On HOD-supercompactness *†

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Abstract

During his Fall 2005 set theory seminar, Woodin asked whether V-supercompactness implies HOD-supercompactness. We show, as he predicted, that that the answer is no.

1 Introduction

In connection with his work on suitable extender sequences, which give rise to inner models that can basically have any large cardinal whatsoever, Woodin introduced the concept of N-supercompactness. Suppose N is a proper class inner model of V and let $j : V \to M$ be an elementary embedding. Then we let $j(N) = \cup_{\alpha<ORD} j(V_\alpha^N)$. Note that we then get that $j \restriction N : N \to j(N)$ is a Σ_1-elementary embedding. If N is definable then $j \restriction N$ is also fully elementary.

Definition 1.1 (Woodin, [4]) *Suppose N is a proper class inner model of V, and κ is a supercompact cardinal. Then κ is N-supercompact if for all strong limit cardinals λ, there exists an embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$, and $j(N) \cap V_\lambda = N \cap V_\lambda$.*

In conjunction with Definition 1.1, Woodin asked the following question.

Question 1 (Woodin, [5]) *Suppose κ is supercompact. Is κ HOD-supercompact?*

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We intend to show, as Woodin predicted, that the answer to Question 1 is consistently no (if $V = \text{HOD}$ then the two concepts coincide, so one can only hope for a consistency proof). Note that any supercompact cardinal is L-supercompact. But HOD-supercompactness is a non-trivial concept. It is worth noting that Woodin showed that if κ is an extendible cardinal then κ is HOD-supercompact (see [4]). Before giving the proof of the theorem, we introduce the necessary terminology and review some background material.

If κ is a cardinal then $\text{Add}(\kappa, 1)$ is the poset for adding 1 Cohen subset to κ. The next partial ordering is designed to code sets of ordinals into the power set function and hence, into HOD. Suppose $\kappa < \lambda$ are cardinals and A is a subset of κ. Let λ_α be the $\alpha + 1$ st successor cardinal strictly greater than λ. Let $\mathbb{S}_{\kappa,\lambda}(A) = \prod_{\alpha \in A} \text{Add}(\lambda_\alpha, \lambda_\alpha^{++})$. If GCH holds between κ and $\lambda^{+\kappa}$ then in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $\alpha \in A \leftrightarrow 2^{\lambda_\alpha} = \lambda_\alpha^{++}$. This implies that in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $A \in \text{HOD}$.

Laver, in [3], showed that if κ is a supercompact cardinal then there is a function $f : \kappa \to V_\kappa$ such that for any set x and any cardinal λ such that $|TC(x)| \leq \lambda$ there is an embedding $j : V \to M$ such that $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$. When κ is a supercompact cardinal and f is such a function then f is called a Laver function.

A forcing notion $\mathbb{P}$ admits a closure point at δ if it factors as $\mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is non-trivial, $|\mathbb{Q}| \leq \delta$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed"(this notion is due to Hamkins). δ-strategic closure certainly follows from just δ-closure. In this paper, we do not use posets that are δ-closed but are not δ-strategically closed. Therefore, there is no need to explain what δ-strategic closure is. The following theorem will be used in the proof of the main theorem of this paper.

Theorem 1 (Hamkins, [1]) *If $V \subseteq V[G]$ admits a closure point at δ and $j : V[G] \to M[j(G)]$ is an ultrapower embedding in $V[G]$ with $\delta = \text{cp}(j)$, then $j \restriction V : V \to M$ is a definable class in V.*

2 The Theorem

Theorem 2 *Suppose κ is a supercompact cardinal. Then there is a forcing extension of V in which κ is supercompact but not* HOD*-supercompact.*

Proof: Without loss of generality we can assume that GCH holds in V.

Before presenting the forcing, let us give some motivation for its definition. We want to preserve the supercompactness of κ but turn it into a non HOD-supercompact cardinal. Suppose we have defined a partial ordering $\mathbb{P}$ that achieves this. Let G be a generic for $\mathbb{P}$. To preserve the supercompactness of κ, we would like $\mathbb{P}$ to be a Reverse Easton Iteration (see [2] for more on this kind of iteration). Because of this $\mathbb{P}$ will admit a closure point, which also means that Theorem 1 applies. Let $W = V[G]$. By replacement and the fact that HOD is first order definable, we can find λ such that $\text{HOD} \cap W_\lambda = \text{HOD}^{W_\lambda}$ (here HOD is the HOD of W). Let $j : W \to M$ be a λ-supercompactness embedding. Then $\text{HOD} \cap W_\lambda \neq j(\text{HOD}) \cap W_\lambda$. Because of Theorem 1, there is $i : V \to N$

such that j is the lift of i. Then $j(G)$ is N-generic and $M = N[j(G)]$. Because $\mathrm{HOD} \cap W_\lambda \neq j(\mathrm{HOD}) \cap W_\lambda$ and $\mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$, there must be $a \in W_\lambda$ such that a is OD in M but not in W. Our forcing should make sure that this happens. Thus, $i(\mathbb{P})$ must add a set to W_λ and make it OD in M but the witness that the set is OD in M should be outside W_λ. To achieve this, $\mathbb{P}$ has to do the same below κ. A Laver function can be used to organize the iteration and make sure that the new set will be OD in M but not in W_λ.

Let f be a Laver function for κ. Let $S = \{\delta : \delta$ is a measurable cardinal, $f''\delta \subseteq \delta$ and $f(\delta) > \delta\}$. $\mathbb{P}$, our partial ordering, is a length κ Reverse Easton Iteration followed by $\mathrm{Add}(\kappa, 1)$. Let $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1)$. Since we described the support of the iteration, we only need to describe the posets used in the iteration. Let $\mathbb{P}_\kappa = \langle\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha\rangle : \alpha < \kappa\rangle$. Then $\dot{\mathbb{Q}}_\alpha$ is a term for trivial forcing as long as α is not in S. If α is in S then we let $\dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\alpha, 1) * \mathbb{S}_{\alpha, f(a)}(\dot{X})$ where $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$. Thus, forcing with $\mathbb{Q}_\alpha$ introduces a new subset of α and $\mathbb{Q}_\alpha$ itself codes this subset into the powerset function above $f(a)$. This means that the subset of α added by $\mathbb{Q}_\alpha$ is coded into HOD of $V^{\mathbb{P}}$. We claim that $\mathbb{P}$ is as desired. Let G be V-generic for $\mathbb{P}_\kappa$ and g be $V[G]$-generic for $\mathrm{Add}(\kappa, 1)$.

Lemma 2.1 *$V[G][g] \vDash$ "κ is supercompact".*

Proof: Let $\lambda > \kappa$ be a strong limit cardinal of cofinality strictly bigger than κ and $j : V \to M$ be a λ-supercompactness embedding such that $j(f)(\kappa) = \lambda$. Since it is easy to see that the stage κ forcing in $M^{\mathbb{P}_\kappa}$ is non-trivial, we have that $j(\mathbb{P}) = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, 1) * \mathbb{S}_{\kappa, j(f)(\kappa)}(\dot{X}) * \dot{\mathbb{Q}} * j(\dot{\mathrm{Add}}(\kappa, 1)) = \mathbb{P} * \dot{\mathrm{Add}}(\kappa, 1) * \mathbb{S}_{\kappa,\lambda}(\dot{X}) * \dot{\mathbb{Q}} * j(\dot{\mathrm{Add}}(\kappa, 1))$ where $\dot{X}$ is the name for the generic subset of κ added by $Add(\kappa, 1)$. Since the first stage of forcing in $\mathbb{S}_{\kappa,\lambda}(\dot{X})$ is greater than λ, we may rewrite $j(\mathbb{P}) = \mathbb{P} * \dot{\mathrm{Add}}(\kappa, 1) * \mathbb{S}_{\kappa,\lambda}(\dot{X}) * \dot{\mathbb{P}}_{tail} * j(\dot{\mathrm{Add}}(\kappa, 1))$ where the first stage of forcing in $\mathbb{P}_{tail}$ is greater than λ. Laver's original argument of [3] now shows that for any cardinal $\gamma < \lambda$, $V[G][g] \vDash$ "κ is γ supercompact". This completes our proof of Lemma 2.1.

□

Lemma 2.2 *In $V[G][g]$, κ is not* HOD*-supercompact.*

Proof: Towards a contradiction assume that κ is HOD-supercompact in $V[G][g]$. Let $W = V[G][g]$ and let HOD be $\mathrm{HOD}^{V[G][g]}$. Fix a strong limit cardinal λ such that $\mathrm{HOD}^{W_\lambda} = \mathrm{HOD} \cap W_\lambda$. Let $j : W \to M$ be a λ-supercompactness embedding such that $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$. By Theorem 1, $i = j \restriction V$ is in V and j is the lift of i. Let $N = j(V) = \cup_{\alpha<ORD} i(V_\alpha)$. We have that if H is the N-generic for $i(\mathbb{P})$ then $M = N[H]$. We also have that $H \cap \mathbb{P}_\kappa = G$. Let g' be the generic for $\mathrm{Add}(\kappa, 1)$ given by H ($\mathbb{Q}_\kappa$ in N is non-trivial, as otherwise j cannot be a supercompactness embedding). Then in $N[H]$, g' is ordinal definable. But because $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$, $g' \in \mathrm{HOD}^{W_\lambda}$. Thus,

g' is ordinal definable in W_λ. But $W_\lambda = V_\lambda[G][g]$. This along with the fact that g' is added by a homogeneous forcing over $V_\lambda[G]$ imply that g' is in $V_\lambda[G]$. However, this cannot happen as g' is a $V[G]$-generic object for $\mathrm{Add}(\kappa, 1)$. This completes the proof of Lemma 2.2.

□

Lemma 2.1 and Lemma 2.2 complete the proof of Theorem 2.

□

We finish the paper with a few questions. Woodin showed the following theorem:

Theorem 3 (Woodin, [4]) *Suppose δ is* HOD*-supercompact and for a proper class of singular strong limit cardinals δ such that $\mathrm{cf}(\delta) > \omega$, $(\delta^+)^{\mathrm{HOD}} = \delta^+$. Moreover, suppose that for some $\gamma < \kappa$ there is an embedding $j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$ with critical point $\geq \kappa$. Then $j \in \mathrm{HOD}$.*

We do not know whether Theorem 3 is true when instead of HOD-supercompactness supercompactness is assumed.

Question 2 *Is it consistent that there is a supercompact cardinal κ, there is a proper class of singular strong limit cardinals δ such that $\mathrm{cf}(\delta) > \omega$ and $\delta^+ = (\delta^+)^{\mathrm{HOD}}$, there is a $\gamma > \kappa$ and $j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$ with critical point $\geq \kappa$ but $j \notin \mathrm{HOD}$?*

Question 3 (Woodin, [5]) *Is it consistent that there is a supercompact cardinal κ but for some strong limit cardinal $\delta > \kappa$, $\mathrm{cf}(\delta) > \omega$ and $(\delta^+)^{\mathrm{HOD}} < \delta^+$?*

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