with the class of strongly compact cardinals while strong compactness of any strongly compact cardinal κ is indestructible under κ-directed closed posets that force GCH at κ.

1 Introduction

Magidor in his seminal paper *How large is the first strongly compact cardinal? or A study on identity crises* (see [24]), showed that it is consistent

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that the least strongly compact cardinal can be the least measurable cardinal. This phenomena is called *identity crisis*, and we say that strongly compact cardinals suffer from identity crisis. Later, Kimchi and Magidor ([22]) extended this result by showing that it is consistent relative to n supercompact cardinals that the first n measurable cardinals are the first n strongly compact cardinals. Since then the identity crisis of strongly compact cardinals have been studied extensively and many results have appeared in print. Apter and Cummings showed that the class of strong cardinals can be nonempty yet coincide with the class of strongly compact cardinals. Apter and Gitik showed that the first strongly compact cardinal can be the first measurable cardinal and fully indestructible (see [8]) (even fully indestructible strongly compact cardinals suffer from identity crisis). Apter and the author extended this result to two strongly compact cardinals; however, they failed to get full indestructibility for the second strongly compact cardinal. Here are the more formal presentations of these results along with few other results on identity crisis.

Theorem 1 *The following theories are relatively consistent with n supercompact cardinals.*

1. (**Kimchi-Magidor, [22]**) *The first n-strongly compact cardinals are the first n measurable cardinals.*
2. (**Apter-Gitik, [8]**) The first strongly compact cardinal is the least measurable and fully indestructible.
3. (**Apter-S, [12]**) *The first two strongly compact cardinals κ_0 and κ_1 are the first two measurable cardinals, κ_0 is fully indestructible, and κ_1 is indestructible under κ_1-directed closed (κ_1, ∞)-distributive partial orderings.*
4. **(Apter-Cummings, [6])** *The first n measurable cardinals $\langle \kappa_i : i < n\rangle$ are the first n strongly compact cardinals, each κ_i is κ_i^+-supercompact, and $2^{\kappa_i} = \kappa_i^{++}$.*
5. **(Apter-S, [11])** *The first n measurable Woodin cardinals are the first n strongly compact cardinals.*

Theorem 2 (Apter-Cummings, [7]) *It is consistent relative to proper class of suppercompact cardinals that the class of strong cardinals coincides with the class of strongly compact cardinals.*

There are many other results of this kind that have appeared in print. The interested reader should consult [1], [3], [5], [6], [8], and [11]. The common theme in all of the results in Theorem 1, which has its origins in Magidor's original work (see [24]), is to characterize strong compactness, a global property, with local properties like measurability or limited amount of supercompactness and etc. Thus far, the available methods have been successful only when the goal is to characterize the first n strongly compact cardinals via local properties. All attempts to extend such characterizations to ω cardinals have failed. The main problem of the field is the following;

Main Open Problem. Can the first ω measurable cardinals be the first ω strongly compact cardinals?

Theorem 2 is different from the rest in that the characterization of strong compactness is via strongness which is a global property but is consistency wise weaker then strongly compact cardinals. We note in passing that it is not a trivial matter to show that strongly compact cardinals have higher consistency strength than strong cardinals. One needs core model machinery to evaluate lower bounds of the consistency strength of strongly compactness (see [26]). In fact, identity crisis is one of the reasons behind the difficulty of evaluating the consistency strength of strong compactness inside the large cardinal hierarchy (it is not known, for instance, that strongly compact cardinals are stronger consistency wise than superstrong cardinals). Characterizing strongly compact cardinals via global properties that are weaker than strongly compacts can be tricky as well as many global properties when coupled with strong compactness imply that there are many strongly compact cardinals in the universe (see [7] and Proposition 4 of Section 7).

In this paper, our goal is to investigate the indestructibility properties of strong compactness in the models satisfying the theories of Theorem 1 and Theorem 2. The following is part of our Main Theorem 1 (see section 3 for the more general version).

Theorem 3 *It is consistent relative to n supercompact cardinals that the first n measurable cardinals are the first n strongly compact cardinals while the strong compactness of any strongly compact cardinal is indestructible under Levy collapses.*

Few words on the motivations behind Theorem 3 are probably in order. All the results on identity crisis that deal with more than one strongly compact cardinal are a combination of product forcing and iterated forcing. Our goal has been to remove the product forcing component of these arguments and instead, use an iteration for the entire forcing. Theorem 3 is an application of this method and it cannot be proved using the product forcing technique used before (see the discussion in section 3.2). Thus, the main technical contribution of the paper is our poset. At this point, however, our poset or its modifications don't seem to be helpful in resolving the Main Open Problem.

The indestructibility phenomena for strong compactness in the universes where strong compactness suffers from identity crisis is a mysterious one. As long as we care only about one strongly compact cardinal, everything is under control as illustrated by 2 of Theorem 1. This is mainly because iterations of Prikry forcing can be used in those situations (see [24]). However, such iterations cannot work with more than one strongly compact cardinal as we cannot iterate Prikry forcing above a strongly compact. The only other available method, Reverse Easton Iterations, are much harder to control and at the moment we do not know how to get full indestructibility for strongly compact cardinals in models where the first two strongly compact cardinals are the first two measurable cardinals.

We organized the paper as follows. In 2, we explain our notation and list all the known results and their modifications that we need. In 3.1, we define and prove the existence of a *nice universal Laver function*. In 3.2, we define our forcing and prove some basic properties of it. In 3.3, we give the proof of the Main Theorem 1. In 4, we use the ideas involved in the proof of Main Theorem 1 to generalize Theorem 2. In 5, we make some concluding remarks.

2 Preliminary Material

Some authors, especially those associated with California school, write $p \leq q$ for "p extends q", and some especially those associated with the Israel school, write $p \geq q$ for "p extends q". The author has worked with people of both schools and there have been many confusions involving notation. This prompted the author to use the notation used in [9] and [15]. Thus, when forcing, $p \Vdash q$ will mean that "p extends q". If G is V-generic over $\mathbb{P}$, we will abuse notation somewhat and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$.

For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. Iterations are sequences $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha < \kappa\rangle$ where $\dot{\mathbb{Q}}_\alpha \in V^{\mathbb{P}_\alpha}$ is the poset used at stage α. If $\alpha \leq \beta$ then we let

1. $\mathbb{P}^{\alpha,\beta} \in V^{\mathbb{P}_\alpha}$ be the iteration in the interval $[\alpha, \beta]$.
2. $\mathbb{P}^{>\alpha,\beta} \in V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha}$ be the iteration in the interval $(\alpha, \beta]$.
3. $\mathbb{P}^{\alpha,<\beta} \in V^{\mathbb{P}_\alpha}$ be the iteration in the interval $[\alpha, \beta)$.
4. $\mathbb{P}^{>\alpha,<\beta} \in V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha}$ be the iteration in the interval (α, β).

If β is the length of the iteration then we let $\mathbb{P}^{\alpha,\beta} = \mathbb{P}^{\alpha}$ and $\mathbb{P}^{>\alpha,\beta} = \mathbb{P}^{>\alpha}$. Thus, $\mathbb{P} = \mathbb{P}_\alpha * \mathbb{P}^\alpha = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \mathbb{P}^{>\alpha}$. If $G \subseteq \mathbb{P}$ is a generic object then we define G_α, $G^{\alpha,\beta}$, $G^{>\alpha,\beta}$, $G^{\alpha,<\beta}$, $G^{>\alpha,<\beta}$, G^α and $G^{>\alpha}$ accordingly. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x, and $i_G(\dot{x})$ or $\dot{x}_G$ will be the interpretation of $\dot{x}$ using G. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If κ is a regular cardinal, $\mathrm{Add}(\kappa, 1)$ is the standard partial ordering for adding a single Cohen subset of κ. If $\mathbb{P}$ is an arbitrary partial ordering, $\mathbb{P}$ is κ-distributive if for every sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\kappa} D_\alpha$ is dense open. Equivalently, $\mathbb{P}$ is κ-distributive if and only if $\mathbb{P}$ adds no new subsets of κ. $\mathbb{P}$ is κ-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if every two elements p_ρ and p_ν have a common upper bound of the form p_σ) there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), then player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ^+-directed closed, then $\mathbb{P}$ is κ-strategically closed. Also, if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$. $\mathbb{P}$ is $\prec\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages (again choosing the trivial condition at stage 0), then player II has a strategy which ensures the game can always be continued.

Suppose $\kappa < \lambda$ are regular cardinals. A partial ordering $\mathbb{S}(\kappa, \lambda)$ that will be used in this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality κ to λ. Specifically, $\mathbb{S}(\kappa, \lambda) = \{s : s$ is a bounded subset of λ consisting of ordinals of cofinality κ so that for every $\alpha < \lambda$, $s \cap \alpha$ is non-stationary in $\alpha\}$, ordered by end-extension. Two things which can be shown (see [13]) are that $\mathbb{S}(\kappa, \lambda)$ is δ-strategically closed for every $\delta < \lambda$, and if G is V-generic over $\mathbb{S}(\kappa, \lambda)$, in $V[G]$, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality κ has been introduced. It is also virtually immediate that $\mathbb{S}(\kappa, \lambda)$ is κ-directed closed.

Suppose $\kappa < \lambda$ are regular cardinals and λ is an inaccessible cardinal. A partial ordering $\mathbb{Q}(\kappa, \lambda)$ that will also be used in this paper is the partial ordering for adding a club to λ which is disjoint from the set of inaccessibles $< \lambda$. Specifically, $\mathbb{Q}(\kappa, \lambda) = \{s : s$ is a bounded club subset of (κ, λ) such that whenever $\eta \in (\kappa, \lambda)$ is inaccessible, $s \cap \eta < \eta\}$ ordered by end-extension. It is immediate that $\mathbb{Q}(\kappa, \lambda)$ is κ^+-directed closed and $\mathbb{Q}(\kappa, \lambda)$ is $< \lambda$-strategically closed. Moreover, for any $\eta < \lambda$ and any condition $p \in \mathbb{Q}(\kappa, \lambda)$ there is an extension q of p such that $\{r \in \mathbb{Q}(\kappa, \lambda) : r \Vdash q\}$ is

η-directed closed.

We mention that we are assuming familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. An interested reader may consult [21] for more information. Following [21], we let $\mathcal{P}_\kappa(\lambda) = \{x : x \subseteq \lambda \wedge |x| < \kappa\}$. We say κ is generically measurable if it carries a normal κ-complete precipitous ideal (generic large cardinals were first considered by Foreman, see [18] and [17]).

Suppose κ is a supercompact cardinal. Then f is a Laver function for κ if whenever X is a set and $\lambda > \kappa$ is such that $|TC(X)| \leq \lambda$ then then there is an elementary embedding $j : V \to M$ witnessing that κ is λ supercompact and $j(f)(\kappa) = X$. Laver (see [23]) showed that each supercompact cardinal has a Laver function. In this paper we will need *universal Laver function*: f is a universal Laver function if for any supercompact cardinal κ, $f \restriction \kappa : \kappa \to V_\kappa$ and $f \restriction \kappa$ is a Laver function for κ. Laver's original proof, suitably modified, also shows that there is a universal Laver function (see [2]).

Suppose κ is a measurable cardinal (supercompact cardinal, strongly compact cardinal and etc.) Then we say κ's measurability (supercompactness, strong compactness and etc.) is fully indestructible or Laver indestructible if whenever $\mathbb{P}$ is a κ-directed closed poset, κ remains measurable (supercompact, strongly compact, and etc.) in $V^{\mathbb{P}}$. Laver showed that if κ is supercompact then after doing, what is sometimes refereed to, Laver preparation, κ's supercompactness becomes fully indestructible (see [23]).

We will need the following concepts and theorem all due to Hamkins. A forcing notion $\mathbb{P}$ admits a closure point at δ if it factors as $\mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is non-trivial, $|\mathbb{Q}| \leq \delta$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed"(this notion is due to Hamkins). δ-strategic closure certainly follows from just δ-closure. In this paper, we do not use posets that are δ-closed but are not δ-strategically closed. Therefore, there is no need to explain what δ-strategic closure is.

Theorem 4 (Hamkins, [19]) *If $V \subseteq V[G]$ admits a closure point at δ and $j : V[G] \to M[j(G)]$ is an ultrapower embedding in $V[G]$ with $\delta = \mathrm{cp}(j)$, then $j \restriction V : V \to M$ is a definable class in V.*

We will also make a heavy use of term partial ordering. This concept is due to Laver and first appeared in [16]. Given a poset $\mathbb{P}$ and a poset $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$, we let $\mathbb{Q}^*$ be the partial ordering with the domain

$$\{\tau : \tau \in V^{\mathbb{P}} \text{ is a term such that } \Vdash_{\mathbb{P}} \tau \in \dot{\mathbb{Q}} \text{ and for any } \pi \in V^{\mathbb{P}} \text{ such that } \pi \text{ has a smaller rank than } \tau \text{ there is } p \in \mathbb{P},\, p \Vdash \tau \neq \pi\}$$

We then let $\tau \Vdash_{\mathbb{Q}^*} \pi$ if $\Vdash_{\mathbb{P}} \tau \Vdash_{\dot{\mathbb{Q}}} \pi$. It is clear that $\mathbb{Q}^* \in V$. We write $t(\dot{\mathbb{Q}}/\mathbb{P})$ for the term partial ordering associated with $\dot{\mathbb{Q}}$ with respect to $\mathbb{P}$. The following proposition is easy to verify:

Proposition 1 (Term forcing argument) *Suppose $\mathbb{P}$ and $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ are as above. Then*

1. *(see [16]). Suppose $G \subseteq \mathbb{P}$ and $H \subseteq t(\dot{\mathbb{Q}}/\mathbb{P})$ are V-generic. Then the filter generated by the set $\{\tau_G : \tau \in H\}$ is a $V[G]$-generic filter over $\dot{\mathbb{Q}}_G$.*
2. *If for some κ, $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-strategically closed or κ-directed closed" then in V, $t(\dot{\mathbb{Q}}/\mathbb{P})$ is κ-strategically closed or κ-directed closed.*

We present two by now standard methods of lifting ground model embeddings to generic extensions. We will be using them repeatedly and therefore, it is best if we give them descriptive names and refer back to them whenever we need.

The counting argument. Suppose $j : V \to M$ is an embedding, $\mathbb{P} \in M$ is a poset such that $M \vDash$ "$\mathbb{Q}$ is $< \lambda$-strategically closed" and the cardinality of the set $\{D \subseteq \mathbb{P} : D \in M$ is a dense set $\}$ is $\leq \lambda$. Then there is $g \in V$ which is M-generic for $\mathbb{P}$. For further details see Fact 1 on page 8 of [14].

The transferring argument. Suppose $j : V \to M$ is an extender embedding given by some (κ, λ)-extender, $\mathbb{P} \in V$ is a poset such that $V \vDash$ "$\mathbb{Q}$ is (κ, ∞)-distributive" and $G \subseteq \mathbb{P}$ is a V-generic for $\mathbb{P}$. Let $H \subseteq j(\mathbb{P})$ be the filter generated by the set j"G. Then H is an M-generic filter for $j(\mathbb{P})$ and j lifts to $j^* : V[G] \to M[H]$. For further details see Fact 2 on page 7 of [14].

3 Indestructibility, identity crisis and measurable cardinals.

In this section, we investigate the indestructibility properties of the first n strongly compact cardinals in models where they are the first n measurable cardinals. More specifically we prove the following theorem.

Theorem 5 (Main Theorem 1) *It is consistent relative to n supercompact cardinals that the first n measurable cardinals $\langle \kappa_i : i < n \rangle$ coincide with the first n strongly compact cardinals while the strong compactness of any κ_i is indestructible under κ_i-directed closed posets $\mathbb{P}$ that create only finitely many measurables and force GCH at each one of them.*

Examples of partial orderings that are covered by Main Theorem 1 are the Levy collapses and adding Cohen subsets (not too many, though). Of course, there are many more. Essentially, κ_i's strong compactness is indestructible under any partial ordering of the form $\mathbb{R} * \mathbb{Q} * Add(\kappa_i^+, 1)$ where $\mathbb{R}$ is any κ_i-directed closed poset and $\mathbb{Q} \in V^{\mathbb{R}}$ is the partial ordering that adds clubs disjoint from inacessibles to measurables of $V^{\mathbb{R}}$ different from κ_i. It was previously not know how to get a model where the first two strongly compact cardinals coincide with the first two measurable cardinals while both strongly compacts are indestructible under Levy collapses. We can also borrow Apter-Gitik theorem (see 2 of Theorem 1) and get a model in which the first n strongly compacts coincide with the first n measurable cardinals, the first strongly compact is fully indestructible while others have the indestructibility properties of Main Theorem 1. It will be clear from the proof that in our model all measurable cardinals are fully indestructible.

In the following subsections we give the proof of Main Theorem 1. Here is how the proof is organized. In 3.1 we define *nice universal Laver function* and prove that it exists. In 3.2, we define our poset and establish some basic properties of it. In 3.3, we show that the poset of 3.2 is as desired.

3.1 A special universal Laver function

We say f is a *special universal Laver function* if

1. $dom(f)$ consists only of measurable cardinals.
2. If $\lambda \in dom(f)$, then $f(\lambda) = \langle\langle\lambda_i : i \leq k\rangle, X\rangle$ where $1 \leq k < \omega$, $\lambda = \lambda_0 < \lambda_1 < ... < \lambda_k$ are cardinals such that there are no inaccessible cardinals in the interval $(\lambda_{k-1}, \lambda_k]$ and $|TC(\{X\})| \leq \lambda_k$.

   For $\lambda \in dom(f)$, let $n(\lambda)$ be such that $f(\lambda) = \langle\langle\lambda_i : i \leq n(\lambda)\rangle, X\rangle$. Also, $f^0(\lambda) = \langle\lambda_i : i \leq n(\lambda)\rangle$, $f^1(\lambda) = X$, and $f^0(\lambda)_i = \lambda_i$.

3. If for some λ the set $\{\beta < \lambda : f(\beta) \notin V_\lambda\}$ is unbounded in λ then $\lambda \notin dom(f)$.
4. If $\lambda \in dom(f)$ then $f^0(\lambda)_i \notin dom(f)$ for any $0 < i \leq n(\lambda)$ and $f``(\lambda, f^0(\lambda)_i) \subseteq V_{f^0(\lambda)_i}$ for all $i \leq n(\lambda)$.
5. If $\lambda \in dom(f)$ and there is $\beta \in \lambda \cap dom(f)$ such that for some $i \leq n(\beta)$, $f^0(\beta)_i > \lambda$ then $f(\beta)_{k-1} < f^0(\lambda)_{n(\lambda)} < f^0(\beta)_k$ where k is the least such that $\lambda < f^0(\beta)_k$ (this actually follows from 4).
6. if κ is a supercompact cardinal then $f``\kappa \subseteq V_\kappa$ and $\kappa \notin dom(f)$

7. if κ is a supercompact cardinal, $\langle\langle\lambda_i : i \leq k\rangle, X\rangle$ is some sequence such that $\langle\lambda_i : i \leq k\rangle$ is increasing, $\lambda_0 = \kappa$, there are no measurable cardinals in the interval $(\lambda_{k-1}, \lambda_k]$, and $|TC(\{X\})| \leq \lambda_k$ then for any $\lambda \geq \lambda_k$ there is $j : V \to M$ witnessing that κ is λ-supercompact, $j(f)(\kappa) = \langle\langle\lambda_i : i \leq k\rangle, X\rangle$, and if F is the graph of f then $j(F) \cap H_\lambda = F \cap H_\lambda$

In the next theorem we show that it is consistent that there is a special universal Laver function. Notice that property 7 is the only part that is somewhat unclear. We call it the *coherence* property. The reason for the other requirements is that we would like to make the definition of our poset clearer. Other than that we could have chosen to work with any Laver function with the coherence property and distill it through 1-6 while defining our poset. Also, the theorem isn't stated in its optimal form, but that is all we need in this paper. Also, the only reason that we want to show that there is a special universal Laver function is to prove Main Theorem 1 from the stated hypothesis. If one wants to assume a cardinal which is Woodin with respect to supercompact cardinals, then for any universal Laver function, there are many supercompact cardinals that satisfy the coherence property.

Theorem 6 *Assume GCH and suppose V has supercompact cardinals. There is then a partial ordering $\mathbb{P} \in V$ such that all supercompact cardinals of V remain supercompact in $V^{\mathbb{P}}$, GCH holds in $V^{\mathbb{P}}$ and there is a special universal Laver function in $V^{\mathbb{P}}$.*

Proof: Let $\mathbb{P} \in V$ be the canonical poset that forces GCH. $\mathbb{P}$ is a Reverse Easton Iteration that adds a Cohen subset to every regular cardinal κ at stage κ, i.e., $\dot{\mathbb{Q}}_\kappa = (Add(\kappa, 1))^{V^{\mathbb{P}_\kappa}}$ if $\Vdash_{\mathbb{P}_\kappa}$ "$\check{\kappa}$ is regular" and otherwise $\dot{\mathbb{Q}}_\kappa$ is trivial. Note that because GCH already holds, $\Vdash_{\mathbb{P}_\kappa}$ "$\check{\kappa}$ is regular" iff κ is regular in V. Moreover, standard arguments show that $\mathbb{P}$ preserves all cardinals and cofinalitis. Let $G \subseteq \mathbb{P}$ be a V-generic. For each $V[G]$-cardinal λ, let $g_\lambda = G^{\lambda,\lambda}$ be $V[G_\lambda]$-generic object for $\dot{\mathbb{Q}}_\lambda$. Let now $F : ORD \to V[G]$ be the partial function given by $F(\alpha) = f_\alpha$ where $f_\alpha : \alpha^+ \to \mathcal{P}(\alpha)$ is the canonical function induced by g_α.

Claim. In $V[G]$, for all supercompact cardinals κ and $\lambda > \kappa$ there is $j : V[G] \to M$ witnessing that κ is λ-supercompact, κ is not λ-supercompact in M and $j(F) \cap (H_\lambda)^{V[G]} = F \cap (H_\lambda)^{V[G]}$ (we identify F with its graph).

Proof. Suppose κ is a supercompact cardinal of V. We first show the claim for singular cardinals of cofinality $> \kappa$. Let λ be such a cardinal. Let $j : V \to M$ be a λ-supercompactness embedding such that λ isn't supercompact in M. Then standard arguments show that j lifts to $j^* : V[G_\lambda] \to M[G_\lambda * g_{\lambda^+}][H]$ where $j^* \in V[G_\lambda * g_{\lambda^+}]$ and H is a generic for $j(\mathbb{P}_\lambda)^{>\lambda^+}$. Because $\mathbb{P}^\lambda$ is λ-directed closed, we have that $(\mathcal{P}_\kappa(\lambda))^{V[G]} = (\mathcal{P}_\kappa(\lambda))^{V[G_\lambda]}$. Let $\nu = \{X \in V[G_\lambda] : j"\lambda \in j^*(X)\}$ be the ultrafilter derived from j^*. Then $\nu \in V[G_\lambda * g_{\lambda^+}]$. Note that j^* is an ultrapower embedding, i.e., for any set $a \in M[G_\lambda * g_{\lambda^+}][H]$ there is $f \in V[G_\lambda]$ such that $f : (\mathcal{P}_\kappa(\lambda))^{V[G_\lambda]} \to V[G_\lambda]$ and $a = [f]_\nu$. Because $(H_{\lambda^+})^{V[G_\lambda]} = (H_{\lambda^+})^{V[G]}$, we must have that $Ult(V[G_\lambda], \nu)$ agrees with $Ult(V[G], \nu)$ on sets of rank $j(\lambda)$. In particular, $(H_\lambda)^{V[G]} = (H_\lambda)^{M[G_\lambda * g_{\lambda^+}][H]}$. We then immediately get that if $j_\nu : V[G] \to Ult(V[G], \nu)$ then $j_\nu(F) \cap (H_\lambda)^{V[G]} = F \cap (H_\lambda)^{V[G]}$. Moreover, because κ is not supercompact in M, by Theorem 4, κ cannot be supercompact in $M[G_\lambda * g_{\lambda^+}][H]$ and hence, in $Ult(V[G], \nu)$.

It is now easy to show that the coherence property holds for any λ. Fix such a λ. Let $\eta > \lambda$ be a singular cardinal of cofinality $> \kappa$ and let $j : V[G] \to M$ witness that $j(F) \cap (H_\eta)^{V[G]} = F \cap (H_\eta)^{V[G]}$ and κ isn't supercompact in M. Let $\nu = \{X \subseteq \mathcal{P}_\kappa(\lambda) : j"\lambda \in X\}$. Then we have $i_\nu : V[G] \to Ult(V[G], \nu)$ and $k : Ult(V[G], \nu) \to M$ such that $\mathrm{cp}(k) > \lambda$ and $j = k \circ i_\nu$. It then follows that $i_\nu(F) \cap (H_\lambda)^{V[G]} = F \cap (H_\lambda)^{V[G]}$ and κ isn't supercompact in $Ult(V[G], \nu)$. *Q.E.D.*

We now define our special Laver function f. The general idea is Laver's original idea. We let $W = V[G]$ and we use F to choose the minimal counterexamples. Suppose for some measurable α we have defined $f \restriction \alpha$ and we want to decide whether $\alpha \in dom(f)$ and if it is then we also want to define $f(\alpha)$. If α is supercompact we let $f(\alpha)$ be undefined. If the set $\{\beta < \alpha : f(\beta) \notin W_\alpha\}$ is unbounded in α then we let $f(\alpha)$ be undefined. If there is $\beta < \alpha$ such that $f^0(\beta)_i = \alpha$ for some $i \leq n(\beta)$ then we let $f(\alpha)$ be undefined. Suppose now that α isn't supercompact, the set $\{\beta < \alpha : f(\beta) \notin W_\alpha\}$ is bounded below α and there is no $\beta < \alpha$ such that $f^0(\beta)_i = \alpha$ for some $i \leq n(\beta)$. Let $\gamma = \sup(\{\beta < \alpha : f(\beta) \notin W_\alpha\})$. Let $f^* : \alpha \to W_\alpha$ be the function given by $f(\xi) = 0$ if $\xi \leq \gamma$ and $f^*(\xi) = f(\xi)$ otherwise. Suppose there are λ, an increasing sequence $\langle \lambda_i : i \leq n \rangle$ of cardinals and a set X such that $\lambda \geq \lambda_n$, $TC(\{X\}) \leq \lambda$, $\lambda_0 = \alpha$, there are no inaccessible cardinals in the interval $(\lambda_{n-1}, \lambda_n]$, and there is no supercompactness measure μ over $\mathcal{P}_\alpha(\lambda)$ such that $j_\mu : W \to Ult(W, \mu)$ is such that $j_\mu(F) \cap (H_\lambda)^W = F \cap (H_\lambda)^W$ and $j_\mu(f^*)(\alpha) = \langle \langle \lambda_i : i \leq n \rangle, X \rangle$. We then let λ be the least such cardinal and $\langle \langle \lambda_i : i \leq n \rangle, X \rangle$ be the f_λ-least sequence witnessing the above statement. Suppose there is $\beta < \alpha$ such that $f^0(\beta)_i > \alpha$ for some least i, and either $\lambda_n \geq f(\beta)_i$ or $X \notin W_{f(\beta)_i}$ then we let $f(\alpha)$ be undefined. Otherwise we let $f(\alpha) = \langle \langle \lambda_i : i \leq n \rangle, X \rangle$.

It is not hard to see that f is a special universal Laver function. It is clear that whenever κ is a supercompact cardinal then $f"\kappa \subseteq V_\kappa$ (because by reflection witnesses are always in V_κ). Our definition of f was specifically designed to accommodate 1-5 in the definition of special universal Laver function. Thus, it remains to verify 7. Let $W = V[G]$. Suppose 7 is not true for κ. Then we have a least cardinal λ and f_λ-least $\langle\langle\lambda_i : i \leq n\rangle, X\rangle$ such that $n \geq 1$ and no supercompactness measure μ over $\mathcal{P}_\kappa(\lambda)$ is such that $j_\mu : W \to M$ witnesses that $j_\mu(f)(\kappa) = \langle\langle\lambda_i : i \leq n\rangle, X\rangle$ and for any $i \leq n$, $j(f) \cap (H_\lambda)^W = f \cap (H_\lambda)^W$ (we identify f with its graph). Let μ be a supercompactness measure over $\mathcal{P}_\kappa(\lambda^{++})$ such that $j_\mu : W \to M$ witnesses that $j_\mu(F) \cap (H_{\lambda^{++}})^W = F \cap (H_{\lambda^{++}})^W$ but κ is not λ^{++}-supercompact cardinal in M. It is easy to see that κ must be in the domain of $j_\mu(f)$. Because $j_\mu(F)(\lambda) = f_\lambda$, we in fact have that $j_\mu(f)(\kappa) = \langle\langle\lambda_i : i \leq \omega\rangle, X\rangle$. The only problem now is that μ was a λ^+-supercompactness measure. We overcome this by letting μ^* be the λ-supercompacness measure derived from j_μ. Then an easy factorization argument shows that in fact μ^* witnesses 7 (see [23] or [21] for more details).

□

3.2 The poset

In this subsection, we define our partial ordering. From now on until the end of section 3 we assume that we have n supercompact cardinals $\langle\kappa_i : i < n\rangle$. We also assume that there are no inaccessible cardinals in V above κ_{n-1}. Moreover, as it is a folklore result, we also assume without losing generality, that GCH holds in V. By Theorem 6, without losing generality, we can also assume we have a special universal Laver function f.

Before we go on, we give a little bit of motivation. Our partial ordering, just like many of the partial orderings used in the similar contexts, iteratively destroys the measurable cardinals other than κ_is. Unlike the previous partial orderings, our final partial ordering will be an iteration of length κ_{n-1} and this requires "postponing" the stages at which we kill measurable cardinals. To illustrate the problem lets take the well known Kimchi-Magidor construction. They start with n-supercompact cardinals $\langle\lambda_i : i < n\rangle$ and in their final model the only measurable cardinals are λ_i which also preserve their strong compactness. The ad hoc assumptions are that each λ_i's supercompactness is fully indestructible and also there are no measurable cardinals above λ_{n-1}. The partial ordering used is a product $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1 \times \mathbb{P}_2 \times ...\mathbb{P}_n$. $\mathbb{P}_i$ is the Reverse Easton Iteration of length λ_i that adds non-reflecting stationary sets to every measurable cardinal in the interval $(\lambda_{i-1}, \lambda_i)$, $(\lambda_{-1} = \omega)$ consisting of points of cofinality λ^+_{i-1}. The proof that λ_i remains strongly compact cardinal in the final model is a downward induction. Because of indestructibility, λ_i is supercompact cardinal in $V^{\mathbb{P}_{i+1} \times \mathbb{P}_{i+2} \times \ldots \times \mathbb{P}_n}$. One then uses various lifting arguments to show that λ_i remains strongly compact after forcing with $\mathbb{P}_i$. Lets now take the representative case $n = 2$ and lets imagine that $\mathbb{P} = \mathbb{P}_0 * \mathbb{P}_1$ is an iteration. Then if $j : V \to M$ is an embedding witnessing some degree of supercompactness of κ_0 then $j(\mathbb{P}_0) = \mathbb{P}_0 * \mathbb{S}(\omega, \kappa_0) * \mathbb{P}_{tail}$. Now we have no way of finding a generic for $\mathbb{P}_{tail}$. The reason is that on the V side we have a forcing that looks like $(\mathbb{P}_{tail})_{\kappa_1}$ namely $\mathbb{P}_1$ but $\mathbb{P}_1 \in V^{\mathbb{P}_0}$ whereas $(\mathbb{P}_{tail})_{\kappa_1} \in V^{\mathbb{P}_0 * \mathbb{Q}(\omega,\kappa_0)}$. Also, $(\mathbb{P}_{tail})_{\kappa_1}$ and $\mathbb{P}_1$ are not quite the "same" as one adds non-reflecting stationary sets of cofinality ω while the other of cofinality κ_0 (This part is less worrisome, as one could add non-reflecting stationary sets of unspecified cofinality. This idea is due to Apter, but we will not use it as it seems to create other problems in our situation.).

Our solution to the first problem is to just not do any forcing at stages that potentially look like κ_0 and we postpone the stage at which cardinals that "look like" κ_0 get killed (one way that cardinals potentially look like κ_0 is that they are in the domain of f. Of course κ_0 is not in the domain of f but when j is some embedding that we would like to lift then κ is in the domain of $j(f)$.). We will use f to decide what cardinals "look like" κ_0. The second problem is handled similarly; we will arrange it so that $(\mathbb{P}_{tail})_{\kappa_1}$ adds clubs consisting of ordinals $> \kappa_0$ and disjoint from inaccessibles. The reason that we want to use iteration instead of product is that we want to prove that in our final model the strong compactness of κ_is is indestructible. It is not possible to achieve such indestructibility by a product forcing as the one above. To see this suppose that in $V^{\mathbb{P}_0 \times \mathbb{P}_1}$ both κ_0 and κ_1 are indestructible. Then $V^{\mathbb{P}_0 \times \mathbb{P}_1} = V^{\mathbb{P}_1 \times \mathbb{P}_0}$. But by [20], κ_1 is superdestructible in $V^{\mathbb{P}_1 \times \mathbb{P}_0}$ as $\mathbb{P}_0$ has size $< \kappa_1$.

Our partial ordering is a Reverse Easton Iteration of length κ_{n-1}. We start by defining the first κ_0 steps. We let $\mathbb{Q}_0 = Add(\omega_1, 1)$. Suppose we have defined $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$. We have to describe what $\dot{\mathbb{Q}}_\alpha$ is.

Case 1. Either α is non-measurable and there is no $\beta \in \alpha \cap dom(f)$ such that $f^0(\beta)_{n(\beta)} = \alpha$, or α is measurable and there is $\beta \in \alpha + 1 \cap dom(f)$ such that for some $i < n(\beta)$, $f^0(\beta)_i = \alpha$

Then, we let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing.

Case 2. α is a cardinal such that there is $\beta \in \alpha \cap dom(f)$ such that $\alpha = f^0(\beta)_{n(\beta)}$.

Suppose that $f(\beta) = \langle \langle \lambda_i : i \leq n(\beta) \rangle, X \rangle$. Suppose $\lambda_{i_0} < \lambda_{i_1} < ... < \lambda_{i_k}$ are the measurable cardinals of the sequence $\langle \lambda_i : i \leq n(\beta) \rangle$. Suppose $X \neq \dot{\mathbb{Q}} \in V^{\mathbb{P}_\alpha}$ for some β-directed closed poset $\mathbb{Q}$ such that in $V^{\mathbb{P}_\alpha * \mathbb{Q}}$, if $\eta \in [\lambda_0, \lambda_{n(\beta)})$ is a measurable cardinal then GCH holds at η. Then we let $\dot{\mathbb{Q}}_\alpha = \mathbb{Q}(\lambda^+_{i_0 - 1}, \lambda_{i_0}) * \mathbb{Q}(\lambda^+_{i_1 - 1}, \lambda_{i_1}) * ... * \mathbb{Q}(\lambda^+_{i_k - 1}, \lambda_{i_k})$ where $\lambda_{-1} = \omega$. Suppose now that $\dot{\mathbb{Q}}$ is β-directed closed and is such that in $V^{\mathbb{P}_\alpha * \mathbb{Q}}$, if $\eta \in [\lambda_0, \lambda_{n(\beta)})$ is a measurable cardinal then GCH holds at η. Then we let $\delta_0 < \delta_1 < ... < \delta_m$ be the measurable cardinals of $V^{\mathbb{P}_\alpha * \mathbb{Q}}$ that are in the interval $[\beta, f^0(\beta)_{n(\beta)})$. We let $\delta_{-1} = \omega$ if $\delta_0 = \beta$ and $\delta_{-1} = \beta$ if $\delta_0 > \beta$. We then let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}} * \dot{\mathbb{S}}$ where $\dot{\mathbb{S}} = \mathbb{Q}(\delta_{-1}^+, \delta_0) * \mathbb{Q}(\delta_1^+, \delta_2) * ... * \mathbb{Q}(\delta_{m-1}^+, \delta_m)$.

Case 3. α is a measurable cardinal such that *Case 1* fails.

Suppose first that there is no $\beta \in dom(f) \cap \alpha$ such that $f^0(\beta)_{n(\beta)} > \alpha$. Then let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}(\omega, \alpha)$. If there is a $\beta \in dom(f) \cap \alpha$ such that $f^0(\beta)_{n(\beta)} > \alpha$ then let β be the least such and let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}(f^0(\beta)_i^+, \alpha)$ where i is the largest such that $f^0(\beta)_i < \alpha$. Note that because *Case 3* fails, we must have that $\alpha < f^0(\beta)_{n(\beta)-1}$.

This finishes the definition of $\mathbb{P}_{\kappa_0}$. Let $\lambda = \kappa_{n-1}^{++}$ and let $j : V \to M$ be an embedding witnessing κ_0's κ_{n-1}^{++}-supercompactness and such that $j(f)^0(\kappa_0) = \langle \kappa_i : i < n \rangle^\frown \langle \kappa_{n-1}^+ \rangle$ and if F is the graph of f then $j(F) \cap H_{\kappa_{n-1}^+} = F \cap H_{\kappa_{n-1}^+}$. Let $\mathbb{P} = j(\mathbb{P}_{\kappa_0})_{\kappa_{n-1}}$. $\mathbb{P}$ is our final partial ordering. Before showing that $\mathbb{P}$ works, we list few useful properties of $\mathbb{P}$.

Proposition 2 (Properties of $\mathbb{P}$) *Suppose $\lambda < \kappa_{n-1}$ and $\mathbb{P} = \mathbb{P}_\lambda * \dot{\mathbb{Q}}_\lambda * \mathbb{P}^{>\lambda}$. Then*

1. *$\mathbb{P}$ is independent of the choice of j. Moreover, suppose $k : V \to M$ witnesses that κ_i is κ_{n-1}^{++}-supercompact, $k(f)^0(\kappa_i) = \langle \kappa_m : i \leq m < n \rangle^\frown \langle \kappa_{n-1}^+ \rangle$ and if F is the graph of F then $k(F) \cap H_{\kappa_{n-1}^+} = F \cap H_{\kappa_{n-1}^+}$. Then $k(\mathbb{P}_{\kappa_i})_{\kappa_{n-1}^+} = \mathbb{P}$.*

2. *For all $i < n$, $\dot{\mathbb{Q}}_{\kappa_i}$ is the trivial forcing and $\mathbb{P}^{\kappa_i}$ is κ_i^+-directed closed.*

3. *The set $\{\beta > \lambda : \dot{\mathbb{Q}}_\beta$ is not (λ, ∞)-distributive in $V^{\mathbb{P}_\beta}\}$ is finite.*

4. *If $\beta \in dom(f)$ then*

   (a) $\mathbb{P}_{\beta+1} \subseteq V_\beta$ and $\mathbb{P}_{\beta+1}$ has β-cc.

   (b) $\mathbb{P}^{>f^0(\beta)_0, <f^0(\beta)_{n(\beta)}} = \mathbb{P}^{>\beta, <f^0(\beta)_{n(\beta)}}$ is β-strategically closed.

   (c) $\dot{\mathbb{Q}}_{f(\beta)^0_{n(\beta)}}$ is (γ, ∞)-distributive for any $\gamma < \beta$.

   (d) If $f"\beta \subseteq V_\beta$ then $\mathbb{P}^{>f^0(\beta)_{n(\beta)}}$ is β-strategically closed in $V^{\mathbb{P}^{f^0(\beta)_{n(\beta)}}}$.

   Thus, if $f"\beta \subseteq V_\beta$ then in $V^{\mathbb{P}}$, β is a cardinal, for any $\gamma < \beta$, $2^\gamma \leq \beta$, and if β is a limit of closure points of f then β is inaccessible.

5. *The only measurable cardinals of $V^{\mathbb{P}}$ are $\langle \kappa_i : i < n \rangle$.*

Proof:

1. Let $i : V \to N$ be another embedding such that $i(F) \cap H_{\kappa_{n-1}^{++}} = F \cap H_{\kappa_{n-1}^{++}}$ and $i(f)^0(\kappa_0) = \langle \kappa_i : i < n \rangle^\frown \langle \kappa_{n-1}^+ \rangle$. We have that $\mathbb{P}^{\kappa_0}$ depends only on $j(F) \cap H_{\kappa_{n-1}} = i(F) \cap H_{\kappa_{n-1}}$. Thus $\mathbb{P}$ is independent of the choice of j. The rest is similar.

2. This is because $j(f)^0(\kappa_0)_i = \kappa_i$ and hence we are in *Case 3*. Also, in M, κ_0 is the least γ such that $j(f)^0(\gamma)_{n(\gamma)} > \kappa_i$. Thus, all posets used between $[\kappa_i, \kappa_{i+1}]$ are κ_i^+-directed closed.

3. Notice that if $\alpha > \lambda$ is such that $\dot{\mathbb{Q}}_\alpha$ is not (λ, ∞)-distributive then there must be some $\beta \leq \lambda$ such that $f^0(\beta)_{n(\beta)} = \alpha$ and for some $i < n(\beta)$, $\lambda \in [f^0(\beta)_i, f^0(\beta)_{i+1}]$. It is then enough to show that there can be only finitely many $\beta < \lambda$ such that $f(\beta)_{n(\beta)} \geq \lambda$ but for some $i < n(\beta)$, $f^0(\beta)_i \leq \lambda$. Towards a contradiction, suppose there are infinitely many such β. Let $\langle \beta_i : i < \omega \rangle$ be the first ω many of them in increasing order. Then for each i there is $k_i < n(\beta_i)$ such that $f^0(\beta_i)_{k_i} \leq \lambda$. Because f is a special universal Laver function and if $i < j$ then $f^0(\beta_j)_{k_j} < f^0(\beta_i)_{n(\beta_i)}$, we must have that for $i < j$, $f^0(\beta_j)_{n(\beta_j)} < f^0(\beta_i)_{n(\beta_i)}$. Then, $\langle f^0(\beta_i)_{n(\beta_i)} : i < \omega \rangle$ is a decreasing sequence of ordinals. Contradiction!

4. Follows from the definitions.

5. We now show that all measurable cardinals of V different from κ_is are not measurable in $V^{\mathbb{P}}$. Suppose λ is a V-measurable cardinal. Suppose there is (unique) $\beta < \lambda$ such that $f^0(\beta)_i = \lambda$ for some $i < n(\beta)$. Then at stage $f^0(\beta)_{n(\beta)}$ we force with a poset $\dot{\mathbb{Q}} * \dot{S}$ such that either $\dot{\mathbb{Q}}$ kills the measurability of λ or $\dot{S}$ adds a club disjoint from inaccessibles. If we add a club to λ which is disjoint from inaccessibles then λ's measurability can never be resurrected. Suppose, then, that the measurability of λ is killed by $\dot{\mathbb{Q}}$. If we ever in the future resurrect λ's measurability then we will also kill it by adding a club disjoint from inaccessibles in which case it will never again be resurrected. By 4, λ's measurability cannot be resurrected by $\mathbb{P}$ as there is some $\alpha \geq \lambda$ such that $\mathbb{P}^\alpha$ is λ^{++}-strategically closed.

   Now suppose λ is a measurable cardinal of $V^{\mathbb{P}}$ different from κ_is. Then by Theorem 4 λ is measurable in V. But we already showed that all such cardinals are not measurable in $V^{\mathbb{P}}$, contradiction. Next we show that κ_i remains measurable cardinal in $V^{\mathbb{P}}$.

Claim. For $i < n$, κ_i is a measurable cardinal in $V^{\mathbb{P}}$.

Proof. Fix i. Let $j : V \to M$ be an ultrapower embedding via a measure on κ_i that has Mitchell order 0. It is enough to show that κ_i is a measurable cardinal in $V^{\mathbb{P}_{\kappa_i}}$ as $\dot{\mathbb{Q}}_{\kappa_i}$ is trivial and the rest of the forcing is κ_i^+-directed closed. Let H be a V-generic for $\mathbb{P}_{\kappa_i}$. We have that $j(\mathbb{P}_{\kappa_i}) = \mathbb{P}_{\kappa_i} * \dot{\mathbb{Q}} * \mathbb{P}_{tail}$ where $\dot{\mathbb{Q}}$ is the forcing at stage κ and $\mathbb{P}_{tail}$ is the rest of the forcing. Since κ_i is not measurable in M, there is no stage in $j(\mathbb{P}_{\kappa_i})$ that adds an unbounded subset of κ_i. Moreover, because $f \upharpoonright \kappa_i \subseteq V_{\kappa_i}$, there is no stage in $\mathbb{P}_{tail}$ that adds a bounded subset of κ_i. Therefore, $\dot{\mathbb{Q}}$ is trivial and $\mathbb{P}_{tail}$ is κ_i^+-strategically closed in $M[H]$. Using the counting argument in $V[H]$ we get an M-generic object $h \in V[H]$ for $\mathbb{P}_{tail}$. We can then extend j to $j^* : V[H] \to M[H][h]$. Thus, κ_i is a measurable cardinal in $V[H]$. *Q.E.D.*

□

3.3 The proof of Main Theorem 1.

We want to show that for any $i < n$ if $\mathbb{R} \in V^{\mathbb{P}}$ is a partial ordering which is κ_i-directed closed, forces GCH at measurable cardinals of $V^{\mathbb{P}*\mathbb{R}}$ that are $\geq \kappa_i$ and in $V^{\mathbb{P}*\mathbb{R}}$ there are only finitely many measurables then κ_i is strongly compact in $V^{\mathbb{P}*\mathbb{R}}$. Note that by Theorem 4, V-measurables are the only possible candidates for being measurable in $V^{\mathbb{P}*\mathbb{R}}$. We simplify our life and the reader's life by making the unnecessary assumption that $n = 2$. This case is a good representative case and the general case is just like it only more involved in terms of notation. Having said this, we simplify our life even further by verifying only the indestructibility of κ_0. It should be clear that this is indeed the hard case. Let then $\kappa = \kappa_0$ and $\delta = \kappa_1$. Fix a singular strong limit cardinal $\lambda > \delta, rank(\mathbb{P} * \dot{\mathbb{R}})$ of cofinality $> \max(\delta, \left|\mathbb{P} * \dot{\mathbb{R}}\right|)$. We want to show that κ is λ-strongly compact in $V^{\mathbb{P}*\dot{\mathbb{R}}}$. We make one further simplification and assume that κ and δ are the only possible measurable cardinals of $V^{\mathbb{P}*\mathbb{R}}$. Again, this simplifications are unnecessary and they only make the proof more transparent.

Let $G_0 * G_1 * G_2$ be a V-generic for $\mathbb{P}_\kappa * \mathbb{P}^\kappa * \mathbb{R}$. Let $j : V \to M$ be an embedding witnessing that κ is λ-supercompact such that $j(f)(\kappa) = \langle\langle\kappa, \delta, \lambda\rangle, \dot{\mathbb{R}}\rangle$ and if F is the graph of f then $j(F) \cap H_\lambda = F \cap H_\lambda$. Then we have that $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2 * \mathbb{P}_{tail}$ where $\mathbb{Q}_0 = j(\mathbb{P}_\kappa)^{\kappa,\delta} = \mathbb{P}^\kappa$, $\mathbb{Q}_1 = j(\mathbb{P}_\kappa)^{>\delta,<\lambda}$, $\mathbb{Q}_2$ is the forcing done at stage λ, and $\mathbb{P}_{tail}$ is the rest of the partial ordering. We then have that $\mathbb{Q}_0 = \mathbb{P}^\kappa$, $\mathbb{Q}_1$ is trivial and $\mathbb{Q}_2 = \mathbb{R} * \mathbb{S}$ where $\mathbb{S} = \mathbb{S}_0 * \mathbb{S}_1$ is such that if κ (δ) remains measurable in $V^{\mathbb{P}*\mathbb{R}}$ then $\mathbb{S}_0 = \mathbb{Q}(\omega, \kappa_0)$ ($\mathbb{S}_1 = \mathbb{Q}(\kappa, \delta)$) and if κ (δ) doesn't remain measurable in $V^{\mathbb{P}*\mathbb{R}}$ then $\mathbb{S}_0$ ($\mathbb{S}_1$) is trivial.

Claim. Either $\mathbb{S}_0$ or $\mathbb{S}_1$ is not trivial.

Proof. If both $\mathbb{S}_0$ and $\mathbb{S}_1$ are trivial then standard arguments show that j can be lifted to $j : V[G_0 * G_1 * G_2] \to M[j(G_0 * G_1 * G_2)]$ and hence, κ is a supercompact cardinal in $V^{\mathbb{P}*\mathbb{R}}$, which is nonsense. *Q.E.D.*

We thus have that $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{P}^\kappa * \mathbb{R} * \mathbb{S} * \mathbb{P}_{tail}$ where $\mathbb{S}$ is nontrivial. The hard case is, of course, the one that both $\mathbb{S}_0$ and $\mathbb{S}_1$ are not trivial. Lets assume the hard case holds. If both $\mathbb{S}_0$ and $\mathbb{S}_1$ are nontrivial then $\mathbb{R}$ preserves the measurability of both κ and δ. Because $\mathbb{P}_\kappa * \mathbb{P}^\kappa * \mathbb{R}$ has a gap with respect to κ and κ is measurable in $V^{\mathbb{P}*\mathbb{R}}$, it must be the case that, by Theorem 4, there is $j_0 : M \to M_0$ such that $j_0 \in M$ lifts to $j_0^* : M[G_0 * G_1 * G_2] \to M_0[j_0(G_0 * G_1 * G_2)]$ and j_0^* is an ultrapower embedding in $M[G_0 * G_1 * G_2]$. Because δ is a measurable cardinal in $M[G_0 * G_1 * G_2]$, it must be the case that $j_0(\delta) = \delta$. This means that δ is a measurable cardinal in $M_0[j_0^*(G_0 * G_1 * G_2)]$. Therefore, using Theorem 4 in $M_0[j_0^*(G_0 * G_1 * G_2)]$, we get that there must be $j_1 : M_0 \to M_1$ such that $j_1 \in M_0$ and j_1 lifts to $j_1^* : M_0[j_0^*(G_0 * G_1 * G_2)] \to M_0[j_1^*(j_0^*(G_0 * G_1 * G_2))]$. Let $k = j_1 \circ j_0 \circ j$. Then $k : V \to M_1$. Note that because j_0^* and j_1^* are ultrapower embeddings and λ has cofinality $> \delta$, we must have that $j_0^*(\lambda) = j_1^*(\lambda) = \lambda$. Also, for the same reason, $k(\kappa) = j(\kappa)$. This means that $j_1(j_0(j``\lambda))$ covers $k``\lambda$ in M_1 and has size $< k(\kappa)$ in M_1. Thus, k is a strong compactness embedding (that such k is a strong compactness embedding was first observed by Magidor). k is what we will lift to $V[G_0 * G_1 * G_2]$.

We have that $k(\mathbb{P}) = \mathbb{P}_\kappa * \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2 * \mathbb{P}_{tail}$ where $\mathbb{Q}_0$ is the partial ordering between (κ, λ), $\mathbb{Q}_1 = j_1(j_0(\mathbb{R}))$, $\mathbb{Q}_2 = j_1(j_0(\mathbb{S}))$, and $\mathbb{P}_{tail}$ is the rest of the forcing. We now describe how to find generic objects for $\mathbb{Q}_0$, $\mathbb{Q}_1$, $\mathbb{Q}_2$ and $\mathbb{P}_{tail}$. Notice that $j_1^*(j_0^*(G_0 * G_1 * G_2))$ is a generic for $\mathbb{P}_\kappa * \mathbb{Q}_0 * \mathbb{Q}_1$. Thus, we only need to find generic objects for $\mathbb{Q}_2$ and $\mathbb{P}_{tail}$.

By our assumption, $\mathbb{Q}_2 = \mathbb{Q}(\omega, j_0(\kappa)) * \mathbb{Q}(j_0(\kappa), j_1(\delta)) \in M_1[j_1^*(j_0^*(G_0 * G_1 * G_2))]$. Also, by our assumption, $M[G_0 * G_1 * G_2] \models 2^\kappa = \kappa^+$ and $M_0[j_0^*(G_0 * G_1 * G_2)] \models 2^\delta = \delta^+$. Because j_0^* is an ultrapower embedding in $M[G_0 * G_1 * G_2]$, using the counting argument in $M[G_0 * G_1 * G_2]$, we can get an $M_0[j_0^*(G_0 * G_1 * G_2)]$-generic object $g_0 \in M[G_0 * G_1 * G_2]$ for $\mathbb{Q}(\omega, j_0(\kappa))$. Because g_0 comes from a small forcing relative to δ, we can lift j_1^* to

$j_1^{**}: M_0[j_0^*(G_0*G_1*G_2)][g_0] \to M_1[j_1^*(j_0^*(G_0*G_1*G_2))][g_0]$. Notice that j_1^{**} is still an ultrapower embedding in $M_0[j_0^*(G_0*G_1*G_2)][g_0]$ and $M_1[j_1^*(j_0^*(G_0*G_1*G_2))][g_0] \models 2^\delta = \delta^+$. This means that we can use the counting argument in $M_0[j_0^*(G_0*G_1*G_2)][g_0]$, to get an $M_1[j_1^*(j_0^*(G_0*G_1*G_2))][g_0]$-generic object g_1 for $\mathbb{Q}(j_0(\kappa), j_1(\delta))$. Then $g_0 * g_1$ is a $M_1[j_1^*(j_0^*(G_0*G_1*G_2))]$-generic object.

We now describe how to find a generic object for $\mathbb{P}_{tail}$. We will use an argument that appeared in [7]. The argument mixes the term forcing argument with counting and transfer arguments. Let $\mathbb{P}^* = t(j(\mathbb{P}_\kappa)^{>\lambda}/j(\mathbb{P}_\kappa)^{\kappa,\lambda}) \in M^{\mathbb{P}_\kappa}$. Then $\mathbb{P}^*$ is λ^+-strategically closed partial ordering in $M[G_0]$ and because j is an ultrapower embedding witnessing λ-supercompactness and $\mathbb{P}_\kappa$ is κ-cc, $M[G_0]$ is λ-closed in $V[G_0]$. This means that we have only λ^+-many dense subset of $\mathbb{P}^*$ in $V[G]$ and by counting argument applied in $V[G]$ we can get $H \in V[G]$ which is $M[G]$-generic for $\mathbb{P}^*$. We can now use the transfer argument and transfer H all the way to M_1 but this is not as obvious as it sounds because our embeddings j_0 and j_1 where rather mysterious embeddings. Here is what we do.

Let H^* be the filter generated by j_0^{*}"H. We would like to see that H^* is $M_0[j_0^*(G_0)]$-generic for $j_0^*(\mathbb{P}^*)$. Fix $f \in M[G_0 * G_1 * G_2]$ such that $j_0^*(f)(\kappa) = D$ is a dense subset of $j_0^*(\mathbb{P}^*)$ in $M_0[j_0^*(G_0)]$. But then it is not hard to see that f is essentially a function $f: \kappa \to M[G_0]$. The hard case is when $f \notin M[G_0]$ in which cases it is added by $G_1 * G_2$. We then assume that the hard case holds and let $\dot{f} \in M[G_0]$ be the name of f. We can then let $g: (\mathbb{P}^\kappa * \mathbb{R}) \times \kappa \to M[G_0]$ be given by $g(p, \alpha) = b$ if $p \Vdash_{\mathbb{P}^\kappa * \mathbb{R}} \dot{f}(\check{\alpha}) = \check{b}$. Note that $g(p,\alpha) \in M[G_0]$ and $g(p,\alpha)$ is always a dense subset of $\mathbb{P}^*$ in $M[G_0]$. We have that $\mathbb{P}^*$ is (λ^+, ∞)-distributive in $M[G_0]$ (because it is λ^+-strategically closed) and $|\mathbb{P}^\kappa * \mathbb{R}| < \lambda$ in $M[G_0]$. This means that $D^* = \cap_{p \in \mathbb{P}^\kappa * \mathbb{R}, \alpha < \kappa} g(p,\alpha)$ is a dense subset of $\mathbb{P}^*$ in $M[G_0]$. Let $r \in D^* \cap H$. We then have that in $M[G_0]$, for any $\alpha < \kappa$, $\Vdash_{\mathbb{P}^\kappa * \mathbb{R}} \check{r} \in \dot{f}(\check{\alpha})$. Applying j_0^*, we get that in $M[j_0^*(G_0)]$, $\Vdash_{\mathbb{P}^\kappa * \mathbb{R}} j_0^*(\check{r}) \in j_0^*(\dot{f})(\check{\kappa})$. This then implies that $j_0^*(r) \in D$ and hence, $j_0^*(r) \in H^* \cap D$. Using the same argument, we can transfer H^* one more time and get $M_1[j_1^*(j_0^*(G_0))] = M_1[j_0^*(G_0)]$-generic object H_1^{**} for $j_1^*(j_0^*(\mathbb{P}^*))$. Then using the term forcing argument, we get H^{***} which is a $M_1[j_1^*(j_0^*(G_0 * G_2 * G_2))][g_0 * g_1]$-generic object for $\mathbb{P}_{tail}$ (recall that $j_1^*(j_0^*(\mathbb{P}^*)) = t(\mathbb{P}_{tail}/k(\mathbb{P})^{\kappa,\lambda}) \in M_1^{\mathbb{P}_\kappa}$).

To finish the lifting process we need to find a generic for $k(\mathbb{P}^\kappa * \mathbb{R})$. We combine the counting argument, master condition argument, term forcing argument and the transfer argument to do this. First we get a term $\tau \in M^{j(\mathbb{P}_\kappa)}$ such that $\Vdash_{j(\mathbb{P}_\kappa)}$ "for every $\dot{p} \in \dot{h}, \tau \Vdash_{j(\mathbb{P}^\kappa * \mathbb{R})} j(\dot{p})$", where $\dot{h}$ is the name for the generic object associated with $\mathbb{P}^\kappa * \mathbb{R}$. Note that because j"$\dot{h} \in M^{j(\mathbb{P}_\kappa)}$ and $M \models$ " $\Vdash_{j(\mathbb{P}_\kappa)}$ "j"$\dot{h} \subseteq j(\mathbb{P}^\kappa * \mathbb{R})$ is a directed set of size $< \check{\lambda}$

and $j(\mathbb{P}^\kappa * \mathbb{Q}_\lambda)$ is $\check{\lambda}^+$-directed closed"", there must be a name τ as desired. Thus, in M, $\Vdash_{j(\mathbb{P}_\kappa)}$ "for every $\dot{p} \in \dot{h}, \tau \Vdash_{j(\mathbb{P}^\kappa * \mathbb{R})} j(\dot{p})$".

Next we let $\mathbb{P}^* = t((\mathbb{P}^\kappa * \mathbb{R})/\mathbb{P}_\kappa)$. Again $\mathbb{P}^*$ is κ-directed closed partial ordering in V and $j(\mathbb{P}^*)$ is λ^+-directed closed partial ordering in M. Because $\mathbb{P}_\kappa$ has κ chain condition, cardinality of $j(\mathbb{P}^*)$ in V is λ^+ and moreover, there are only λ^+-many dense subsets of $j(\mathbb{P}^*)$ available in M. Thus, using counting argument in V, we can construct an M-generic $K \in V$ for $j(\mathbb{P}^*)$ with an additional property that our term τ is in K. Using the transfer argument (more precisely its modification presented above), we can now transfer K all the way to M_1. Let K^* be the resulting M_1-generic for $k(\mathbb{P}^*)$. But $k(\mathbb{P}^*) = t(k(\mathbb{P}^\kappa * \mathbb{R})/k(\mathbb{P}_\kappa))$. Therefore, using the term forcing argument, we now get K^{**} which is $M_1[j_1^*(j_0^*(G_0 * G_1 * G_2))][g_0 * g_1][H^{**}]$-generic for $k(\mathbb{P}^\kappa * \mathbb{R})$. To finish, we need to verify that $k"G_1 * G_2 \subseteq K^{**}$. Fix $p \in G_1 * G_2$. Recall the definition of $\mathbb{S}$ at the begining of our proof; it was the second part of the poset used at stage λ in $j(\mathbb{P}_\kappa)$. Then in $M[G_0 * G_1 * G_2]$, we have that $\Vdash_{\mathbb{S} * j(\mathbb{P}_\kappa)^{>\lambda}}$ "$\tau \Vdash_{j(\mathbb{P}^\kappa * \mathbb{R})} j(\check{p})$". By elementarity of $j_1^* \circ j_0^* : M[G_0 * G_1 * G_2] \to M_1[j_1^*(j_0^*(G_0 * G_1 * G_2))]$, we have that $M_1[j_1^*(j_0^*(G_0 * G_1 * G_2))] \vDash \Vdash_{\mathbb{Q}_2 * k(\mathbb{P}_\kappa)^{>\lambda}}$ "$j_1^*(j_0^*(\tau)) \Vdash_{k(\mathbb{P}^\kappa * \mathbb{R})} k(\check{p})$". But $j_1^*(j_0^*(\tau)) \in K^{**}$. Therefore, $k(p) \in K^{**}$. We thus have that k lifts to $k^* : V[G_0 * G_1 * G_2] \to M_1[j_1^*(j_0^*(G_0 * G_1 * G_2))][g_0 * g_1][H^{**}][K^{**}]$. This means that κ is strongly compact in $V^{\mathbb{P} * \mathbb{R}}$, and this completes the proof of Main Theorem 1 in the case when $n = 2$. It is not hard to generalize this case to arbitrary integer n. *Q.E.D.*

We note that in the model constructed the measurability of each κ_i is fully indestructible.

4 Indestructibility, identity crisis and strong cardinals.

In this section, we add indestructibility to Apter-Cummings model (see Theorem 2) and we also extend a result of Apter that appeared in [4].

Theorem 7 (Main Theorem 2) *The following theories are consistent relative to a proper class of supercompact cardinals.*

1. *There is a proper class of strong cardinals, the class of strong cardinals coincides with the class of strongly compact cardinals, and strong compactness of any strongly compact cardinal κ is indestructible under κ-directed closed partial orderings that force GCH at κ (eg, Levy collapse, adding Cohen subsets, and etc).*

2. *There are no supercompact cardinals, there is a proper class of strongly compact cardinals, and all strongly compact cardinals are fully indestructible.*

The proof of Main Theorem 2 uses the ideas involved in the proof of Main Theorem 1 in addition to ideas used in [4], [6] and [12]. In particular, to show 2 of Main Theorem 2, we will use resurrectability idea used by Apter in [4]. Main Theorem 2 answers some questions asked in [4] and [12].

4.1 The proof of 1 of Main Theorem 2.

Because the proof is very similar to the proof of Main Theorem 1 we will be sketchy at times. We start with the usual harmless assumption that GCH holds in V and we also assume that there is no measurable limit of supercompact cardinals. We fix a universal Laver function f. If κ is a measurable cardinal, we let $\nu_\kappa = \sup\{\lambda < \kappa : \lambda \text{ is a supercompact cardinal }\}$. Then $\nu_\kappa < \kappa$ for every measurable cardinal κ. The poset $\mathbb{P}$ then, as the reader might have guessed, is the following; $\mathbb{P}$ is a Reverse Easton Iteration in which a non-trivial poset is used only at the strong cardinals that are not a member of s. Within the set of strong cardinals, if κ is strong but $f(\kappa)$ is not a $\mathbb{P}_\kappa$-name for a κ-directed closed partial ordering that forces GCH at κ then $\dot{\mathbb{Q}}_\kappa = \mathbb{S}(\nu_\kappa^+, \kappa)$. If κ is a strong cardinal such that $f(\kappa) = \dot{\mathbb{R}} \in V^{\mathbb{P}_\kappa}$ is a name for a κ-directed closed partial ordering such that $2^\kappa = \kappa^+$ in $V^{\mathbb{P}_\kappa * \mathbb{R}}$ then $\dot{\mathbb{Q}}_\kappa = \dot{\mathbb{R}} * \dot{\mathbb{S}}$ where $\dot{\mathbb{S}} \in V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}}$ is the trivial forcing if κ is not a measurable cardinal in $V^{\mathbb{P}_\kappa * \dot{\mathbb{R}}}$ and $\dot{\mathbb{S}} = \dot{\mathbb{S}}(\nu_\kappa^+, \kappa)$ otherwise. We then claim that $V^{\mathbb{P}}$ is as desired.

Let $s = \langle \kappa_\alpha : \alpha \in Ord \rangle$ be the sequence of supercomact cardinals in the increasing order. Then it is not hard to see that in $V^{\mathbb{P}}$, if κ is not a member of s then κ is not strong in V. To see this, first note that by Theorem 4, all strong cardinals of $V^{\mathbb{P}}$ must be strong cardinals of V. But any strong cardinal κ of V which is not a member of s gets killed at stage κ by either adding a non-reflecting stationary set or by a κ-directed closed partial ordering which destroys the measurability of it. As $\mathbb{P}^\kappa$ is κ^{++}-strategically closed, we can never resurrect the measurability of κ after stage κ.

Claim 1. For all α, κ_α is a strong cardinal in $V^{\mathbb{P}}$.

Proof. The proof is just like the proof of the same claim in [5]. Fix α and let $\lambda > \kappa_\alpha$ be a non-measurable inaccessible cardinal. Let $j : V \to M$ be an embedding witnessing that κ_α is λ-strong while in M, κ is not strong. Then consider $j(\mathbb{P}_{\kappa_\alpha})$. Because κ is not strong in M, there are no strong cardinal in M between κ and λ. Hence, $j(\mathbb{P}_{\kappa_\alpha})^{\kappa,\lambda}$ is trivial. Thus, $j(\mathbb{P}_{\kappa_\alpha}) = \mathbb{P}_{\kappa_\alpha} * \mathbb{Q}$ where $\mathbb{Q}$ is the partial ordering between $(\lambda, j(\kappa))$. We now use the factorization argument used in the same claim of [5]. This argument is originally due to Woodin. Let μ be the measure given by $A \in \mu \leftrightarrow \lambda \in j(A)$, and let $i = i_\mu : V \to Ult(V, \mu) = N$. Let k be the usual factor map $k : N \to M$ given by $k([f]_\mu) = j(f)(\lambda)$. Let $G \subseteq \mathbb{P}_{\kappa_\alpha}$ be V-generic. Note that if $\bar{\lambda}$ is such that $k(\bar{\lambda}) = \lambda$ then the stages between $[\kappa_\alpha, \bar{\lambda}]$ of $i(\mathbb{P}_{\kappa_\alpha})$ are trivial as $k([\kappa_\alpha, \bar{\lambda}]) = [\kappa_\alpha, \lambda]$. Then using the counting argument in $V[G]$ we get an $M[G]$-generic $h \in V[G]$ for $\bar{Q}$ where $\bar{\mathbb{Q}}$ is such that $k(\bar{\mathbb{Q}}) = \mathbb{Q}$. Using the transferring argument, we can then transfer h along k and get an $M[G]$-generic object $g \in V[G]$ for $\mathbb{Q}$. (Here are more details but see [5] for even more details. Let g be the filter generated by $k"h$. We claim that g is $M[G]$-generic. To see this, let $D \in M[G]$ be a dense subset of $\mathbb{Q}$. Then there is a function $f \in V[G]$ such that $D = j(f)(a)$ for some $a \in [\lambda]^{<\omega}$. Let $\bar{D} = \cap_{b \in [\bar{\lambda}]^{<\omega}} i(f)(b)$. Then $\bar{D}$ is a dense subset of $\bar{\mathbb{Q}}$ as $\bar{\mathbb{Q}}$ is (λ, ∞)-distributive in $N[G]$. Let $p \in \bar{D} \cap h$. Then $k(p) \in D \cap g$.). This then allows us to lift j to $j : V[G] \to M[G][g]$. If now H is a $V[G]$ generic for $\mathbb{P}^{\kappa_\alpha}$ then using the transfer argument we can lift j further to $j : V[G][H] \to M[G][g][j(H)]$ (the transfer argument applies as $\mathbb{P}^{\kappa_\alpha}$ is (κ_α, ∞)-distributive. *Q.E.D.*

Claim 2. For all α, κ_α's strong copactness is indestructible under κ_α-directed closed partial orderings that force GCH at κ_α.

Proof. Suppose not. Fix α such that $\kappa = \kappa_\alpha$ is not so indestructible. Fix $\mathbb{R} \in V^{\mathbb{P}}$ which is κ-directed closed and forces GCH at κ. Let λ be a non-measurable inaccessible cardinal $> (rank(\mathbb{R}))^{V^{\mathbb{P}}}$ such that κ is not λ-strongly compact in $V^{\mathbb{P} * \mathbb{R}}$. Then in fact κ is not λ strongly compact in $V^{\mathbb{P}_\lambda * \mathbb{R}}$. Let $j : V \to M$ be a λ-supercompactness embedding in V such that $j(f)(\kappa) = \mathbb{P}^{\kappa,\lambda} * \mathbb{R}$ and κ is not λ-supercompact in M. Then $j(\mathbb{P}_\lambda * \mathbb{R}) = \mathbb{P}_\kappa * \mathbb{P}^{\kappa,\lambda} * \mathbb{R} * \mathbb{S} * \mathbb{P}_{tail} * j(\mathbb{P}^{\kappa,\lambda} * \mathbb{R})$ where $\mathbb{S}$ is trivial if κ is not measurable in $M^{\mathbb{P}_\lambda * \mathbb{R}}$ and $\mathbb{S} = (\mathbb{S}(\nu_\kappa^+, \kappa))^{M^{\mathbb{P} * \mathbb{R}}}$ otherwise, and $\mathbb{P}_{tail}$ is the part of the forcing between $(\lambda, j(\kappa))$. (Note that $j(\mathbb{P}_\kappa)^{>\kappa,\lambda}$ is trivial as there are no strong cardinals in the interval (κ, λ)). As in the proof of Main Theorem 1, $\mathbb{S}$ has to be non-trivial (otherwise we could lift the entire embedding to $V^{\mathbb{P}_\lambda * \mathbb{R}}$ showing that κ is λ-supercompact in $V^{\mathbb{P}_\lambda * \mathbb{R}}$, which cannot happen). Thus, $\mathbb{S}$ must be nontrivial and therefore, κ must be a measurable cardinal in $M^{\mathbb{P}_\lambda * \mathbb{R}}$. As in the proof of Main Theorem 1, using Theorem 4, there is an embedding $i : M \to N$ that lifts to $M^{\mathbb{P}_\lambda * \mathbb{R}}$ and becomes an ultrapower embedding by a normal measure on κ. Let then $k = i \circ j$. k witnesses that κ is λ-strongly compact and k is what we will lift. Let $G_0 * G_1 * G_2 \subseteq \mathbb{P}_\kappa * \mathbb{P}^{\kappa,\lambda} * \mathbb{R}$ be V-generic. At this point, we will be very sketchy as we essentially repeat what we did in the proof of Main Theorem 1. Let $k(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2 * \mathbb{Q}_3 * \mathbb{P}_{tail}$ where $\mathbb{Q}_0 = i(\mathbb{P}_\kappa)^\kappa$, $\mathbb{Q}_1 = i(\mathbb{P}^{\kappa,\lambda})$, $\mathbb{Q}_2 = i(\mathbb{R})$, $\mathbb{Q}_3 = i(\mathbb{S})$ and $\mathbb{P}_{tail}$ is the rest of the partial ordering. We now start describing the generics for $\mathbb{Q}_i$s and $\mathbb{P}_{tail}$.

We first fix a name for a master condition τ with the property that $M \vDash\Vdash_{j(\mathbb{P}_\kappa)}$ " for all $\dot{p} \in \dot{h}$, $\tau \Vdash_{j(\mathbb{P}^{\kappa,\lambda} * \mathbb{R})} j(\dot{p})$" where $\dot{h}$ is the canonical name for the generic for $\mathbb{P}^{\kappa,\lambda} * \mathbb{R}$. We then lift i to $i^* : M[G_0 * G_1 * G_2] \to N[i^*(G_0 * G_1 * G_2)]$. Thus, $i^*(G_0 * G_1 * G_2)$ is an N-generic for $\mathbb{P}_\kappa * \mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{Q}_2$. Next, we use the counting argument in $M[G_0 * G_1 * G_2]$ to get an $N[i^*(G_0 * G_1 * G_2)]$-generic $H \in M[G_0 * G_1 * G_2]$ for $\mathbb{Q}_3$ (this is possible because $2^\kappa = \kappa^+$ in $M[G_0 * G_1 * G_2]$). Then, we use the counting argument in $V[G_0]$ to get an $M[G_0]$-generic g for $t(j(\mathbb{P}_\kappa)^{>\lambda}/(\mathbb{P}^{\kappa,\lambda} * \mathbb{R} * \mathbb{S}))$. Using the modification of the transfer argument used in the proof of Main Theorem 1, we get an $N[G_0]$-generic g^* for $t(\mathbb{P}_{tail}/i(\mathbb{P}^{\kappa,\lambda} * \mathbb{R} * \mathbb{S}))$. Using the term forcing argument, this then gives $N[i^*(G_0 * G_1 * G_2)][H]$-generic object g^{**} for $\mathbb{P}_{tail}$. Next, we use the counting argument in V and get an M-generic $K \in V$ for $t(j(\mathbb{P}^{\kappa,\lambda} * \mathbb{R})/j(\mathbb{P}_\kappa))$ such that $\tau \in K$. We then, using the transferring argument, get an N-generic K^* over $t(k(\mathbb{P}^{\kappa,\lambda} * \mathbb{R})/k(\mathbb{P}_\kappa))$. Using the term forcing argument, we get an $N[i^*(G_0 * G_1 * G_2)][H][g^{**}]$-generic K^{**} for $k(\mathbb{P}^{\kappa,\lambda} * \mathbb{R})$. Using the same argument as in the proof of Main Theorem 1, we get that $k"G_1 * G_2 \subseteq K^{**}$. This then allows us to lift k to $k : V[G_0 * G_1 * G_2] \to N[i^*(G_0 * G_1 * G_2)][H][g^{**}][K^{**}]$. We thus get a contradiction, as k now witnesses that κ is λ-strongly compact in $V[G_0 * G_1 * G_2]$. *Q.E.D.*

4.2 The proof of 2 of Main Theorem 2.

In this section we give the proof of 2 of Main Theorem 2. One of the ideas is to use the trick used by Apter in [4]. The trick is essentially the resurrectability phenomenon. In [4], Apter using this trick managed to get indestructibility under posets that look like $\mathbb{Q} * Add(\kappa, 1)$. Unfortunately, his poset cannot be iterated and it works only for one strongly compact. We use the trick according to the following intuition; whenever the partial ordering is κ-directed closed but not (κ, ∞)-distributive, we should be able to prove indestructibility under it by resurrecting the supercompactness.

Our proof will again be very similar to the previous two proofs and therefore, there is no need to be meticulous. We start with a model where GCH already holds and there are no measurable limits of supercompact cardinals. Again, for a measurable cardinal κ, ν_κ is defined as before. We also fix a universal Laver function f. Our partial ordering $\mathbb{P}$ is again a proper class Reverse Easton Iteration in which nontrivial forcing is done only at non-supercompact strong cardinals. If κ is a strong cardinal then we do the following.

Case 1. If $f(\kappa) = \dot{\mathbb{R}}$ where $\dot{\mathbb{R}} \in V^{\mathbb{P}_\kappa}$ is κ-directed closed poset.

If $\mathbb{R}$ is not κ-distributive then we let $\dot{\mathbb{Q}}_\kappa = \mathbb{R}$. If $\mathbb{R}$ is κ-distributive then we let $\dot{\mathbb{Q}}_\kappa = \mathbb{R} * \mathbb{Q}(\nu_\kappa^+, \kappa)$.

Case 2. Otherwise.

In this case, we let $\dot{\mathbb{Q}}_\kappa = \mathbb{Q}(\nu_\kappa^+, \kappa)$.

Claim 1. There are no supercompact cardinals in $V^{\mathbb{P}}$.

Proof. Suppose not. By Theorem 4, all supercompact cardinals of $V^{\mathbb{P}}$ are supercompact in V. Let κ be a supercompact cardinal in V. Suppose κ is κ^+-supercompact in $V^{\mathbb{P}}$. Then κ is κ^+-supercompact in $V^{\mathbb{P}_\kappa}$. Let $j : V \to M$ be an embedding in the ground model that lifts to $V^{\mathbb{P}_\kappa}$ where it witnesses that κ is κ^+-supercompact. Because of GCH, κ is strong in $M^{j(\mathbb{P}_\kappa)}$ and hence, in M. Also, κ cannot be supercompact in M as otherwise, in V, we would have a measurable limit of supercompact cardinals. Thus, $(\dot{\mathbb{Q}}_\kappa)^{j(\mathbb{P}_\kappa)} \neq \emptyset$. If $j(f)(\kappa)$ is such that we are not in *Case 1* above, then $\dot{\mathbb{Q}}_\kappa = \mathbb{Q}(\nu_\kappa^+, \kappa)$ which means that κ cannot be a measurable cardinal in $V^{\mathbb{P}_\kappa}$. Thus, suppose we are in *Case 1.* If $j(f)(\kappa) = \mathbb{R}$ where $\mathbb{R}$ is κ-directed closed but not (κ, ∞)-distributive then it adds a subset of κ which is not in $V^{\mathbb{P}_\kappa}$. It must be then that $\mathbb{R}$ is κ-directed closed and (κ, ∞)-distributive. But then $\dot{\mathbb{Q}}_\kappa = \mathbb{R} * \mathbb{Q}(\nu_\kappa^+ * \kappa)$. *Q.E.D.*

Claim 2. Each supercompact cardinal κ remains fully indestructible strongly compact cardinal in $V^{\mathbb{P}}$.

Proof. Fix κ a supercompact cardinal of V and let $\mathbb{R} \in V^{\mathbb{P}}$ be a κ-directed closed poset. Fix some non-measurable inaccessible $\lambda > rank(\mathbb{R})^{V^{\mathbb{P}}}$. We want to show that κ is λ-strongly compact in $V^{\mathbb{P}*\mathbb{R}}$. It is enough to show that κ is λ-strongly compact in $V^{\mathbb{P}_\lambda * \mathbb{R}}$. Let $j : V \to M$ be λ-supercompactness embedding such that $j(f)(\kappa) = \mathbb{P}^{\kappa,\lambda} * \mathbb{R}$ and κ is not λ-supercompact in M. Suppose that $\mathbb{R}$ is κ-directed closed but not κ-distributive. Then standard arguments show that κ is in fact a supercompact cardinal in $V^{\mathbb{P}_\lambda * \mathbb{R}}$ (this is just because $\dot{\mathbb{Q}}_\kappa = \mathbb{P}^{\kappa,\lambda} * \mathbb{R}$). This is what we were calling resurrectability trick. If $\mathbb{R}$ is κ-directed closed and κ-distributive then we have $\dot{\mathbb{Q}}_\kappa = \mathbb{P}^{\kappa,\lambda} * \mathbb{R} * \mathbb{Q}(\nu_\kappa, \kappa)$. We then let $i : M \to N$ be an embedding given by a normal measure on κ which has Mitchell order 0. Let $k = i \circ j$. Using the arguments just like those used in the proof of Main Theorem 1 and part 1 of Main Theorem 2, we lift k to $V^{\mathbb{P}_\lambda * \mathbb{R}}$ (again, that such a k witnesses strong compactness, was first observed by Magidor). *Q.E.D.*

5 Concluding Remarks

We conjecture that in some sense Main Theorem 1 is best possible. The problem is that using Laver preparation to force indestructibility produces many cardinals that are not measurable yet are resurrectable. Here is what we mean.

Observation. Suppose κ is indestructible supercompact and there is a measurable cardinal above. Then we claim that there are cardinals $\delta < \kappa$ such that δ is not measurable yet after some δ-directed closed forcing they become measurable. In fact, the forcing can just be $Add(\delta, 1)$. To see this, let λ be the least measurable cardinal above κ. Let $\mathbb{P}$ be the Reverse Easton Iteration that adds a Cohen subset to every inaccessible cardinal of the interval (κ, λ). This then destroys the measurability of λ while preserves the supercompactness of κ. But by adding a Cohen subset to λ we can resurrect the measurability of λ. This means, by reflection, that there are many cardinals $\delta < \kappa$ that have the same property, i.e. they become measurable after just adding one Cohen subset.

We then conjecture that the same must be true for strongly compact cardinals.

Question 1. Suppose $\kappa_0 < \kappa_1$ are two measurable cardinals such that κ_0 is strongly compact cardinal which is indestructible under κ_0-directed closed partial orderings that force GCH at κ_0. Is there $\delta \neq \kappa_i$ such that δ is generically measurable? Is there $\delta \neq \kappa_i$ such that δ is resurrectably measurable?

We do not even know the answer to the following question.

Question 2. Can the first strongly compact cardinal, the first measurable cardinal and the first generically measurable cardinal coincide?

It is interesting to note that getting indestructibility for strong compactness becomes more and more difficult as it starts suffering more and more from identity crisis. In 2 of Main Theorem 2, we get the full indestructibility but the identity crisis is mild. In 1 of Main Theorem 2, we get indestructibility under κ-directed closed posets that force GCH at κ and identity crisis is in somewhat intermediate stage (i.e., strong compactness is lined up with strongness). In Main Theorem 1, we get indestructibility under κ-directed posets that force GCH not only at κ but at other measurable cardinals as well. In the model of Main Theorem 1, identity crisis is at its maximum. It should also be noted that, in showing indestructibility for strong compactness suffering from identity crisis, major difficulties arise only when we target more than one strongly compact cardinal.

The following questions remain open. It is remarkable that the questions 3-6 have positive answers for $n = 1$ while are open problems for $n = 2$.

Question 3. Can the first two strongly compact cardinals be the first two measurable cardinals yet be fully indestructible?

Question 4. Can the first two strongly compact cardinals $\kappa_0 < \kappa_1$ be the first two measurable cardinals yet be indestructible under posets forcing GCH at κ_0 and κ_1 but $2^{\kappa_i} = \kappa_i^{++}$?

Question 5. Can the first two strongly compact cardinals be the first two measurable cardinals and the second strongly compact cardinal be indestructible under $Add(\kappa, \kappa^{++})$?

Question 6. Can there be a proper class of measurable cardinals the first two of which are the first two strongly compact cardinals?

We also take the opportunity to answer a question asked in [5]. Apter and Cummings showed the following proposition.

Proposition 3 (Apter, Cummings) *If κ is a superstrong cardinal and a strongly compact cardinal then there is a normal measure μ on κ such that the set of strongly compact cardinals below κ has μ measure one.*

It follows from Proposition 3 that the least superstrong cardinal cannot be the least strongly compact cardinal. Apter and Cummings also asked if the least strongly compact cardinal can be the least Shelah cardinal. We give a negative answer to this question;

Proposition 4 *If κ is a Shelah cardinal and a strongly compact cardinal then there is a normal measure μ on κ such that the set of strongly compact cardinals below κ has μ measure one.*

Proof: We first show that κ must be a limit of strongly compact cardinals. Suppose not. Let $\eta < \kappa$ be such that there are no strongly compact cardinals in the interval $[\eta, \kappa)$. Then for every $\alpha < \kappa$ let $g(\alpha) =$ the least inaccessible above α if α isn't measurable or $\alpha < \eta$ and $g(\alpha) = \sup\{\beta + 1 : \alpha$ is β-strongly compact $\}$ if α is measurable. Clearly $g(\alpha) > \alpha$ for all $\alpha < \kappa$. Also, note that for all $\alpha < \kappa$, $g(\alpha) < \kappa$. This is because if $g(\alpha) \geq \kappa$ then α is $< \kappa$ strongly compact and κ is strongly compact. This means that α is strongly compact contradicting our assumption. Thus, $g : \kappa \to \kappa$. Let $f : \kappa \to \kappa$ be defined by $f(\alpha) =$ the least inaccessible above $g(\alpha)$. Let $j : V \to M$ be such that $V_{j(f)(\kappa)} \subseteq M$. In particular, κ is $< j(f)(\kappa)$-strongly compact in M. Thus, by definition of g, we have that $j(g)(\kappa) \geq j(f)(\kappa)$, a contradiction. It must then be the case that κ is a limit of strongly compact cardinals. For each $\alpha < \kappa$ let $h(\alpha)$ be the least strongly compact cardinal above α. Then $h : \kappa \to \kappa$. Let $j : V \to M$ be such that $V_{j(h)(\kappa)} \in M$ and $\mathrm{cp}(j) = \kappa$. Then in M, κ is $< j(h)(\kappa)$ strongly compact and $j(h)(\kappa)$ is strongly compact. This implies that in M, κ is strongly compact. Let then $\mu = \{A : \kappa \in j(A)\}$. It is then clear that the set of strongly compact cardinals below κ has μ measure one.

□

However, the strongly compact cardinals can be characterized by superstrong cardinals.

Theorem 8 (Apter-S, [10]) *It is consistent relative to n supercompact cardinals that the first n strongly compact cardinals are the first n measurable limits of superstrong cardinals and there is no cardinal κ which κ^+-supercompact.*

We also mention a problem that might be easier to solve than the Main Open Problem.

Question 7. For $n > 2$, are the theories "$ZF+$ the first n-measurable cardinals are the first n-supercompact cardinals" and "$ZF+$ the first n-measurable cardinals are the first n-strongly compact cardinals" consistent where $n \in [1, \omega]$? (for $n = 1, 2$ see [9]).

It is conceivable that in ZFC, the first ω-measurable cardinals $\langle \kappa_i : i < \omega \rangle$ cannot be the first ω-strongly compact cardinals. Whether this is the case or not probably depends on the reflection properties of $H_{\kappa_\omega^+}$ and H_{κ_ω}.

Question 8. Suppose $\langle \kappa_i : i < \omega \rangle$ are strongly compact cardinals. What kind of reflection properties does $H_{\kappa_\omega^+}$ have?

There are few positive results on *Question 2.* First the following is a folklore fact.

Fact. If $\langle \kappa_i : i < \omega \rangle$ are strongly compact cardinals and $\kappa_\omega = \sup \langle \kappa_i : i < \omega \rangle$ then every stationary subset of κ_ω^+ reflects.

Next, there is the following beautiful result of Magidor and Shelah.

Theorem 9 (Magidor and Shelah, [25]) *If $\langle \kappa_i : i < \omega \rangle$ are strongly compact cardinals and $\kappa_\omega = \sup \langle \kappa_i : i < \omega \rangle$ then there are no κ_ω^+-Aronszjan trees.*

Our final word is optimistic in nature. We do think that the Main Open Problem should be within the scope of current knowledge. It is a difficult problem, one whose ultimate solution might just lie elsewhere then the places that were suspected in the past. Understanding the combinatorics of λ^+ where λ is a limit of strongly compact cardinals might eventually lead to its negative resolution.

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