On the prewellorderings associated to the directed systems of mice.*†

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Abstract

Working under AD, we investigate the length of prewellorderings given by the iterates of $\mathcal{M}_{2k+1}$, which is the minimal proper class mouse with $2k+1$ many Woodin cardinals. In particular, we answer some questions from [2].

In recent years, there have been many interactions between inner model theory and descriptive set theory. While the connection between the two areas was established early on in 1960s, the bulk of modern interactions seem to go back to the work of Martin, Steel and Woodin carried out in late 80s and early 90s. In particular, Steel's computation of $\mathrm{HOD}^{L(\mathbb{R})}$ below Θ (see [21]), Woodin's subsequent computation of $\mathrm{HOD}^{L(\mathbb{R})}$ (see [20]) and Woodin's computation of $\mathrm{HOD}^{L[x][g]}$ (largely unpublished) have been of crucial importance for the results that followed.

In this paper, we investigate the prewellordering associated with the directed system generated by $\mathcal{M}_{2k+1}$ where $k \in \omega$. Our intended application is the computation of the sup of the

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lengths of $\Game^{2k+1}(\omega \cdot n - \underset{\sim}{\Pi}^1_1)$-prewellorderings. We show that the sup is κ^1_{2k+3}. This generalizes Hjorth's computation of $\Game^1(\omega \cdot n - \underset{\sim}{\Pi}^1_1)$-prewellorderings. See Section 3 for the statement of the main theorem of this paper.

We assume familiarity with descriptive set theory and with inner model theory. The standard reference for descriptive set theory is [11] and while for inner model theory there isn't any standard reference the reader is assumed to be familiar with [19]. Lastly, the results presented in this paper were proven in Berlin during the Spring of 2006 and are the first steps I took in inner model theory. I wish to thank my adviser John Steel who brought this project to my attention.

0.1 On descriptive set theory

We assume AD throughout this paper and for us, $\mathbb{R}$ is the Baire space ω^ω. We let u_n be the nth uniform indiscernible and $s_n = \langle u_i : i \leq n \rangle$. We let $s_0 = \emptyset$. Under AD, $u_n = \aleph_n$ (see [5]).

Recall that for $x \in \mathbb{R}$, $C_{2n}(x)$ is the largest countable $\Sigma^1_{2n}(x)$ set and $Q_{2n+1}(x)$ is the largest countable $\Pi^1_{2n+1}(x)$-bounded set. Alternatively,

$$C_{2n}(x) = \{y \in \mathbb{R} : y \text{ is } \Delta^1_{2n}(x) \text{ in a countable ordinal } \}$$

and

$$Q_{2n+1}(x) = \{y \in \mathbb{R} : y \text{ is } \Delta^1_{2n+1}(x) \text{ in a countable ordinal } \}.$$

The first equality is due to Harrington and Kechris (see [1]) and the second one is due to Kechris, Martin and Solovay (see [7]).

Following [11], we let pointclass stand for any collection of sets of reals (that is, we are not requiring closure under anything). If Γ is a pointclass then $\breve{\Gamma}$ is the dual pointclass and $\Delta_\Gamma = \Gamma \cap \breve{\Gamma}$.

A relation $\leq$ is a *prewellordering* if it is transitive, reflexive, connected and wellfounded. Given a set of reals A, ϕ is a norm on A if $\phi : A \to \mathrm{Ord}$. For each norm ϕ on A, we let $\leq^\phi$

be the binary relation on A given by $x \leq^{\phi} y$ iff $\phi(x) \leq \phi(y)$. Then $\leq^{\phi}$ is a prewellordering of A. The opposite is true as well, given a prewellordering $\leq$ of A there is an associated norm ϕ defined on A such that $\leq = \leq^{\phi}$. If Γ is a pointclass then ϕ is a Γ-norm if there are relations $\leq^{\phi}_{\Gamma} \in \Gamma$ and $\leq^{\phi}_{\breve{\Gamma}} \in \breve{\Gamma}$ such that for every $y \in dom(\phi)$ and for any $x \in \mathbb{R}$,

$$[x \in dom(\phi) \wedge \phi(x) \leq \phi(y)] \leftrightarrow x \leq^{\phi}_{\Gamma} y \leftrightarrow x \leq^{\phi}_{\breve{\Gamma}} y.$$

If Γ is a pointclass, we let

$$d(\Gamma) = \sup\{\leq^{*} : \leq^{*} \in \Gamma \text{ and } \leq^{*} \text{ is a prewellordering }\}.$$

A sequence of norms $\vec{\phi} = \langle \phi_i : i < \omega \rangle$ on A is a *scale* on A if whenever $\langle x_i : i < \omega \rangle \subseteq A$ is a sequence of reals converging to x such that for each i the sequence $\langle \phi_i(x_k) : k < \omega \rangle$ is eventually constant then $x \in A$ and for each i, $\phi_i(x) \leq \lambda_i$ where λ_i is the eventual value of $\langle \phi_i(x_k) : k < \omega \rangle$. We write $x_i \to x(mod\vec{\phi})$ if $\langle x_i : i < \omega \rangle$ converges to x in the above sense. $\vec{\phi}$ is a Γ-scale on A if there are relations $R \in \Gamma$ and $S \in \breve{\Gamma}$ such that for all $y \in A$, for any $x \in \mathbb{R}$ and for any $n < \omega$

$$[x \in A \wedge \phi_n(x) \leq \phi_n(y)] \leftrightarrow R(n, x, y) \leftrightarrow S(n, x, y).$$

We say Γ has the prewellordering property if every set in Γ has a Γ-norm. We say Γ has the scale property if every set in Γ has a Γ-scale. For more on prewellordering property and scale property see [11].

Suppose κ is a cardinal. $T \subseteq \cup_{n<\omega} \omega^n \times \kappa^n$ is a tree if whenever $s \in T$ then $s \upharpoonright i \in T$ for any $i < lh(s)$. For $(x, f) \in \omega^\omega \times \kappa^\omega$ is a branch of T if $(x \upharpoonright i, f \upharpoonright i) \in T$ for any $i < \omega$. $[T]$ is the set of branches of T. $p[T]$ is the projection of $[T]$ on the first coordinate, i.e., $x \in p[T]$ iff there is $f \in \kappa^\omega$ such that $(x, f) \in T$.

A set of reals A is κ-Suslin if there is a tree $T \subseteq \cup_{n<\omega} \omega^n \times \kappa^n$ such that $A = p[T]$. A is Suslin if it is κ-Suslin for some κ. Given a scale $\vec{\phi}$ on A one can construct a tree T such that

$p[T] = A$. More precisely, let T be the set of pairs (s, f) such that there is some real $x \in A$ such that $s \lhd x$ and $f(i) = \phi_i(x)$ for each $i < lh(f)$. Given a tree T such that $p[T] = A$, one can get a scale $\vec{\phi}$ on A by considering the leftmost branches of T. Thus, carrying a scale and being Suslin are equivalent.

Finally, we say that κ is a *Suslin cardinal* if there is set of reals A which is κ-Suslin but A is not η-Suslin for any $\eta < \kappa$. We let $S(\kappa)$ be the pointclass of κ-Suslin sets. It is not hard to show that $S(\kappa)$ is closed under projections (see [11]). For more on trees and Suslin sets see [11]. For a complete characterization of Suslin cardinals see [4].

Under AD, for each n and real z, $\Pi^1_{2n+1}(z)$ and $\Sigma_{2n+2}(z)$ have the scale property. The sup of $\undertilde{\Pi}^1_{2n+1}$ prewellorderings and $\undertilde{\Sigma}_{2n+2}$ prewellorderings play an important role in descriptive set theory. Following [11], we let

$$\delta^1_{2n+1} = d(\undertilde{\Pi}^1_{2n+1}) = \delta(\Pi^1_{2n+1})$$

and

$$\delta^1_{2n} = \delta(\undertilde{\Sigma}^1_{2n}).$$

It turns out that under AD,

$$\delta^1_{2n} = (\delta^1_{2n+1})^+$$

and δ^1_{2n+1} is a successor cardinal whose predecessor is denoted by κ^1_{2n+1} (see [11]). It is shown in [11] that

$$\undertilde{\Sigma}^1_{2n+3} = S(\kappa^1_{2k+1}).$$

Also, $\kappa^1_3 = \aleph_\omega$, $\delta^1_3 = \aleph_{\omega+1}$ and $\delta^1_4 = \aleph_{\omega+2}$.

$\Game$ is the *game quantifier*. Given a finite or an ω sequence of reals $(x_i : i < \omega)$ we let $\langle x_i : i < \omega \rangle$ be the real coding it. Given a set of reals $A \subseteq \mathbb{R}^2$ we let $\Game^{\mathbb{R}} A$ be the set

$$x \in \Game^{\mathbb{R}} A \leftrightarrow \exists x_0 \forall x_1 \exists x_2 \forall x_3 \cdots \exists x_{2n} \forall x_{2n+1} \cdots ((x, \langle x_i : i < \omega \rangle) \in A).$$

Equivalently,

$$\Game^{\mathbb{R}} A = \{x \in \mathbb{R} : \text{player } I \text{ has a winning strategy in } G_{A_x}\}.$$

A set is $\omega \cdot n - \underset{\sim}{\Pi}^1_1$ if there is a sequence $\langle A_\alpha : \alpha < \omega \cdot n \rangle \subseteq \underset{\sim}{\Pi}^1_1$ such that

$$x \in A \leftrightarrow \text{the least } \alpha \text{ such that } x \notin A_\alpha \text{ is odd.}$$

Equivalently sets in $\omega \cdot n - \underset{\sim}{\Pi}^1_1$ constitute the first $\omega \cdot n$ levels of *difference hierarchy* for $\underset{\sim}{\Pi}^1_1$.

1 On inner model theory

Recall that if $\mathcal{M}$ is a premouse then $\mathcal{G}_\kappa(\mathcal{M})$ is a two play game that has $< \kappa$ moves. In this game, player I plays the successor steps which amounts to choosing an extender and applying it to the earliest model it makes sense to apply. Player II plays limit stages and her job is to choose a well-founded cofinal branch of the resulting iteration tree. II wins if all the models produced in the game are well founded. Σ is then called an iteration strategy for $\mathcal{M}$ if it is a winning strategy for player II.

If $\mathcal{M}$ is a mouse and $\xi \leq o(\mathcal{M})$, then we let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a cutpoint of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (crit(E), lh(E)]$. We say ξ is a strong cutpoint if there is no E on $\mathcal{M}$ such that $\xi \in [crit(E), lh(E)]$.

If $\mathcal{T}$ is an iteration tree, i.e., a play of the game, then, following the notation of [10], $\mathcal{T}$ has the form

$$\mathcal{T} = \langle T, deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} | \alpha + 1 < \eta \rangle \rangle.$$

Recall that D is the set of *dropping* points. Recall also that if η is limit then

$$\vec{E}(\mathcal{T}) = \cup_{\alpha<\eta}(\dot{E}^{\mathcal{M}_\alpha \restriction lh(E_\alpha)}),$$

$$\mathcal{M}(\mathcal{T}) = \cup_{\alpha<\eta}\mathcal{M}_\alpha \restriction lh(E_\alpha),$$

$$\delta(\mathcal{T}) = \sup_{\alpha<\eta} lh(E_\alpha).$$

If b is a branch of $\mathcal{T}$ then $\mathcal{M}_b^{\mathcal{T}}$ is the branch model of the tree. Then if $\alpha \leq_T \beta$ then $i_{\alpha,\beta}^{\mathcal{T}} : \mathcal{M}_\alpha^* \rightarrow \mathcal{M}_\beta^{\mathcal{T}}$ is the iteration map if $[\alpha, \beta]_{\mathcal{T}} \cap D = \emptyset$ and $i_{\alpha,b}^{\mathcal{T}} : \mathcal{M}_\alpha^* \rightarrow \mathcal{M}_b^{\mathcal{T}}$ is the iteration map if $\alpha \in b$ and $b - \alpha \cap D = \emptyset$.

It is by now a standard fact that if b and c are cofinal branches of $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$ then $\mathcal{R} \vDash$ "$\delta(\mathcal{T})$ is Woodin" (see [19]). Moreover, if $\mathcal{Q}$ is a mouse over $\mathcal{M}(\mathcal{T})$ (this in particular means that $\mathcal{Q}$ has no extenders overlapping with $\delta(\mathcal{T})$) such that $\mathcal{Q} \vDash$ "$\delta(\mathcal{T})$ is Woodin" yet definably over $\mathcal{Q}$ there is a counterexample to Woodiness of $\delta(\mathcal{T})$ then there is at most one cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ (see [19]). The following lemma, which builds upon the proof of the aforementioned fact is one of the most important ingredients available to us and will be used in this paper many times. It is essentially due to Martin and Steel, see Theorem 2.2 of [9].

Lemma 1.1 (Uniqueness of branches) *Suppose $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of limit length and s is a cofinal subset of $\delta(\mathcal{T})$. Then there is at most one cofinal branch b such that there is $\alpha \in b$ with the property that $i_{\alpha,b}^{\mathcal{T}}$ exists and $s \subseteq ran(i_{\alpha,b}^{\mathcal{T}})$.*

Proof. Towards a contradiction, suppose there are two cofinal branches b and c such that for some α, β, both $i_{\alpha,b}^{\mathcal{T}}$ and $i_{\beta,c}^{\mathcal{T}}$ exist and $s \subseteq ran(i_{\alpha,b}^{\mathcal{T}}) \cap ran(i_{\beta,c}^{\mathcal{T}})$. Without loss of generality we can assume that α and β are the least ordinals with this property, $\alpha \leq \beta$ and that b and c diverge at α or earlier, i.e., if γ is the least ordinal in $b \cap c$ then $\gamma \leq \alpha$. By [9], we can assume that $b = \langle \alpha_n : n < \omega \rangle$, $c = \langle \beta_n : n < \omega \rangle$, $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Let then ξ be the least ordinal in $ran(i_{\alpha,b}^{\mathcal{T}}) \cap ran(i_{\beta,c}^{\mathcal{T}})$. Let n be the least such that $crit(i_{\alpha_n,b}^{\mathcal{T}}) > \xi$. This means that $crit(E_{\alpha_{n+1}-1}^{\mathcal{T}}) > \xi$ and that $lh(E_{\alpha_n}^{\mathcal{T}}) < \xi$. By the proof of Theorem 2.2 of [9], this means that for some $m \geq 1$, $\xi \in [crit(E_{\beta_m-1}^{\mathcal{T}}), lh(E_{\beta_m-1}^{\mathcal{T}}))$. This then implies that $\xi \notin ran(i_{\beta_{m-1},c}^{\mathcal{T}})$, which is a contradiction. $\square$

The proof of Lemma 1.1 gives the following as well.

Lemma 1.2 *Suppose $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ of limit length and b, c are two cofinal branches of $\mathcal{T}$ such that $i_b^{\mathcal{T}}$ and $i_c^{\mathcal{T}}$ exist. Suppose that for some α,*

$$i_b^{\mathcal{T}}(\alpha) = i_c^{\mathcal{T}}(\alpha) < d(\mathcal{T}).$$

Then $i_b^{\mathcal{T}} \restriction \alpha = i_c^{\mathcal{T}} \restriction \alpha$. Moreover, if $\xi \in b$ is the least such that $crit(E_\xi^{\mathcal{T}}) > i_b^{\mathcal{T}}(\alpha)$ then $b \cap \xi = c \cap \xi$.

If $\mathcal{M}$ is a mouse and $\mathcal{T}$ is a tree then we say $\mathcal{T}$ is above η if all extender used in $\mathcal{T}$ have critical point $> \eta$. If Σ is an (ω_1, ω_1)-iteration strategy for $\mathcal{M}$ and $\vec{\mathcal{T}}$ is a stack of trees on $\mathcal{M}$ according Σ with last model $\mathcal{N}$ then we let $\Sigma_{\mathcal{N}, \vec{\mathcal{T}}}$ be the strategy of $\mathcal{N}$ induced by Σ. We say Σ has the Dodd-Jensen property if whenever $\mathcal{N}$ is an iterate of $\mathcal{M}$ via Σ and $\pi : \mathcal{M} \rightarrow \mathcal{W} \trianglelefteq \mathcal{N}$ is (fine structural) embedding then the iteration from $\mathcal{M}$ to $\mathcal{N}$ doesn't drop, $\mathcal{W} = \mathcal{N}$ and if $i : \mathcal{M} \rightarrow \mathcal{N}$ is the iteration embedding then for every α, $i(\alpha) \leq \pi(\alpha)$. If Σ has the Dood-Jensen property and $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are two stacks on $\mathcal{M}$ with last model $\mathcal{N}$ such that $i^{\vec{\mathcal{T}}}$ and $i^{\vec{\mathcal{U}}}$ exist then $i^{\vec{\mathcal{T}}} = i^{\vec{\mathcal{U}}}$ and $\Sigma_{\mathcal{N}, \vec{\mathcal{T}}} = \Sigma_{\mathcal{N}, \vec{\mathcal{U}}}$. Lastly, we let

$$I(\mathcal{M}, \Sigma) = \{\mathcal{N} : \text{there is a stack } \vec{\mathcal{T}} \text{ on } \mathcal{M} \text{ according to } \Sigma \text{ with last model } \mathcal{N} \text{ and } i^{\vec{\mathcal{T}}} \text{ exists } \}.$$

Next we introduce $\mathcal{S}$-constructions which were first fully introduced in [15] where they were called P-constructions. Such constructions are due to Steel and hence, we change the terminology and call them $\mathcal{S}$-constructions. These constructions allow one to translate mice over some set A to mice over some set B provided A and B are somehow close. The complete proof of the following proposition is essentially the proof of Lemma 1.5 of [15].

Proposition 1.3 *Suppose $\mathcal{M}$ is a sound mouse and δ is a strong cutpoint cardinal of $\mathcal{M}$. Suppose further that $\mathcal{N} \in \mathcal{M}|\delta + 1$ is such that $\delta \subseteq \mathcal{N} \subseteq H_\delta^{\mathcal{M}}$ and there is a partial ordering $\mathbb{P} \in L_\omega[\mathcal{N}]$ such that whenever $\mathcal{Q}$ is a mouse over $\mathcal{N}$ such that $H_\delta^{\mathcal{Q}} = \mathcal{N}$ then $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $\mathcal{Q}$. Then there is a mouse $\mathcal{S}$ over $\mathcal{N}$ such that $\mathcal{M}|\delta$ is generic over $\mathcal{S}$ and $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$.*

It is clear what $\mathcal{S}$ must be. Because $\mathbb{P}$ is a small forcing with respect to the critical points of the extenders of $\mathcal{M}$ that have indices bigger than δ, all such extenders can be put on a sequence of some mouse over $\mathcal{N}$. This is exactly what S-constructions do. An S-construction of $\mathcal{M}$ over $\mathcal{N}$ is a sequence of $\mathcal{N}$-mice $\langle \mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha \leq \eta \rangle$ such that

1. $\mathcal{S}_0 = L_\omega[\mathcal{N}]$,

2. if $\mathcal{M}|\delta$ is generic over $\bar{\mathcal{S}}_\alpha$ for a forcing in $L_\omega[\mathcal{N}]$ then $\bar{\mathcal{S}}_\alpha[\mathcal{N}] = \mathcal{M}|(\omega \times \alpha)$ and

   (a) if $\mathcal{M}||(\omega \times \alpha)$ is active then $\mathcal{S}_\alpha$ is the expansion of $\bar{\mathcal{S}}_\alpha$ by the last extender of $\mathcal{M}||(\omega \times \alpha)$ and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,

   (b) if $\mathcal{M}||(\omega \times \alpha)$ is passive then $\mathcal{S}_\alpha = \bar{\mathcal{S}}_\alpha$ and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,

3. if λ is limit then $\bar{S}_\lambda = \cup_{\alpha<\lambda} \mathcal{S}_\alpha$.

By the proof of Lemma 1.5 of [15], the $\mathcal{S}$-construction described in 1-3 cannot fail as long as the hypothesis of 2 holds. Thus, we always have a last model of $\mathcal{S}$-construction which might be some $\bar{\mathcal{S}}_\alpha$ instead of $\mathcal{S}_\alpha$. We let $\mathcal{S}$ be the last model of $\mathcal{S}$ construction. Then by the proof of Lemma 1.5 of [15], $\mathcal{S}[\mathcal{M}|\delta] \trianglelefteq \mathcal{M}$ and if the hypothesis of 2 never fails then in fact, $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$. Moreover, $\mathcal{S}$ inherits whatever iterability $\mathcal{M}$ has above δ. $\mathcal{S}$-constructions are used in various places in inner model theory. A particularly important application for us is the following lemma.

Lemma 1.4 *Suppose $\mathcal{M} \vDash ZFC - Powerset$ is a mouse and η is a strong cutpoint non-Woodin cardinal of $\mathcal{M}$. Suppose $\gamma > \eta$ is a cardinal of $\mathcal{M}$ and $\mathcal{N} = L[\vec{E}]^{\mathcal{M}|\gamma}$. Suppose $L_\omega(\mathcal{N}|\eta) \vDash$ "η is Woodin". Let $\langle \mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha < \nu \rangle$ be the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$. Then for some $\alpha < \nu$, $\mathcal{S}_\alpha \vDash$ "η isn't Woodin".*

Proof. Let $\mathcal{S}$ be the last model of the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$. Suppose η is a Woodin cardinal of $\mathcal{S}$. Then $\mathcal{M}|\eta$ is generic for the η-generator version of the extender algebra of $L_\omega(\mathcal{N}|\eta)$. we also have that $\mathcal{M}|\eta$ is generic over $\mathcal{S}$ for the η-generator version of the extender algebra at η and hence, $\mathcal{S}[\mathcal{M}|\eta] = \mathcal{M}|(\eta^+)^{\mathcal{M}}$. Thus, η isn't Woodin in $\mathcal{S}[\mathcal{M}|\eta]$. Let $f : \eta \to \eta$ be the function in $\mathcal{M}$ witnessing that η isn't Woodin. Then because the η-generator version of extender algebra is η-cc, there is $g \in \mathcal{S}$ which dominates f. Let E be the extender that witnesses that η is Woodin for g. Then if E^* is the resurrection of E then E^* witnesses the Woodiness of η for f in $\mathcal{M}$, contradiction! $\square$

If $\mathcal{S}$ is the output of the $\mathcal{S}$-construction of $\mathcal{M}$ over $\mathcal{N}$ then we write $\mathcal{S}^{\mathcal{M}}(\mathcal{N})$ for $\mathcal{S}$.

Before moving on, we introduce one last notation. Given a model M of a fragment of ZFC with a unique Woodin cardinal, we let $\mathbb{B}^M$ be the extender algebra of M at its unique Woodin cardinal. If $G \subseteq \mathbb{B}^M$ then we let x_G be the set naturally coded by G.

2 DST and IMT together

We let $\mathcal{M}_n$ be the minimal proper class mouse with n Woodin cardinals. $\mathcal{M}_n^\#$ is the minimal mouse with last extender and with n Woodin cardinals. Clearly, $\mathcal{M}_n$ is the result of iterating the last measure of $\mathcal{M}_n^\#$ through the ordinals. We let $\mathcal{M}_0 = L$. In [18], Steel and Woodin computed the descriptive set theoretic complexity of the reals of $\mathcal{M}_n$. They showed that

$$C_{2n+2}(x) = \mathbb{R}^{\mathcal{M}_{2n}(x)}$$

and

$$Q_{2n+3}(x) = \mathbb{R}^{\mathcal{M}_{2n+1}(x)}.$$

We let

$$S_n(x) = \begin{cases} C_{n+2}(x) : & n \textit{ is even} \\ Q_{n+2}(x) : & n \textit{ is odd} \end{cases}$$

It is then clear that

$$S_n(x) = \mathbb{R}^{\mathcal{M}_n(x)}.$$

We also let $\mathcal{M}_\omega$ be the minimal proper class mouse with ω Woodin cardinals and $\mathcal{M}_\omega^\#$ be the minimal mouse with ω Woodin cardinals and with a last extender. Then $\mathcal{M}_\omega$ is the result of iterating the last measure of $\mathcal{M}_\omega^\#$ through the ordinals. Woodin showed that if $\mathcal{M}_\omega^\#$ exists then AD holds in $L(\mathbb{R})$, and Steel and Woodin showed that if $\mathcal{M}_\omega^\#$ exists then for every countable transitive set a, then letting $\Gamma = (\Sigma_1^2)^{L(\mathbb{R})}$,

$$C_\Gamma(a) = \mathbb{R}^{\mathcal{M}_\omega(x)}.$$

See [19] for the proof of both results.

Let Σ be the canonical iteration strategy of $\mathcal{M}_\omega$. Let

$$\mathcal{F} = \{\mathcal{P} : \text{there is a } \Sigma\text{-iterate } \mathcal{N} \text{ of } \mathcal{M}_\omega \text{ such that } \mathcal{P} = \mathcal{N}|(\nu^{+\omega})^N \text{ where } \nu \text{ is a successor cardinal of } \mathcal{N} \text{ which is less than the least } \mathcal{N}\text{-cardinal which is strong to the least Woodin of } \mathcal{N}\}.$$

Then by the above mentioned result of Steel and Woodin, for every $\mathcal{P} \in \mathcal{F}$, $\Sigma_\mathcal{P} \in L(\mathbb{R})$. We can define $\leq_\mathcal{F}$ on $\mathcal{F}$ by $\mathcal{P} \leq_\mathcal{F} \mathcal{Q}$ iff there is α such that $\mathcal{Q}|\alpha \in I(\mathcal{P}, \Sigma_\mathcal{P})$. Notice that if $\mathcal{P} \leq_\mathcal{F} \mathcal{Q}$ and α is such that $\mathcal{Q}|\alpha \in I(\mathcal{P}, \Sigma_\mathcal{P})$ then for some $\nu < \alpha$, $\alpha = (\nu^+)^\mathcal{Q}$. If $\mathcal{P} \leq_\mathcal{F} \mathcal{Q}$ then we let $i_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}|\alpha$ be the iteration embedding.

Notice that $\leq_\mathcal{F}$ is directed and hence, we can let $\mathcal{M}_\infty$ be the direct limit of $(\mathcal{F}, \leq_\mathcal{F})$. Steel showed that

$$L(\mathbb{R}) \vDash \mathcal{M}_\infty = V_\delta^{\text{HOD}}$$

where $\delta = \delta(\underset{\sim}{\Sigma}_1^2)$ (see [19] or [21]). Lets make this a theorem.

Theorem 2.1 (Steel, [21]) $L(\mathbb{R}) \vDash \mathcal{M}_\infty = V_\delta^{\text{HOD}}$.

Woodin extended this result to compute the full HOD of $L(\mathbb{R})$. We refer the reader to [20] for more on Woodin's work on $\text{HOD}^{L(\mathbb{R})}$. It is important to note that the existence of $\mathcal{M}_\omega^\#$, which is a tiny bit stronger than $AD^{L(\mathbb{R})}$, is unnecessary and all the results in this paper can be proved only from $AD^{L(\mathbb{R})}$. Nevertheless, it is convenient and aesthetically more pleasant to assume that $\mathcal{M}_\omega^\#$ exists and we will do so whenever we wish. Experts will have no problem seeing how to remove this assumption. We refer the reader to [19] for an expanded version of this short summary of inner model theory. [19] also proves most of the results stated in this section without assuming the existence of $\mathcal{M}_\omega^\#$ but just $AD^{L(\mathbb{R})}$.

3 The main theorem

By a result of Martin (see [8]) and Neeman (see [12]), for $k \geq 1$, a set of reals A is $\Game^k(\omega\cdot n - \underset{\sim}{\Pi}^1_1)$ iff there is $m \in \omega$, a real z and a formula ϕ such that

$$x \in A \leftrightarrow \mathcal{M}_{k-1}(x, z) \vDash \phi[x, z, s_m].$$

We let $\Gamma_{k,m}(z)$ be the set of reals A such that there is a formula ϕ such that

$$x \in A \leftrightarrow \mathcal{M}_{k-1}(x, z) \vDash \phi[x, z, s_m].$$

We let $\Gamma_{k,m} = \Gamma_{k,m}(0)$ and $\underset{\sim}{\Gamma}_{k,m} = \cup_{z\in\mathbb{R}}\Gamma_{k,m}(z)$. Also, we let $\underset{\sim}{\Gamma}_k = \cup_{m<\omega}\underset{\sim}{\Gamma}_{k,m}$.

In [3], Hjorth computed the sup of the lengths of $\underset{\sim}{\Gamma}_{1,m}$-prewellorderings. He showed that

$$\delta(\underset{\sim}{\Gamma}_{1,m}) \leq u_{m+2}.$$

and therefore,

$$\kappa^1_3 = \aleph_\omega = \delta(\underset{\sim}{\Gamma}_1)^1$$

In this paper, assuming AD, we compute $\delta(\underset{\sim}{\Gamma}_{k,m})$. First let

$$a_{k,m} = \delta(\underset{\sim}{\Gamma}_{k,m}).$$

[1] It is not hard to see that the standard prewellordering of the $\{x^\# : x \in \mathbb{R}\}$ has length κ^1_3, i.e, let $\phi(n, m, x^\#) = \tau_n^{L[x]}(x, s_m)$ where $\langle \tau_n : n < \omega\rangle$ is some enumeration of the terms in the appropriate language. Then ϕ has length $u_\omega = \kappa^1_3$ and for each m letting ϕ_m be the prwellordering given by $\phi_m(n, x^\#) = \phi(n, m.x^\#)$, we have that $\phi_m \in \Gamma_{1,m+1}$. Thus, we indeed have an equality.

Here is our main theorem.

Theorem 3.1 (Main Theorem) *Assume AD and let k be an integer. Then*

$$\sup_{n<\omega} a_{2k+1,m} = \kappa^1_{2k+3}.$$

We will prove the theorem using directed systems of mice. Our proof relies on a generalization of Woodin's analysis of $\mathrm{HOD}^{L[x][g]}$. The proof is divided into subsections. The proof presented here suggests further applications of the directed systems in descriptive set theory and we will end with a discussion of projects that are left open. We start with introducing the direct limit associated with $\mathcal{M}_n$'s.

3.1 The directed system associated to $\mathcal{M}_n$.

In this section, we analyze the length of the prewellordering given by the iterates of $\mathcal{M}_{2n+1}$. As it turns out, the even case, i.e., the prewellordering associated to $\mathcal{M}_{2n}$'s, doesn't give much beyond Theorem of [19]. Nevertheless, we make all the definitions for arbitrary n. The prewellordering associated with the iterates of $\mathcal{M}_{n+1}$ that we are interested in is the following.

For any iterate $\mathcal{P}$ of $\mathcal{M}_{n+1}$ we let $\delta^{\mathcal{P}}$ be the least Woodin of $\mathcal{P}$. Let Σ be the canonical iteration strategy of $\mathcal{M}_{n+1}$. If $\mathcal{P} \in I(\mathcal{M}_{n+1}, \Sigma)$ and $\mathcal{Q} \in I(\mathcal{P}, \Sigma_{\mathcal{P}})$ then we let $i_{\mathcal{P},\mathcal{Q}}$ be the iteration embedding. We then define a prewellordering R^+_n of the set

$$\{(\mathcal{P}, \alpha) : \mathcal{P} \in I(\mathcal{M}_{n+1}, \Sigma) \wedge \alpha < \delta^{\mathcal{P}}\}$$

by $(\mathcal{P}, \alpha) R^+_n (\mathcal{Q}, \beta)$ iff $\mathcal{Q} \in I(\mathcal{P}, \Sigma_{\mathcal{P}})$ and $i_{\mathcal{P},\mathcal{Q}}(\alpha) \leq \beta$. Clearly R^+_n is a prewellordering. One problem with R^+_n is that it is a prewellordering of uncountable objects and hence, cannot be regarded as a prewellordering of the reals. Here is how one can find an equivalent prewellordering of countable objects.

We let $\mathcal{W}_n = \mathcal{M}_{n+1} | (\delta^{+\omega})^{\mathcal{M}_{n+1}}$. we define the equivalent of R^+_n on the set

$$\mathcal{J}^+_n = \{(P, \alpha) : \mathcal{P} \in I(\mathcal{W}_n, \Sigma_{\mathcal{W}_n}) \wedge \alpha < \delta^{\mathcal{P}}\}.$$

We set $(\mathcal{P},\alpha)R_n^+(\mathcal{Q},\beta)$ iff $\mathcal{Q} \in I(\mathcal{P},\Sigma_{\mathcal{P}})$ and $i_{\mathcal{P},\mathcal{Q}}(\alpha) \leq \beta$. It is not hard to see that R_n^+ is essentially the old R_n^+. Two questions then immediately come up: 1. What is the length of R_n^+? and 2. What is the complexity of R_n^+?

Because $x \to \mathcal{M}_n^\#(x)$ is a Π^1_{n+2} (see [18]), we get that $\mathcal{J}_n^+$ is $\Sigma^1_{n+3}(\mathcal{W}_n)$ (i.e., $\Sigma^1_{n+3}(x)$ for any code x of $\mathcal{W}_n$). The complexity essentially comes from the fact that we require $i_{\mathcal{P},\mathcal{Q}}$ be the correct iteration embedding and to say that we need to refer to $x \to \mathcal{M}_n^\#(x)$ operator. It is then clear that $|R_n^+| < \delta^1_{n+3}$.

To prove our main theorem we need to somehow internalize R_n^+ to $\mathcal{M}_n(x)$ where x is any real coding $\mathcal{W}_n$. Notice that $\mathcal{M}_n(x)$ doesn't know the strategy of $\mathcal{W}_n$ and hence, it doesn't know how to define its own version of R_n^+. We will define an enlargement of R_n^+ which $\mathcal{M}_n(x)$ can define and we will show that the enlargement has the same length as R_n^+. We now introduce concepts that we will need in order to complete the internalization R_n^+. Most of these concepts have their origins in Woodin's unpublished work on $\mathrm{HOD}^{L[x][g]}$. Various sources have expositions of similar concepts. For example, [20] has most of what we need excepts for the full hod limit. None of these concepts appeared for projective mice such as $\mathcal{M}_n$ and here we take a moment to develop these ideas. We start with suitability.

Definition 3.2 (n-suitable) *$\mathcal{P}$ is* n-suitable *if there is δ such that*

1. *$\mathcal{P} \vDash ZFC - Replacement$,*

2. *$\mathcal{P} \vDash$ "δ is the only Woodin cardinal",*

3. *$o(\mathcal{P}) = \sup_{i<\omega}(\delta^{+i})^{\mathcal{P}}$,*

4. *for every strong cutpoint cardinal η of $\mathcal{P}$, $S_n(\mathcal{P}|\eta) = \mathcal{P}|(\eta^+)^{\mathcal{P}}$.*

If $\mathcal{P}$ is n-suitable then we let $\delta^{\mathcal{P}}$ be the δ of Definition 3.2. Clearly $\mathcal{W}_n$ is a n-suitable premouse. Moreover, if $\mathcal{Q} \in I(\mathcal{W}_n,\Sigma_{\mathcal{W}_n})$ then $\mathcal{Q}$ is n-suitable because $i_{\mathcal{W}_n,\mathcal{Q}}$ can be lifted to

$i : \mathcal{M}_{n+1} \to \mathcal{M}_n(\mathcal{Q})$. Sometimes we will just say that $\mathcal{P}$ is n-suitable implying that it is n-suitable for some n.

To approximate the iteration strategy of $\mathcal{W}_n$ inside $\mathcal{M}_n(x)$, the notion of s-iterability is used. We now work towards introducing it. Given an iteration tree $\mathcal{T}$ on an n-suitable $\mathcal{P}$, we say $\mathcal{T}$ is *correctly guided* if for every limit $\alpha < lh(\mathcal{T})$, if b is the branch of $\mathcal{T} \restriction \alpha$ chosen by $\mathcal{T}$ and $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{M}_n(\mathcal{M}(\mathcal{T} \restriction \alpha))$. $\mathcal{T}$ is *short* if there is a well-founded branch b such that $\mathcal{T}^\frown\{\mathcal{M}_b^\mathcal{T}\}$ is correctly guided. $\mathcal{T}$ is *maximal* if $\mathcal{T}$ is not short.

Suppose $\mathcal{P}$ is n-suitable. We say $\langle \mathcal{T}_i, \mathcal{P}_i : i < m \rangle$ is a *finite correctly guided stack* on $\mathcal{P}$ if

1. $\mathcal{P}_0 = \mathcal{P}$,
2. $\mathcal{P}_i$ is n-suitable and $\mathcal{T}_i$ is a correctly guided tree on $\mathcal{P}_i$ below $\delta^{\mathcal{P}_i}$,
3. for every i such that $i + 1 < m$ either $\mathcal{T}_i$ has a last model and $i^\mathcal{T}$-exists or $\mathcal{T}$ is maximal, and
   (a) if $\mathcal{T}_i$ has a last model then $\mathcal{P}_{i+1}$ is the last model of $\mathcal{T}_i$,
   (b) if $\mathcal{T}_i$ is maximal then $\mathcal{P}_{i+1} = \mathcal{M}_n(\mathcal{M}(\mathcal{T}_i))|(\delta(\mathcal{T}_i)^{+\omega})^{\mathcal{M}_n(\mathcal{M}(\mathcal{T}_i))}$.

We say $\mathcal{Q}$ is the last model of $\langle \mathcal{T}_j, \mathcal{P}_j : i < k \rangle$ if one of the following holds:

1. $\mathcal{T}_{k-1}$ has a last model and $\mathcal{Q}$ is the last model of $\mathcal{T}_{k-1}$,
2. $\mathcal{T}_{k-1}$ is short and there is a cofinal well-founded branch b such that $\mathcal{Q}(b, \mathcal{T})$ exists and is iterable and $\mathcal{Q} = \mathcal{M}_b^\mathcal{T}$,
3. $\mathcal{T}_{k-1}$ is maximal and

$$\mathcal{Q} = \mathcal{M}_n(\mathcal{M}(\mathcal{T}_{k-1}))|(\delta(\mathcal{T}_{k-1})^{+\omega})^{\mathcal{M}_n(\mathcal{M}(\mathcal{T}_{k-1}))}.$$

We say $\mathcal{Q}$ is a *correct iterate* of $\mathcal{P}$ if there is a correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{Q}$.

Suppose $\mathcal{P}$ is n-suitable and $s = \langle \alpha_0, ..., \alpha_m \rangle$ is a finite sequence of ordinals. Then we let $T^{\mathcal{P}}_{s,k} \subseteq [((\delta^{\mathcal{P}})^{+k})^{\mathcal{P}}]^{<\omega} \times \omega$ be the set

$$(t, \phi) \in T^{\mathcal{P}}_{s,k} \leftrightarrow \phi \text{ is } \Sigma_1 \text{ and } \mathcal{M}_n(\mathcal{P}) \vDash \phi[t, s].$$

$$\gamma^{\mathcal{P}}_s = Hull^{\mathcal{P}}_1(\{T^{\mathcal{P}}_{s,i} : i \in \omega\}) \cap \delta^{\mathcal{P}}.$$

Notice that

$$\gamma^{\mathcal{P}}_s = Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_s \cup \{T^{\mathcal{P}}_{s,i} : i \in \omega\}).$$

Let

$$H^{\mathcal{P}}_s = Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_s \cup \{T^{\mathcal{P}}_{s,i} : i \in \omega\}).$$

If $s = s_m$, then we let $\gamma^{\mathcal{P}}_m = \gamma^{\mathcal{P}}_{s_m}$ and $H^{\mathcal{P}}_m = H^{\mathcal{P}}_{s_m}$. The following is not hard to show.

Lemma 3.3 $\sup_{n<\omega} \gamma^{\mathcal{P}}_n = \delta^{\mathcal{P}}$.

Proof. Suppose not. Let $\gamma = \sup_{n<\omega} \gamma^{\mathcal{P}}_n$. Let $X = Hull^{\mathcal{P}}_1(\gamma \cap \{T^{\mathcal{P}}_{s_m,i} : m, i \in \omega\})$. Let $\mathcal{N}$ be the collapse of X and let $\pi : \mathcal{N} \to \mathcal{P}$ be the inverse of the collapsing map. We have that for each m, i there is $S_{m,i} \in \mathcal{N}$ such that $\pi(S_{m,i}) = T^{\mathcal{P}}_{s_m,i}$. We have that $\gamma = \delta^{\mathcal{S}}$. Notice that for each i, $\cup_{m\in\omega} S_{m,i}$ is a complete and consistent theory and if $\mathcal{R}$ is its model then $\mathcal{R}$ is essentially the hull of ordinals $< (\gamma^{+i})^{\mathcal{N}}$ and ω many indiscernible. Moreover, because of π, we have that π can be extended to $\pi^* : \mathcal{R} \to \mathcal{M}_n(\mathcal{P})$. This implies that $\mathcal{R}$ is well-founded and therefore, it has to be $\mathcal{M}_n(\mathcal{N}|(\gamma^{+i})^{\mathcal{N}})$. This shows that $\mathcal{M}_n(\mathcal{N}|\gamma) \vDash$ "γ is Woodin" which implies that $\mathcal{P} \vDash$ "γ is Woodin". This is a contradiction as $\delta^{\mathcal{P}}$ is the least Woodin of $\mathcal{P}$. $\square$

Definition 3.4 (s-iterability) *Suppose $\mathcal{P}$ is n-suitable and $s = \langle \alpha_i : i < l \rangle$ is an increasing finite sequence of ordinals. $\mathcal{P}$ is* s-iterable *if whenever $\langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ is a finite correctly guided stack on $\mathcal{P}$ with last model $\mathcal{Q}$ then there is a sequence $\langle b_k : k < m \rangle$ such that*

1. for $k < m-1$,

$$b_k = \begin{cases} \emptyset : & \mathcal{T}_k \textit{ has a successor length} \\ cofinal\ well-founded & \\ branch\ such\ that\ \mathcal{M}^{\mathcal{T}}_{b_k} = \mathcal{P}_\kappa : & \mathcal{T}_k \textit{ is maximal} \end{cases}$$

2. if $\mathcal{T}_{m-1}$ has a successor length then $b_{m-1} = \emptyset$, if $\mathcal{T}_{m-1}$ is short then b_{m-1} is the unique cofinal well-founded branch such that $\mathcal{Q}(b_{m-1}, \mathcal{T}_{m-1})$ exists and is iterable, and if $\mathcal{T}_{m-1}$ is maximal then b_{m-1} is a cofinal well-founded branch,

3. letting

$$\pi_k = \begin{cases} i^{\mathcal{T}_k} : & \mathcal{T}_k \textit{ has a successor length} \\ i^{\mathcal{T}_k}_{b_k} : & \mathcal{T}_k \textit{ is maximal} \end{cases}$$

and $\pi = \pi_{m-1} \circ \pi_{m-2} \circ \cdots \pi_0$ then for every l

$$\pi(T^{\mathcal{P}}_{s,l}) = T^{\mathcal{Q}}_{s,l}.$$

Suppose $\mathcal{P}$ is n-suitable, $s = \langle \alpha_i : i < l \rangle$ is an increasing finite sequence of ordinals and $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ is a correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{Q}$. We say $\vec{b} = \langle b_k : k < m \rangle$ witness s-iterability for $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m \rangle$ if 2 above is satisfied. We then let

$$\pi_{\vec{\mathcal{T}},\vec{b},k} = \begin{cases} i^{\mathcal{T}_k} : & \mathcal{T}_k \textit{ has a successor length} \\ i^{\mathcal{T}_k}_{b_k} : & \mathcal{T}_k \textit{ is maximal} \end{cases}$$

and $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi_{\vec{\mathcal{T}},\vec{b},m-1} \circ \pi_{\vec{\mathcal{T}},\vec{b},m-2} \circ \cdots \pi_{\vec{\mathcal{T}},\vec{b},0}$.

Suppose now that $\vec{b}$ and $\vec{c}$ are two s-iterability branches for $\vec{\mathcal{T}}$. Then using Lemma 1.2, it is easy to see that $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{P}}_s = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H^{\mathcal{P}}_s$. Lets record this as a lemma.

Lemma 3.5 (Uniqueness of s-iterability embeddings) *Suppose $\mathcal{P}$ is n-suitable, s is a finite sequence of ordinals and $\vec{\mathcal{T}}$ is a finite correctly guided stack on $\mathcal{P}$. Suppose $\vec{b}$ and $\vec{c}$ are two s-iterability branches for $\vec{\mathcal{T}}$. Then*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H^{\mathcal{P}}_s = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H^{\mathcal{P}}_s.$$

Moreover, if $\vec{\mathcal{T}}$ consists of just one normal tree $\mathcal{T}$, $\mathcal{Q}$ is the last model of $\mathcal{T}$ and b and c witness s-iterability for $\mathcal{T}$ then if $\xi \in b$ is the least such that $crit(E_\xi^{\mathcal{T}}) > \gamma_s^{\mathcal{Q}}$ then $b \cap \xi = c \cap \xi$.

If $\mathcal{P}$ is s-iterable and $\mathcal{T}$ is a normal correctly guided tree then we let $b_s^{\mathcal{T}} = \cap\{b : b$ witnesses the s-iterability of $\mathcal{P}$ for $\mathcal{T}\}$. Here is how s-iterability is connected to iterability. Suppose $\mathcal{P}$ is n-suitable. We say $\mathcal{P}$ has a *correct ω_1-iteration strategy* if whenever $\mathcal{T}$ is a correctly guided tree of limit length and $b = \Sigma(\mathcal{T})$ then $\mathcal{T}^\frown\mathcal{M}_b^{\mathcal{T}}$ is correctly guided.

Lemma 3.6 *Suppose $\mathcal{P}$ is n-suitable and for every m, $\mathcal{P}$ is s_m-iterable. Then $\mathcal{P}$ has a correct iteration strategy.*

Proof. Let $\mathcal{T}$ be a correctly guided tree. If $\mathcal{T}$ is short then using s_m-iterability there must be a branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$-exists and is iterable. In this case we define $\Sigma(\mathcal{T}) = b$. Suppose now $\mathcal{T}$ is maximal with last model $\mathcal{Q}$. Then for each m, let $b_m = b_{s_m}^{\mathcal{T}}$. Notice that $b_m \subseteq b_{m+1}$. Also, because $\sup_{m\in\omega} \gamma_m^{\mathcal{Q}} = \delta^{\mathcal{Q}}$, we have that if $b = \cup_{m\in\omega} b_m$ then b is a cofinal branch. We claim that $\mathcal{M}_b^{\mathcal{T}} = \mathcal{Q}$. Let $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}}$. For all we know $\mathcal{R}$ may not be well-founded. But notice that if $R_m = i_b^{\mathcal{T}}(H_m^{\mathcal{P}})$ then there is $\pi_m : \mathcal{R}_m \rightarrow_{\Sigma_1} H_m^{\mathcal{Q}}$. This is because $i_b^{\mathcal{T}} \restriction \gamma_m^{\mathcal{P}} = \pi_{\mathcal{T},b_m} \restriction \gamma_m^{\mathcal{P}}$ where b_m is any cofinal well-founded branch witnessing s-iterability of $\mathcal{P}$ for $\mathcal{T}$. It then follows that if $\pi = \cup_{m\in\omega}\pi_m$ then $\pi : \cup_{m\in\omega}\mathcal{R}_m \rightarrow \mathcal{Q}$ and because $\cup_{m\in\omega}\mathcal{R}_m = \mathcal{R}$, we have that $\mathcal{R}$ is well-founded. Because for each i and m, $T_{s_m,i}^{\mathcal{Q}} \in ran(\pi)$, using the proof of Lemma 3.3, we get that $\mathcal{R}$ is n-suitable and hence, $\mathcal{R} = \mathcal{Q}$ and $\pi = id$. In this case, then, we define $\Sigma(\mathcal{T}) = b$. It follows from our construction that Σ is a correct iteration strategy. □

Notice that, if $\mathcal{P}$ is s-iterable, $\vec{\mathcal{T}}$ is a correctly guided finite stack on $\mathcal{P}$, and $\vec{b}$ witnesses s-iterability of $\mathcal{P}$ for $\vec{\mathcal{T}}$, then even though $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_s^{\mathcal{P}}$ is independent of $\vec{b}$ it may very well depend on $\vec{\mathcal{T}}$. This observation motivates the following definition.

Definition 3.7 (Strong s-iterability) *Suppose $\mathcal{P}$ is n-suitable and s is a finite sequence of ordinals. Then $\mathcal{P}$ is strongly s-iterable if $\mathcal{P}$ is s-iterable and whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_j, \mathcal{P}_j : j < u\rangle$ and*

$\vec{\mathcal{U}} = \langle \mathcal{U}_j, \mathcal{Q}_j : j < v\rangle$ *are two correctly guided finite stacks on* $\mathcal{P}$ *with common last model* $\mathcal{Q}$*,* $\vec{b}$ *witnesses* s*-iterability for* $\vec{\mathcal{T}}$ *and* $\vec{c}$ *witnesses* s*-iterability for* $\vec{\mathcal{U}}$ *then*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_s^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{c}} \restriction H_s^{\mathcal{P}}.$$

Are there s-iterable $\mathcal{P}$'s? Of course there must be, as otherwise we wouldn't define them, and here is an argument that shows it. Suppose not. Let $s = \langle \alpha_k : k < l\rangle$. Using the fact that there are no s-iterable $\mathcal{P}$'s, we can then get $\vec{B} = \langle B_k : k < \omega\rangle$ such that $B_k = \langle \mathcal{T}_j^k, \mathcal{P}_j^k, \mathcal{Q}_k : j < m_k\rangle$ and

1. $\mathcal{P}_0^0 = \mathcal{W}_n$ and $\mathcal{P}_0^{k+1} = \mathcal{Q}_k$,
2. for every k, $\langle \mathcal{T}_j^k, \mathcal{P}_j^k : j < m_k\rangle$ is a correctly guided finite stack on $\mathcal{P}_0^k$ with last model $\mathcal{Q}_k$,
3. whenever $\langle b_j^k : k < \omega \wedge j < k < m_k\rangle$ is such that
   (a) for $j < m_k - 1$,
   $$b_j^k = \begin{cases} \emptyset : & \mathcal{T}_j^k \textit{ has a successor length} \\ cofinal\ well-founded\ branch & \\ such\ that\ \mathcal{M}_{b_j^k}^{\mathcal{T}} = \mathcal{P}_j^k : & \mathcal{T}_j^k \textit{ is maximal} \end{cases}$$
   (b) if $\mathcal{T}_{m_k-1}^k$ has a successor length then $b_{m_k-1}^k = \emptyset$, if $\mathcal{T}_{m_k-1}^k$ is short then $b_{m_k-1}^k$ is the unique cofinal well-founded branch such that $\mathcal{Q}(b_{m_k-1}^k, \mathcal{T}_{m_k-1}^k)$ exists and is iterable, and if $\mathcal{T}_{m_k-1}^k$ is maximal then $b_{m_k-1}^k$ is a cofinal well-founded branch,[2]

   then letting $\vec{b}_k = \langle b_j^k : j < m_k\rangle$, for some m and every k,
   $$\pi_{\vec{\mathcal{T}}_k,\vec{b}_k}(T_{s,m}^{\mathcal{P}_0^k}) \neq T_{s,m}^{\mathcal{Q}_k}.$$

Let then $\langle b_j^k : k < \omega \wedge j < m_k\rangle$ be the sequence of branches given by $\Sigma_{\mathcal{W}_n}$. Then clearly for every k, $\pi_{\vec{\mathcal{T}}_k,\vec{b}_k}$'s extend to

[2]Notice that s-iterability cannot fail because we cannot find correct branches for short trees as long as we start with $\mathcal{P}_0 = \mathcal{W}_n$.

$$\pi_k : \mathcal{M}_n(\mathcal{P}_0^k) \to \mathcal{M}_n(\mathcal{Q}_k).$$

Let $\beta > \max(s)$ be a uniform indiscernible and let $t = s^\frown\langle\beta\rangle$. Suppose that for some k, $\pi_k(t) = t$. Notice that for every m, $T_{s,m}^{\mathcal{P}_0^k}$ can be defined by

$$(t, \phi) \in T_{s,m}^{\mathcal{P}_0^k} \leftrightarrow \phi \text{ is } \Sigma_1 \text{ and } \mathcal{M}_n(\mathcal{P}_0^k)|\beta \vDash \phi[t, s].$$

Hence, because we are assuming $\pi_k(t) = t$, we get that $\pi_{\vec{\mathcal{T}}_k, \vec{b}_k}(T_{s,m}^{\mathcal{P}_0^k}) = T_{s,m}^{\mathcal{Q}_k}$.

Therefore, we must have that $t <_{lex} \pi_k(t)$. Let $\mathcal{Q}$ be the direct limit of $\langle \mathcal{M}_n(\mathcal{Q}_k) : k < \omega\rangle$ under the maps $\sigma_{k,l} = \pi_l \circ \pi_{l-1} \circ \cdots \pi_k$ and let $\pi_k^* : \mathcal{M}_n(\mathcal{Q}_k) \to \mathcal{Q}$ be the embedding given by the direct limit construction. Now if $t_k = \pi_k^*(t)$, then $\langle t_k : k < \omega\rangle$ is a $\leq_{lex}$-decreasing sequence of finite sequences of ordinals. Because π_k^*'s are iteration embeddings according to $\Sigma_{\mathcal{W}_n}$, we get a contradiction. This completes the proof that for every s there is an s-iterable n-suitable $\mathcal{P}$. Lets make this a lemma.

Lemma 3.8 *For every $s \in Ord^{<\omega}$ and $n \in \omega$ there is an s-iterable n-suitable $\mathcal{P}$. Moreover, for any n-suitable $\mathcal{Q}$ there is a normal correctly guided tree $\mathcal{T}$ with last model $\mathcal{P}$ such that $\mathcal{P}$ is s-iterable.*

Proof. We have already shown that there is an s-iterable n-suitable $\mathcal{P}$. It is then the second clause that needs a proof. Fix a n-suitable $\mathcal{Q}$ and let $\mathcal{P}$ be s-iterable. Comparing $\mathcal{P}$ and $\mathcal{Q}$ produces our desired $\mathcal{T}$. $\square$

Is there a strongly s-iterable $\mathcal{P}$? The proof we have just given shows that there is. Indeed, using the proof given above we have $\mathcal{P}$ which is s-iterable and is a $\Sigma_{\mathcal{W}_n}$-iterate of $\mathcal{W}_n$. Moreover, if $\Lambda = \Sigma_{\mathcal{P}}$ then the branches witnessing s-iterability can be taken to be those given by Λ. It then easily follows from the Dodd-Jensen property of Λ that $\mathcal{P}$ is strongly s-iterable. Lets make this into a lemma as well.

Lemma 3.9 (Strongly s-iterability lemma) *For every s there is a strongly s-iterable $\mathcal{P}$. Moreover, for any n-suitable $\mathcal{Q}$ there is normal correctly guided stack $\mathcal{T}$ with last model $\mathcal{P}$ such that $\mathcal{P}$ is strongly s-iterable.*

Proof. We have already shown that there is a strongly s-iterable $\mathcal{P}$. It is then the second clause that needs a proof. Fix a n-suitable $\mathcal{Q}$ and let $\mathcal{P}$ be a strongly s-iterable. Comparing $\mathcal{P}$ and $\mathcal{Q}$ produces our desired $\mathcal{T}$. $\square$

If $\mathcal{P}$ is strongly s-iterable and $\vec{\mathcal{T}}$ is a correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{Q}$ then we let

$$\pi_{\mathcal{P},\mathcal{Q},s} : H_s^{\mathcal{P}} \to H_s^{\mathcal{Q}}$$

be the embedding given by any $\vec{b}$ which witnesses the s-iterability of $\vec{\mathcal{T}}$, i.e., fixing $\vec{b}$ which witnesses s-iterability for $\vec{\mathcal{T}}$,

$$\pi_{\mathcal{P},\mathcal{Q},s} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_s^{\mathcal{P}}.$$

Clearly, $\pi_{\mathcal{P},\mathcal{Q},s}$ is independent of $\vec{\mathcal{T}}$ and $\vec{b}$.

Notice that $\mathcal{W}_n$ is strongly s_m-iterable for every m. Moreover, if $\vec{\mathcal{T}}$ is any correctly guided stack on $\mathcal{W}_n$ with last model $\mathcal{Q}$ then $\pi_{\mathcal{W}_n,\mathcal{Q},s_m}$ agrees with the correct iteration embedding, i.e., if $i : \mathcal{W}_n \to \mathcal{Q}$ is the iteration embedding according to the canonical iteration strategy of $\mathcal{W}_n$ then

$$\pi_{\mathcal{W}_n,\mathcal{Q},s_m} = i \restriction H_m^{\mathcal{W}_n}.$$

Moreover, since $\cup_{m<\omega} H_m^{\mathcal{W}_n} = \mathcal{W}_n$, we get that

$$\cup_{m<\omega} \pi_{\mathcal{W}_n,\mathcal{Q},s_m} = i.$$

This is how we will approximate Σ inside $\mathcal{M}_n(x)$.

Next let

$$\mathcal{F}_n^+ = \{\mathcal{P} : \mathcal{P} \in I(\mathcal{W}_n, \Sigma_{\mathcal{W}_n}) \text{ as witnessed by some finite stack }\}.$$

We let $\leq_n^+$ be a prewellording of $\mathcal{F}_n^+$ given by $\mathcal{P} \leq_n^+ \mathcal{Q}$ iff $\mathcal{Q} \in I(\mathcal{P}, \Sigma_{\mathcal{P}})$ as witnessed by a finite stack. We then let $\mathcal{M}_{\infty,n}^+$ be the direct limit of $(\mathcal{F}_n^+, \leq_n^+)$ under the iteration maps $i_{\mathcal{P},\mathcal{Q}}$. Notice that $|R_n^+| = \delta^{\mathcal{M}_\infty^+}$. We let $\delta_{\infty,n}^+ = \delta^{\mathcal{M}_{\infty,n}^+}$.

We also let

$$\mathcal{I}_n = \{(\mathcal{P}, s) : \mathcal{P} \text{ is } n\text{-suitable, } s \subseteq Ord^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } s\text{-iterable }\}.$$

and

$$\mathcal{F}_n = \{H_s^{\mathcal{P}} : (\mathcal{P}, s) \in \mathcal{I}_n\}.$$

We define $\leq_n$ on $\mathcal{I}_n$ by: $(\mathcal{P}, s) \leq_n (\mathcal{Q}, t)$ iff $\mathcal{Q}$ is a correctly guided iterate of $\mathcal{P}$ and $s \subseteq t$. Is $\leq_n$ directed? The answer is of course yes and to see that fix $(\mathcal{P}, s), (\mathcal{Q}, t) \in \mathcal{I}_n$. Then we have $\mathcal{R}$ which is strongly $s \cup t$-iterable. Let $\mathcal{S}$ be the result of comparing $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$. Then $(\mathcal{S}, s \cup t) \in \mathcal{I}_n$ and

$$(\mathcal{P}, s) \leq_n (\mathcal{S}, s \cup t) \text{ and } (\mathcal{Q}, t) \leq_n (\mathcal{S}, s \cup t).$$

We can then form the direct limit of $(\mathcal{F}_n, \leq_n)$ under the maps $\pi_{\mathcal{P},\mathcal{Q},s}$. We let $\mathcal{M}_{\infty,n}$ be this direct limit. It is clear that $\mathcal{M}_{\infty,n}^+$ is well-founded. However, it is not at all clear that $\mathcal{M}_{\infty,n}$ is well-founded. We show that not only $\mathcal{M}_{\infty,n}$ is well-founded but that it is also the same as $\mathcal{M}_{\infty,n}^+$.

Before we continue, we fix some notation. If $\mathcal{P} \in I(\mathcal{W}_n, \Sigma_{\mathcal{W}_n})$, then we let $i_{\mathcal{P},\infty} : \mathcal{P} \to \mathcal{M}_{\infty,n}^+$ be the iteration map. For $(\mathcal{P}, s) \in \mathcal{I}_n$, we let $\pi_{\mathcal{P},\infty,s}$ be the direct limit embedding acting on $H_s^{\mathcal{P}}$.

Lemma 3.10 $\mathcal{M}_{\infty,n} = \mathcal{M}_{\infty,n}^+$.

Proof. To show the equality, we define a map $\pi : \mathcal{M}_{\infty,n} \to_{\Sigma_1} \mathcal{M}^+_{\infty,n}$ and show that π is the identity. Let $x \in \mathcal{M}_{\infty,n}$. Let $(\mathcal{P}, s_m) \in \mathcal{I}_n$ be such that for some $y \in H^{\mathcal{P}}_m$, $\pi_{\mathcal{P},\infty,s_m}(y) = x$ and $\mathcal{P}$ is a normal correct iterate of $\mathcal{W}_n$. Then we let

$$\pi(x) = i_{\mathcal{P},\infty}(x).$$

First we need to see that π is independent of the choice of $\mathcal{P}$. Let then $(\mathcal{P}, s_p) \in \mathcal{I}_n$ and $(\mathcal{R}, s_q) \in \mathcal{I}_n$ be such that there are $y \in H^{\mathcal{P}}_p$ and $z \in H^{\mathcal{R}}_q$ such that $\pi_{\mathcal{P},\infty,s_p}(y) = \pi_{\mathcal{R},\infty,s_q}(z) = x$ and both $\mathcal{P}$ and $\mathcal{R}$ are normal iterates of $\mathcal{W}_n$. Let $\mathcal{Q}$ be the outcome of comparing $\mathcal{P}$ and $\mathcal{R}$. Notice that we must have that

$$\pi_{\mathcal{P},\mathcal{Q},s_p}(y) = \pi_{\mathcal{R},\mathcal{Q},s_q}(z).$$

It then follows that

$$i_{\mathcal{Q},\infty}(\pi_{\mathcal{P},\mathcal{Q},s_p}(y)) = i_{\mathcal{Q},\infty}(\pi_{\mathcal{R},\mathcal{Q},s_q}(z)).$$

and hence, π is independent of the choice of $\mathcal{P}$. We then get that π is a Σ_1-elementary and this much is enough to conclude that $\mathcal{M}_\infty$ is well-founded. But we can in fact show that $\pi = id$. For this, fix $x \in \mathcal{M}^+_{\infty,n}|\delta^{\mathcal{M}^+_{\infty,n}}$. Let Q be such that there is $y \in \mathcal{Q}$ such that $x = i_{\mathcal{Q},\infty}(y)$. Let s_m be such that $y \in H^{\mathcal{Q}}_m$. Then if $z = \pi_{\mathcal{Q},\infty,s_m}(y)$ then $\pi(z) = x$. This shows that $\pi \restriction \delta^{\mathcal{M}_{\infty,n}} + 1 = id$.

Now fix $\mathcal{P}$ and let $T^{\infty}_{m,l} = i_{\mathcal{P},\infty}(T^{\mathcal{P}}_{m,l})$. We clearly have that $T^{\infty}_{m,l} \in ran(\pi)$. Let then $S_{m,l} \in \mathcal{M}_{\infty,n}$ be such that $\pi(S_{m,l}) = \mathcal{T}^{\infty}_{m,l}$. Now, let $\mathcal{N} = \mathcal{M}_{\infty,n}$. Then for each l, $\cup_{m<\omega} S_{m,l}$ is a prescription for constructing a model with n Woodin cardinals over $\mathcal{N}|(\delta^{+l})^{\mathcal{N}}$. Moreover, if K is this model then K is the Σ_1-hull of ordinals $< (\delta^{+l})^{\mathcal{N}}$ and ω-indiscernibles. Because of π, it follows that $K = \mathcal{M}^{\#}_n(\mathcal{N}|(\delta^{+l})^{\mathcal{N}})$. This then inductively implies that for every l, $\mathcal{N}|(\delta^{+l})^{\mathcal{N}} = \mathcal{S}|(\delta^{+l})^{\mathcal{S}}$ where $\mathcal{S} = \mathcal{M}^+_{\infty,n}$. Hence, π has to be the identity. $\square$

Before moving on, notice that everything we have done in this section relativizes to arbitrary real x. For any real x, we can define $\mathcal{J}^+_{x,n}$, $\mathcal{I}_{x,n}$, $\mathcal{F}^+_{x,n}$, $\mathcal{F}_{x,n}$, $\leq^+_{x,n}$, $\leq_{x,n}$, $\mathcal{M}^+_{\infty,x,n}$, and $\mathcal{M}_{\infty,x,n}$.

We will then again have that $\mathcal{M}^+_{\infty,x,n} = \mathcal{M}_{\infty,x,n}$ and $\delta^{\mathcal{M}_{\infty,x,n}} < \delta^1_{n+3}$. We let $\delta_{\infty,x,n} = \delta^{\mathcal{M}_{\infty,x,n}}$ and also for $s \in Ord^{<\omega}$, we let

$$\gamma_{\infty,s,x,n} = \sup(\pi_{\mathcal{P},\infty,s}\text{"}\gamma_s^{\mathcal{P}})$$

where $(\mathcal{P}, s) \in \mathcal{I}_{x,n}$. Clearly $\gamma_{\infty,s,x,n}$ is independent of the choice of $\mathcal{P}$. We also let

$$\mathcal{J}_{n,s,z} = \{(\mathcal{P}, \alpha) : (\mathcal{P}, s) \in \mathcal{I}_{n,z} \wedge \alpha < \gamma_s^{\mathcal{P}}\}.$$

We let $R_{n,s,z}$ be a prewellordering of $\mathcal{J}_{n,s,z}$ given by $(\mathcal{P}, \alpha)R_{n,s,z}(\mathcal{Q}, \beta)$ if $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$ and $\pi_{\mathcal{P},\mathcal{Q},s}(\alpha) \leq \beta$. We also let $\mathcal{W}_z = \mathcal{M}_{n+1}(z)|(\delta^{+\omega})^{\mathcal{M}_{n+1}(z)}$ where δ is the least Woodin of $\mathcal{M}_{n+1}(z)$. We let Σ_z be the strategy of $\mathcal{M}_{n+1}(z)$ restricted to stacks on $\mathcal{W}_z$. We now move to internalizing the direct limit construction to $\mathcal{M}_n(x)$ where x is any real coding $\mathcal{W}_n$.

3.2 Internalizing the directed system

Fix a real x that codes $\mathcal{W}_n$ and let δ be the least Woodin of $\mathcal{M}_n(x)$. We will work with this x until the end of this subsection. Notice that because $\mathcal{M}_n(x)|\delta$ is closed under S_n operator, if $\mathcal{T} \in \mathcal{M}_n(x)|\delta$ is a short tree on $\mathcal{M}$ then if b is such that $\mathcal{T}^\frown\mathcal{M}_b^{\mathcal{T}}$ is correctly guided then in fact $b \in \mathcal{M}_n(x)|\delta$. Thus, $\mathcal{M}_n(x)$ knows Σ on short trees of length $< \delta$. How about maximal trees? By a result of Neeman from [13], if $\mathcal{M}_n(x)$ knows the correct iteration strategy of $\mathcal{W}_n$ for trees of length ω then there is a normal iterate $\mathcal{Q} \in HC^{\mathcal{M}_n(x)}$ of $\mathcal{W}_n$ via a tree of length ω such that there is some $\mathcal{Q}$-generic $g \subseteq Coll(\omega, \delta^{\mathcal{Q}})$ such that $g \in \mathcal{M}_n(x)$ and $x \in \mathcal{Q}[g]$. But this is a contradiction as $\mathcal{Q}$ is essentially a real in $\mathcal{M}_n(x)$ while $\mathbb{R}^{\mathcal{M}_n(x)} = S_n(x) \subseteq \mathcal{Q}(x)$. This discussion shows that $\mathcal{M}_n(x)$ doesn't know Σ on all maximal trees. Nevertheless, in the case of $n = 0$, Woodin used s-iterability to track the iteration strategy of $\mathcal{W}_n$ inside $\mathcal{M}_n(x)$. We do that here for arbitrary n. For the purpose of keeping notation simple, while working in this subsection we let $\mathcal{M} = \mathcal{M}_n(x)$ and δ be the least Woodin of $\mathcal{M}$.

Notice that the notions such as suitable, short tree, maximal tree, correctly guided finite stack and etc are all definable over $\mathcal{M}$. That is, all these notions refer to S_n operator and because $\mathcal{M}|\delta$ is closed under S_n operator, we get that, for instance, for $\mathcal{Q} \in \mathcal{M}|\delta$, $\mathcal{Q}$ is suitable iff $\mathcal{M} \vDash$ "$\mathcal{Q}$ is suitable". Notice, however, that s-iterability presents a difficulty as it is not immediately clear how to say "a suitable $\mathcal{P}$ is s-iterable" inside $\mathcal{M}$. When $n = 0$ and $s = \langle a_j : k < l\rangle$, one can just make do with Definition 3.4. This is because the "guiding sets", $T^{\mathcal{P}}_{s,i}$'s, can be identified inside $L[x]$. In general, this doesn't seem to work because we need to correctly identify $T^{\mathcal{P}}_{s,i}$'s. If $\beta > \max(s)$ is a uniform indiscernible then to identify $T^{\mathcal{P}}_{s,i}$ inside $\mathcal{M}$, it is enough to identify $\mathcal{M}_n(\mathcal{P})|\beta$ inside $\mathcal{M}_n(x)$. This is because

$$(t, \phi) \in T^{\mathcal{P}}_{s,m} \leftrightarrow \phi \text{ is } \Sigma_1 \text{ and } \mathcal{M}_n(\mathcal{P})|\beta \vDash \phi[t, s].$$

We then solve the problem by dropping to a smaller set of "good" $\mathcal{P}$'s. This new set of good $\mathcal{P}$'s will nevertheless be dense in the old one. To start, we fix $\kappa < \delta$ which is an inaccessible strong cutpoint cardinal of $\mathcal{M}$ such that $\mathcal{M} \vDash$ "κ is a limit of strong cutpoint cardinals".

We let

$$\mathcal{G}_\kappa = \{\mathcal{P} \in \mathcal{M}|\kappa : \mathcal{P} \text{ is suitable and } \mathcal{M} \vDash \text{" for some strong cutpoint } \eta,\ \delta^{\mathcal{P}} = \eta^+ \text{ and } \mathcal{M}|\eta \text{ is generic over } \mathcal{P} \text{ for the extender algebra at } \delta^{\mathcal{P}}\text{"}\}.$$

If $\mathcal{P} \in \mathcal{G}_\kappa$ then we let $\eta_{\mathcal{P}}$ be the ordinal witnessing that $\mathcal{P} \in \mathcal{G}_\kappa$.

Lemma 3.11 *Suppose $\mathcal{P} \in \mathcal{G}_\kappa$. Then $S(\mathcal{P})^{\mathcal{M}} = \mathcal{M}_n(\mathcal{P})$ and $\delta^{\mathcal{P}} = (\eta^+_{\mathcal{P}})^{\mathcal{M}}$.*

Proof. Let $\eta = \eta_{\mathcal{P}}$. Notice that

$$\mathcal{M}|(\eta^{+\omega})^{\mathcal{M}} =_{via\ \mathcal{S}-constructions} \mathcal{P}[\mathcal{M}|\eta].$$

Hence, $\delta^{\mathcal{P}} = (\eta^+)^{\mathcal{M}}$. But then $\mathcal{S}^{\mathcal{M}}(\mathcal{P})[\mathcal{M}|\eta] = \mathcal{M}$. This means that $\mathcal{S}^{\mathcal{M}}(\mathcal{P})$ is the hull of ordinals $< \delta^{\mathcal{P}}$ and the class of indiscernibles. But this is exactly what $\mathcal{M}_n(\mathcal{P})$ is: it is the unique proper class mouse over $\mathcal{P}$ with n Woodin cardinals which is the hull of a club class of indiscernibles. $\square$

Let $\mathcal{P} \in \mathcal{G}_\kappa$ and $s = \langle \alpha_j : j < l\rangle$. We then write $\mathcal{M} \vDash$ "$\mathcal{P}$ is s-iterable below κ" if whenever $\vec{\mathcal{T}} = \langle \mathcal{T}_j, \mathcal{P}_j : j < k\rangle \in \mathcal{M}|\kappa$ is a correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{Q}$ such that

$\mathcal{Q} \in \mathcal{G}_\kappa$ and whenever $g \subseteq Coll(\omega, |\mathcal{P} \cup \mathcal{Q}|)$ is $\mathcal{M}$-generic, there is $\vec{b} = \langle b_j : j < k \rangle \in \mathcal{M}[G]$ such that for every m,

$$\pi_{\vec{\mathcal{T}},\vec{b}}(T^{\mathcal{P}}_{s,m}) = T^{\mathcal{Q}}_{s,m}$$

where $T^{\mathcal{P}}_{s,m} \subseteq [((\delta^{\mathcal{P}})^{+m})^{\mathcal{P}}]^{<\omega} \times \omega$

$$(t, \phi) \in T^{\mathcal{P}}_{s,m} \leftrightarrow \phi \text{ is } \Sigma_1 \text{ and } \mathcal{S}^{\mathcal{M}}(\mathcal{P}) \vDash \phi[t, s].$$

and $T^{\mathcal{Q}}_{s,m} \subseteq [((\delta^{\mathcal{Q}})^{+m})^{\mathcal{Q}}]^{<\omega} \times \omega$

$$(t, \phi) \in T^{\mathcal{Q}}_{s,m} \leftrightarrow \phi \text{ is } \Sigma_1 \text{ and } \mathcal{S}^{\mathcal{M}}(\mathcal{Q}) \vDash \phi[t, s].$$

Notice that in the light of Lemma 3.11, the definition just given indeed coincides with Definition 3.4 for as long as we stay inside $\mathcal{G}_\kappa$. $\mathcal{M} \vDash$ "$\mathcal{P}$ is srtrongly s-iterable below κ" is defined similarly. Also, notice that even though the requirement that the sequence $\vec{b}$ exists in the generic extension cannot be dropped, the embedding $\pi_{\mathcal{P},\mathcal{Q},s} \in \mathcal{M}$ as it is unique and hence, it is in all generic extensions.

We then let

$$\mathcal{I}_\kappa = \{(\mathcal{P}, s) : \mathcal{P} \in \mathcal{G}_\kappa \wedge \mathcal{M} \vDash \text{“}\mathcal{P} \text{ is strongly } s\text{-iterable below } \kappa\text{”}\}.$$

and

$$\mathcal{F}_\kappa = \{H^{\mathcal{P}}_s : (\mathcal{P}, s) \in \mathcal{I}_\kappa\}.$$

Notice that the proof of Lemma 3.8 can be used to show that for every s there is $\mathcal{P}$ such that $(\mathcal{P}, s) \in \mathcal{I}_\kappa$. Lets make this into a lemma.

Lemma 3.12 *Suppose $\mathcal{P} \in \mathcal{G}_\kappa$ and s is a finite sequence of ordinals. Then there is a normal correct iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q}, s) \in \mathcal{I}_\kappa$*

Clearly, $\mathcal{F}_\kappa \in \mathcal{M}$. We then define $\leq_\kappa$ on $\mathcal{I}_\kappa$ by: $(\mathcal{P}, s) \leq_\kappa (\mathcal{Q}, t)$ iff $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$ and $s \subseteq t$. It is not hard to see that $\leq_\kappa$ is directed.

Lemma 3.13 $\leq_\kappa$ *is directed*

Proof. Fix $(\mathcal{P}, s), (\mathcal{Q}, t) \in \mathcal{I}_\kappa$. Then there is $(\mathcal{R}, s \cup t) \in \mathcal{I}_\kappa$. Simultaneously compare $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ to get $\mathcal{S}^* \in \mathcal{M}|\kappa$. Let $\eta < \kappa$ be a strong cutpoint of $\mathcal{M}$ such that $\mathcal{S}^* \in \mathcal{M}|\eta$. Then iterate $\mathcal{S}^*$ to make $\mathcal{M}|\eta$-generic. This iteration produces $\mathcal{S} \in \mathcal{M}|\kappa$ such that $\delta^{\mathcal{S}} = (\eta^+)^{\mathcal{M}}$. It then follows that $(\mathcal{S}, s \cup t) \in \mathcal{I}_\kappa$.

□

Let then $\mathcal{M}_{\infty,\kappa}$ be the direct limit of $(\mathcal{F}_\kappa, \leq_\kappa)$ under the embeddings $\pi_{\mathcal{P},\mathcal{Q},s}$. We first claim that $\mathcal{M}_{\infty,\kappa}$ is well-founded.

Lemma 3.14 $\mathcal{M}_{\infty,\kappa}$ *is well-founded.*

Proof. The proof is similar to the proof of Lemma 3.10. Let $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle \in \mathcal{M}$ be an enumeration of $\mathcal{G}_\kappa$. We construct a sequence $\langle \mathcal{Q}^0_i, \mathcal{T}^0_i, \mathcal{Q}^1_i, \mathcal{T}^1_i : i < \omega \rangle$ such that

1. $\mathcal{Q}^0_0 = \mathcal{M}$ and $\mathcal{T}^l_i$ is a normal correctly guided tree on $\mathcal{Q}^l_i$ for $l = 0, 1$,
2. $\mathcal{Q}^1_i$ is the last model of $\mathcal{T}^0_i$ and $\mathcal{Q}^0_{i+1}$ is the last model of $\mathcal{T}^1_i$,
3. for every $\alpha < \kappa$, there is $i < \omega$ such that $\mathcal{Q}^0_i$ is a correct iterate of $\mathcal{P}_\alpha$,
4. $\mathcal{Q}^0_i \in \mathcal{G}_\kappa$.

To construct such a sequence, we first fix $\langle \eta_i : i < \omega \rangle$ such that $\sup_{i<\omega} \eta_i = \kappa$. Suppose we have constructed $\langle \mathcal{Q}^0_i, \mathcal{T}^0_i, \mathcal{Q}^1_i, \mathcal{T}^1_i : i \leq k \rangle$. Let $\eta \in [\eta_i, \kappa)$ be a strong cutpoint of $\mathcal{M}$ such that $\langle \mathcal{Q}^0_i, \mathcal{T}^0_i, \mathcal{Q}^1_i, \mathcal{T}^1_i : i \leq k \rangle \in \mathcal{M}|\eta$. Thus, we actually have $\mathcal{Q}^0_{k+1}$. Then let $\mathcal{Q}^1_{k+1}$ be the result of simultaneously comparing all suitable $\mathcal{P}$'s such that $\mathcal{P} \in \mathcal{M}|\eta \cap \mathcal{G}_\kappa$. Notice that $\mathcal{S}$ is a normal correct iterate of every $\mathcal{P} \in \mathcal{M}|\eta \cap \mathcal{G}_\kappa$ including $\mathcal{Q}^0_{k+1}$. Let then $\mathcal{T}^0_{k+1}$ be the normal correctly guided tree on $\mathcal{Q}^0_{k+1}$ with last model $\mathcal{Q}^1_{k+1}$. The problem is that $\mathcal{Q}^1_{k+1}$ may not be in $\mathcal{G}_\kappa$. Let then $\nu \in (\eta, \kappa)$ be a strong cutpoint of $\mathcal{M}$ such that $\mathcal{Q}^1_{k+1} \in \mathcal{M}|\nu$. Iterate $\mathcal{Q}^1_{k+1}$ to make

$\mathcal{M}|\nu$ generic for the extender algebra. Let then $\mathcal{T}^1_{k+1}$ be the resulting tree on $\mathcal{Q}^1_{k+1}$. Clearly $\mathcal{Q}^1_{k+1} \in \mathcal{G}_\kappa$ and the resulting sequence $\langle \mathcal{Q}^0_i, \mathcal{T}^0_i, \mathcal{Q}^1_i, \mathcal{T}^1_i : i < \omega\rangle$ is as desired.

Let then $\sigma_{j,k} = i_{\mathcal{Q}^0_j, \mathcal{Q}^0_k}$ and let $\mathcal{Q}$ be the direct limit of $\langle \mathcal{Q}^0_j, \sigma_{j,k} : j < k < \omega\rangle$. Then the proof of Lemma 3.10 can be used to show that in fact $\mathcal{Q} = \mathcal{M}_{\infty,\kappa}$. $\square$

Next we show that $\delta^{\mathcal{M}_{\infty,\kappa}} = (\kappa^+)^{\mathcal{M}}$. For the purpose of keeping the notation nice, in this subsection we abuse the notation used in the previous subsection and whenever $(\mathcal{P}, s) \in \mathcal{I}_\kappa$, we write $\pi_{\mathcal{P},\infty,s}$ for the direct limit embedding. Thus, $\pi_{\mathcal{P},\infty,s}$ is an embedding that acts on $H^{\mathcal{P}}_s$ and embeds it into the corresponding structure in $\mathcal{M}_{\infty,\kappa}$. For each $s \in Ord^{<\omega}$, let $\gamma_{\infty,s} = \sup(\pi_{\mathcal{P},\infty,s}"\gamma^{\mathcal{P}}_s)$ where $(\mathcal{P}, s) \in \mathcal{I}_\kappa$. Clearly, $\gamma_{\infty,s}$ is independent of the choice of $\mathcal{P}$. Notice that $\delta^{\mathcal{M}_{\infty,\kappa}} = \sup_{s\in Ord^{<\omega}} \gamma_{\infty,s} = \sup_{m<\omega} \gamma_{\infty,s_m}$. Our proof uses an idea that originated in Hjorth's work.

Lemma 3.15 $\delta^{\mathcal{M}_{\infty,\kappa}} = (\kappa^+)^{\mathcal{M}}$.

Proof. First notice that for every $\alpha < \delta^{\mathcal{M}_{\infty,\kappa}}$ there is a surjective map $f : \kappa \to \alpha$. To see this, first fix s such that $\alpha < \gamma_{\infty,s}$ and let $\langle(\mathcal{P}_\beta, \xi_\beta) : \beta < \kappa\rangle$ be an enumeration of the set $\{(\mathcal{P}, \xi) : (\mathcal{P}, s) \in \mathcal{I}_\kappa \wedge \xi < \gamma^{\mathcal{P}}_s\}$. Then let $f(\beta) = \pi_{\mathcal{P}_\beta,\infty,s}(\xi_\beta)$. Clearly $\alpha \subseteq ran(f)$ and f is onto. This observation shows that $\delta^{\mathcal{M}_{\infty,\kappa}} \leq (\kappa^+)^{\mathcal{M}}$.

We therefore need to show that $\delta^{\mathcal{M}_{\infty,\kappa}} \not< (\kappa^+)^{\mathcal{M}}$. Suppose then $\delta^{\mathcal{M}_{\infty,\kappa}} < (\kappa^+)^{\mathcal{M}}$. We can then let $\leq^*$ be a well-ordering of κ of length $\delta^{\mathcal{M}_{\infty,\kappa}}$. Without loss of generality we assume κ is the least such that $\delta^{\mathcal{M}_{\infty,\kappa}} < (\kappa^+)^{\mathcal{M}}$. It then follows that there is a formula ϕ, a sequence $t \in [\kappa]^{<\omega}$ and an integer m such that

$$\alpha \leq^* \beta \leftrightarrow \mathcal{M} \vDash \phi[t, s_m, \alpha, \beta].$$

Now, fix $(\mathcal{P}, s_m) \in \mathcal{I}_\kappa$ such that $t \subseteq \lambda$ where λ is the least measurable cardinal of $\mathcal{P}$. Let $\mathcal{N} = \mathcal{M}_n(\mathcal{P}) = \mathcal{S}^{\mathcal{M}}(\mathcal{P})$. We have that $\mathcal{M}|\eta_{\mathcal{P}}$ is generic over $\mathcal{P}$ for the extender algebra of $\delta^{\mathcal{P}}$. This means that $\mathcal{N}[\mathcal{M}|\eta_{\mathcal{P}}]$ can be reorganized as an x-mouse and in fact, $\mathcal{N}[\mathcal{M}|\eta_{\mathcal{P}}] = \mathcal{M}$.

This then means that there are conditions p which force that $\mathcal{N}[G]$ can be reorganized via $\mathcal{S}$-constructions as a mouse over a real and such that in $\mathcal{N}[G]$, $\delta^{\mathcal{M}_{\infty,\kappa}} < (\kappa^+)^{\mathcal{N}[G]}$. Moreover, among those conditions there are also conditions that force that ϕ defines a well-ordering of κ as above over $\mathcal{N}[G]$. Let then D be the set of conditions p of the extender algebra at $\delta^{\mathcal{P}}$ such that p forces that

1. $\mathcal{N}[G]$ can be reorganized as a premouse over a real,
2. $\mathcal{N}[G] \vDash$ "$\delta^{\mathcal{M}_{\infty,\kappa}} < (\kappa^+)^{\mathcal{N}[G]}$",
3. ϕ defines a well-ordering of κ of length $(\delta^{\mathcal{M}_{\infty,\kappa}})^{\mathcal{N}[G]}$.

Consider now the set B of pairs (p, α) such that $p \in D$ and $\alpha < \lambda$. Notice that whenever $(p, \alpha) \in B$ and G is $\mathcal{P}$-generic such that $p \in G$, α has a rank in the well-ordering given by ϕ over $\mathcal{N}[G]$. We can then for each $\alpha < \lambda$ choose a maximal antichain of conditions p such that $(p, \alpha) \in B$ and for some ξ, p forces that α has rank ξ in the well-ordering given by ϕ. Let $\mathcal{A}_\alpha$ be such an antichain and let $\mathcal{A} = \{(p, \alpha) : p \in \mathcal{A}_\alpha\}$. Notice that without loss of generality we can assume that $\mathcal{A} \in H^{\mathcal{P}}_{m+1}$. We then let $\mathcal{A}^{\mathcal{P}} = \mathcal{A}$.

For $(p, \alpha) \in \mathcal{A}$ let $\xi_{p,\alpha}$ be the rank of α as forced by p. Define $\leq^{\mathcal{P}}$ on $\mathcal{A}$ by $(p, \alpha) \leq^{\mathcal{P}} (q, \beta)$ iff $\xi_{p,\alpha} \leq \xi_{q,\beta}$. Notice that $\left|\leq^{\mathcal{P}}\right|$ is independent of the exact choice of $\mathcal{A}_\alpha$'s and $\left|\leq^{\mathcal{P}}\right| < \gamma^{\mathcal{P}}_{m+1}$.

Define now a relation R on the set $\{(P, \xi) : \mathcal{P} \in \mathcal{G}_\kappa \wedge \xi < \gamma^{\mathcal{P}}_{m+1}\}$ given by

$$R((\mathcal{P}, \xi), (\mathcal{Q}, \nu)) \text{ if whenever } \mathcal{R} \text{ is such that } (\mathcal{P}, s_{m+1}) \leq_\kappa (\mathcal{R}, s_{m+1}) \text{ and}$$
$$(\mathcal{Q}, s_{m+1}) \leq_\kappa (\mathcal{R}, s_{m+1}) \text{ then } i_{\mathcal{P},\mathcal{R},s_{m+1}}(\xi) \leq i_{\mathcal{Q},\mathcal{R},s_{m+1}}(\nu).$$

Clearly R is well-founded and $|R| = \gamma_{\infty,s_{m+1}}$.

Fix now an $\alpha < \kappa$. We say that $(\mathcal{P}, p)$ is a stable code for α if

1. $(\mathcal{P}, s_{m+1}) \in \mathcal{I}_\kappa$,
2. $(p, \alpha) \in \mathcal{A}^{\mathcal{P}}$, $\xi^{\mathcal{P}}_{p,\alpha} = |\alpha|_{\leq^*}$, and whenever $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$ such that $\mathcal{Q} \in \mathcal{G}_\kappa$,

$$\pi_{\mathcal{P},\mathcal{Q},s_{m+1}}(|\alpha|_{\leq^*}) = |\alpha|_{\leq^*},$$

3. if $G \subseteq \mathbb{B}^{\mathcal{P}}$ is a generic object such that $x_G = \mathcal{M}|\eta_{\mathcal{P}}$ then $p \in G$.

Notice that if $(\mathcal{P}, p)$ is a stable code for α then $\xi^{\mathcal{P}}_{p,\alpha} = |\alpha|_{\leq^*}$. This is because of condition 3, i.e., if $G \subseteq \mathbb{B}^{\mathcal{P}}$ is the generic so that $x_G = \mathcal{M}|\eta_{\mathcal{P}}$ then $\mathcal{S}(x_G)^{\mathcal{M}_n(\mathcal{P})[G]} = \mathcal{M}$, $p \in G$ and $(|\alpha|_{\leq^*})^{\mathcal{S}(x_G)^{\mathcal{M}_n(\mathcal{P})[G]}} = |\alpha|_{\leq^*}$.

We claim that for every α there is a stable code for α. Let $\xi = |\alpha|_{\leq^*}$. To see this, suppose not. Let then $\mathcal{P}$ be such that $(\mathcal{P}, s_{m+1}) \in \mathcal{I}_\kappa$, $\alpha < \lambda^{\mathcal{P}}$ and $\mathcal{P}$ is a correct iterate of $\mathcal{W}_n$. Then we can find $p \in \mathcal{P}$ such that $(p, \alpha) \in \mathcal{A}^{\mathcal{P}}$ and $(\mathcal{P}, p)$ satisfies 1 and 3 above. If it satisfies 2 then we are done, and therefore, we assume that $(\mathcal{P}, p)$ doesn't satisfy 2. Let then $(\mathcal{P}_0, p_0) = (\mathcal{P}, p)$ and let $\mathcal{P}_1$ witness the failure of 2. Thus, we have that $\xi = \xi^{\mathcal{P}}_{p,\alpha}$ and $i_{\mathcal{P}_0,\mathcal{P}_1,s_{m+1}}(\xi) > \xi$. But notice that there is $p_1 \in \mathcal{P}_1$ such that $(p_1, \alpha) \in \mathcal{A}^{\mathcal{P}_1}$ and $\xi^{\mathcal{P}_1}_{p_1,\alpha} = \xi$. We then must have that $(\mathcal{P}_1, p_1)$ doesn't satisfy condition 2 above and therefore, we get $(\mathcal{P}_2, p_2)$ such that $\mathcal{P}_2 \in \mathcal{G}_\kappa$ is a correct iterate of $\mathcal{P}_1$, $\pi_{\mathcal{P}_1,\mathcal{P}_2,s_{m+1}}(\xi) > \xi$ and $\xi^{\mathcal{P}_2}_{p_2,\alpha} = \xi$. In this fashion, by successively applying the failure of 2, we get a sequence $\langle \mathcal{P}_i : i < \omega \rangle$ such that for every i, $\mathcal{P}_i$ is a correct iterate of $\mathcal{P}_{i-1}$, $\mathcal{P}_{-1} = \mathcal{W}_n$ and for $i \geq 0$,

$$\pi_{\mathcal{P}_i,\mathcal{P}_{i+1},s_{m+1}}(\xi) > \xi.$$

Let then $\mathcal{Q}$ be the direct limit of $\langle \mathcal{P}_i, i_{\mathcal{P}_i,\mathcal{P}_j} : i < j < \omega \rangle$ and let $\sigma_i : \mathcal{P}_i \to \mathcal{Q}$ be the iteration embedding. Then because $\pi_{\mathcal{P}_i,\mathcal{P}_{i+1},s_{m+1}}$'s agree with $i_{\mathcal{P}_i,\mathcal{P}_j}$, letting $\nu_i = \sigma_i(\xi)$ we get that $\langle \nu_i : i < \omega \rangle$ is a decreasing sequence of ordinals, contradiction! Thus, there is indeed a stable code for α.

Now, for each $\alpha < \kappa$ choose $(\mathcal{P}_\alpha, p_\alpha)$ such that $(\mathcal{P}_\alpha, p_\alpha)$ is a stable code for α. Let $\nu_\alpha = |(p, \alpha)|_{\leq^{\mathcal{P}_\alpha}} < \gamma^{\mathcal{P}_a}_{m+1}$. Then we claim that for any $\alpha, \beta < \kappa$, if $\alpha \leq^* \beta$ then $R((\mathcal{P}_\alpha, \nu_\alpha), (\mathcal{P}_\beta, \nu_\beta))$. Indeed, let $\mathcal{Q} \in \mathcal{G}_\kappa$ be a common correct iterate of $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$. Let $\nu = i_{\mathcal{P}_\alpha,\mathcal{Q},s_{m+1}}(\nu_\alpha)$ and let $\zeta = i_{\mathcal{P}_\beta,\mathcal{Q},s_{m+1}}(\nu_\beta)$. Let $\xi_\alpha = |\alpha|_{\leq^*}$ and $\xi_\beta = |\beta|_{\leq^*}$. We have that $i_{\mathcal{P}_\alpha,\mathcal{Q},s_{m+1}}(\alpha) = \alpha$, $i_{\mathcal{P}_\beta,\mathcal{Q},s_{m+1}}(\beta) = \beta$, $i_{\mathcal{P}_\alpha,\mathcal{Q},s_{m+1}}(\xi_\alpha) = \xi_\alpha$ and $i_{\mathcal{P}_\beta,\mathcal{Q},s_{m+1}}(\xi_\beta) = \xi_\beta$. Because $\xi_\alpha \leq \xi_\beta$, we have that

$$\left|(\pi_{\mathcal{P}_\alpha,\mathcal{Q},s_{m+1}}(p_\alpha),\alpha)\right|_{\leq\mathcal{Q}} \leq \left|(\pi_{\mathcal{P}_\beta,\mathcal{Q},s_{m+1}}(p_\beta),\beta)\right|_{\leq\mathcal{Q}}.$$

Therefore, $\nu \leq \xi$.

This shows that $\alpha \to (\mathcal{P}_\alpha, p_\alpha)$ is an order preserving map of $\leq^*$ into R and hence,

$$|\leq^*| \leq |R| = \gamma_{\infty,s_{m+1}} < \delta^{\mathcal{M}_\infty,\kappa}.$$

$\square$

We finish by remarking that the directed limit of $\mathcal{M}$ at κ is invariant under small forcing. This means that if $\mathbb{P} \in \mathcal{M}|\kappa$ and $g \subseteq \mathbb{P}$ is $\mathcal{M}$-generic then one can, working inside $\mathcal{M}[g]$, construct a directed system, much like we did above, and show that the direct limit of this system is the same as $\mathcal{M}_{\infty,\kappa}$. This mainly follows from Woodin's generic comparison process. The idea has been explained in various places and because of this we will omit it. The idea is as follows. It is enough to show it for g's that are generic for $Coll(\omega,\eta^+)$ where $\eta < \kappa$ is a strong cutpoint. One then fixes a strong cutpoint $\nu < \kappa$ and performs a simultaneous comparison of all suitable pairs in $\mathcal{M}[g]|\nu$. It is then shown that the tree on $\mathcal{W}_n$ is in fact in $\mathcal{M}$. This follows from the homogeneity of the forcing. Let then $\mathcal{P}$ be the last of this comparison. We then get that $\mathcal{P} \in \mathcal{M}$ and it dominates all the suitable mice in $\mathcal{M}[g]|\nu$. This then easily implies that the directed system of $\mathcal{M}[g]$ is dominated by the one in $\mathcal{M}$, and hence, the direct limit of both systems must be the same. For more on the details of the generic comparison we refer the reader to [16], [14] (Section 3.9) and [22].

3.3 The full directed system.

In this subsection, we will establish some lemmas that connect the directed system associated with $\mathcal{M}_\omega$ with the directed system associated with $\mathcal{M}_{2k+1}$. In particular, we will prove Theorem 3.21, originally due to Woodin, which has been widely known yet has remained unpublished for many years. We do not know if the proof of Theorem 3.21 presented here is the same or similar to Woodin's original proof. Woodin's result gives a characterization of κ^1_{2k+1}'s in terms of cardinals of HOD. We remind our readers that we assume that $\mathcal{M}_\omega^\#$ exists and that this assumption is for aesthetical reasons and those familiar with the general theory can easily remove it. Let us also remark that we will use superscript f to indicate that we are dealing with the full directed system, i.e., with the system associated with $\mathcal{M}_\omega^\#$'s. Notice that because of Theorem 2.1, for $\eta < (\delta_1^2)^{L(\mathbb{R})}$, the notation $\mathrm{HOD}^{L(\mathbb{R})}|\eta$ makes sense.

Besides the proof of Theorem 3.21, we will also prove Lemma 3.19 which we will use later on. When we talk about HOD, we mean $\mathrm{HOD}^{L(\mathbb{R})}$. From now on until the end of the next subsection we fix $k \in \omega$. We will often omit superscripts or subscripts that usually would involve k in them. For each real z let ν_z be the least cardinal of HOD_z such that $\mathcal{M}_{2k}(\mathrm{HOD}_z|\nu_z) \vDash$ "ν_z is Woodin". Recall $\mathcal{F}$ of Section 2. We first want to isolate a subset of $\mathcal{F}$ such that the direct limit of this subset will converge to $\mathcal{M}_{2k}(\mathrm{HOD}|\nu_0)|(\nu_0^{+\omega})^{\mathcal{M}_{2k}(\mathrm{HOD}|\nu_0)}$.

For each real z, let η_z be the least cardinal of $\mathcal{M}_\omega(z)$ such that $\mathcal{M}_{2k}(\mathcal{M}_\omega|\eta_z) \vDash$ "η_z is Woodin". Then let $\mathcal{W}_z^f = \mathcal{M}_{2k}(\mathcal{M}_\omega(z)|\eta_z)|(\eta_z^{+\omega})^{\mathcal{M}_{2k}(\mathcal{M}_\omega(z)|\eta_z)}$. We let Σ_z^f be the fragment of (ω_1, ω_1)-strategy of $\mathcal{M}_\omega(z)$ that acts on stacks which are based on $\mathcal{W}_z$. Let

$$\mathcal{F}_z^{+,f} = \{\mathcal{P} : \mathcal{P} \in I(\mathcal{W}_z^f, \Psi_z) \text{ as witnessed by a finite stack }\}.$$

Whenever $\mathcal{P}, \mathcal{Q} \in \mathcal{F}_z^{+,f}$ and $\mathcal{Q} \in I(\mathcal{P}, (\Sigma_z)_\mathcal{P})$, we will let $i^f_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ be the iteration embedding. Notice that in this notation we are omitting z from subscripts and superscripts as it is usually clear what z is. We hope this doesn't cause a confusion.

We can then define $\leq_z^f$ on $\mathcal{F}_z^{+,f}$ by $\mathcal{P} \leq_z^f \mathcal{Q}$ iff $\mathcal{Q} \in I(\mathcal{P}, (\Sigma_z)_\mathcal{P})$. We let $\mathcal{M}_{\infty,z}^{+,f}$ be the direct limit of $(\mathcal{F}_z^{+,f}, \leq_z^{+,f})$ under the iteration maps $i^f_{\mathcal{P},\mathcal{Q}}$. We also let $i^f_{\mathcal{P},\infty} : \mathcal{P} \to \mathcal{M}_{\infty,z}^{+,f}$ be the iteration map. Then clearly $\nu_z = \delta^{\mathcal{M}_{\infty,z}^{+,f}}$.

Next we show that just like $\mathcal{W}_z$, $\mathcal{F}_z^{+,f}$ and $\leq_z^{+,f}$ can be internalized to $\mathcal{M}_{2k}(x)$ where x codes $\mathcal{W}_z^f$. We first make the following definition.

Definition 3.16 *Suppose $\mathcal{P}$ is suitable like and $\mathcal{T}$ is a normal tree on $\mathcal{P}$. We say $\mathcal{T}$ has a* miserable drop *if there is $\alpha < lh(\mathcal{T})$ and ordinal η such that if*

$$\mathcal{M} = \cup\{\mathcal{N} : \mathcal{M}^{\mathcal{T}}_{\alpha}|\eta \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\alpha} \textit{ and } \mathcal{N} \textit{ is a premouse over } \mathcal{M}^{\mathcal{T}}_{\alpha}|\eta\}$$

then the rest of $\mathcal{T}$ is a normal tree on $\mathcal{M}$ above η.

Lemma 3.17 *Suppose $\mathcal{Q}, \mathcal{R} \in \mathcal{F}^{+,f}_z$. Let $\mathcal{T}$ on $\mathcal{Q}$ and $\mathcal{U}$ on $\mathcal{R}$ be the trees constructed via the comparison process in which II uses $(\Sigma_z)_{\mathcal{Q}}$ on the $\mathcal{Q}$-side and II uses $(\Sigma_z)_{\mathcal{R}}$ on the $\mathcal{R}$ side. Then $\mathcal{T}$ and $\mathcal{U}$ have no miserable drops.*

Proof. Suppose towards a contradiction, $\mathcal{T}$ has a miserable drop. Let $\mathcal{Q}^*$ be the last model of $\mathcal{T}$ and $\mathcal{R}^*$ be the last model of $\mathcal{U}$. Then $i^{\mathcal{T}}$ cannot exist and therefore, it follows from the comparison lemma that $\mathcal{R}^* \triangleleft \mathcal{Q}^*$. Let $\alpha < lh(\mathcal{T})$ be the largest such that there is a miserable drop in $\mathcal{M}^{\mathcal{T}}_{\alpha}$. Let η be such that if

$$\mathcal{M} = \cup\{\mathcal{N} : \mathcal{M}^{\mathcal{T}}_{\alpha}|\eta \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\alpha} \text{ and } \mathcal{N} \text{ is a premouse over } \mathcal{M}^{\mathcal{T}}_{\alpha}|\eta\}$$

then the rest of $\mathcal{T}$ is a tree on $\mathcal{M}$ above η. It then follows that $\eta \in \mathcal{R}^*$. Notice that η is a strong cutpoint in $\mathcal{R}^*$ and by fullness of $\mathcal{R}^*$, $\mathcal{M} \trianglelefteq \mathcal{R}^*$. Because $\mathcal{Q}^*$ is an iterate of $\mathcal{M}$ above η, we cannot have that $\mathcal{M} \trianglelefteq \mathcal{Q}^*$, contradiction! $\square$

Our next lemma shows that if $\mathcal{P}, \mathcal{Q} \in \mathcal{F}^{+,f}_z$, then their comparison involves $\mathcal{Q}$-structures that are below S_{2k}-operator.

Lemma 3.18 *Suppose $\mathcal{P}, \mathcal{Q} \in \mathcal{F}^{+,f}_z$. Let $\mathcal{R}$ be the result of their comparison and let $\mathcal{T}$ and $\mathcal{U}$ be the trees on $\mathcal{P}$ and $\mathcal{Q}$ respectively that come from the comparison process. Then for every limit α such that $\alpha + 1 \leq lh(\mathcal{T})$, if b is the branch of $\mathcal{T} \restriction \alpha$ chosen in $\mathcal{T}$ and $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$-exists then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{M}_{2k}(\mathcal{M}(\mathcal{T} \restriction \alpha))$.*

Proof. The real reason for this is the only way to produce normal trees with $\mathcal{Q}$-structures that are beyond S_{2k}-operator is to do a miserable drop. We leave this as an exercise. To see that our claim is true, assume not, and fix α such that $\alpha + 1 \leq lh(\mathcal{T})$ and if b is the branch of $\mathcal{T} \restriction \alpha$

chosen in $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$-exists then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \ntrianglelefteq \mathcal{M}_{2k}(\mathcal{M}(\mathcal{T} \restriction \alpha))$. It then follows that $\mathcal{M}_{2k}(\mathcal{M}(\mathcal{T} \restriction \alpha)) \lhd \mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ and therefore, $\mathcal{M}_{2k}(\mathcal{M}(\mathcal{T} \restriction \alpha)) \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ is Woodin". Notice that it follows from the comparison lemma and the minimality condition on $\mathcal{P}$ that $i^{\mathcal{T}}$ exists. This means that $\alpha + 1 < lh(\mathcal{T})$. But then $lh(E_\alpha^{\mathcal{T}}) > \delta(\mathcal{T} \restriction \alpha)$. Because $\mathcal{R}$ agrees with $\mathcal{M}_\alpha^{\mathcal{T}}$ up to $lh(E_\alpha^{\mathcal{T}})$ and $\mathcal{R} \vDash$ "$lh(E_\alpha^{\mathcal{T}})$ is a cardinal", $\delta(\mathcal{T} \restriction \alpha)$ is a cardinal in $\mathcal{R}$ and moreover, $\mathcal{M}_{2k}(\mathcal{R}|\delta(\mathcal{T} \restriction \alpha)) \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ is Woodin". This means that $\delta(\mathcal{T} \restriction \alpha) = \delta^{\mathcal{R}}$. But notice that $\mathcal{M}_\alpha^{\mathcal{T}} \vDash$ "$crit(E_\alpha^{\mathcal{T}})$ is a limit of cardinals η such that $\mathcal{M}_{2k}(\mathcal{M}_\alpha^{\mathcal{T}}|\eta) \vDash$ "η is Woodin" (otherwise, we must have that $crit(E_\alpha^{\mathcal{T}}) > \delta(\mathcal{T} \restriction \alpha)$ and hence, there is a misearble drop in $\mathcal{T}$). Because of the agreement between $\mathcal{M}_\alpha^{\mathcal{T}}$ and $\mathcal{R}$, we get that there is an $\mathcal{R}$-cardinal $\eta < \delta^{\mathcal{R}}$ such that $\mathcal{M}_{2k}(\mathcal{R}|\eta) \vDash$ "η is Woodin". $\square$

Using miserable drops, we can now define s-iterability for $\mathcal{P} \in \mathcal{F}_z^{+,f}$. First, given an iteration tree $\mathcal{T}$ on $\mathcal{P}$, we say $\mathcal{T}$ is *correctly guided* if $\mathcal{T}$ doesn't have miserable drops and for every limit $\alpha < lh(\mathcal{T})$, if b is the branch of $\mathcal{T} \restriction \alpha$ chosen by $\mathcal{T}$ and $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists then $\mathcal{Q}(b, \mathcal{T} \restriction \alpha) \trianglelefteq \mathcal{M}_{2k}(\mathcal{M}(\mathcal{T} \restriction \alpha))$. $\mathcal{T}$ is *short* if there is a well-founded branch b such that $\mathcal{T}^\frown\{\mathcal{M}_b^{\mathcal{T}}\}$ is correctly guided. $\mathcal{T}$ is *maximal* if $\mathcal{T}$ is not short. One can then proceed and define s-iterability as in Definition 3.4: the only difference is that we require that the trees in the stack be without miserable drops. We define $T_{s,m}^{\mathcal{P}}$, $\gamma_s^{\mathcal{P}}$ and $H_s^{\mathcal{P}}$ as before and we omit z from superscripts and subscripts as that is really part of $\mathcal{P}$. Notice that

$$\sup_{m \in \omega} \gamma_{s_m}^{\mathcal{P}} = \delta^{\mathcal{P}}.$$

For $\mathcal{P}, \mathcal{Q} \in \mathcal{F}_z^{+,f}$, we say $\mathcal{Q}$ is a *correct iterate* of $\mathcal{P}$ if there is a correctly guided finite stack $\vec{\mathcal{T}}$ on $\mathcal{P}$ with last model $\mathcal{Q}$.

Suppose now $\mathcal{P}$ and $\mathcal{Q}$ are two correct iterates of $\mathcal{W}_z^f$. Then using the proof of Lemma 3.3, we can show that the comparison of $\mathcal{P}$ and $\mathcal{Q}$ can be entirely, except possibly the very last step, be carried out in $\mathcal{M}_{2k}(\mathcal{P}, \mathcal{Q})$. That is, one can show that there are correctly guided trees $\mathcal{T}, \mathcal{U} \in \mathcal{M}_{2k}(\mathcal{P}, \mathcal{Q})$ such that $\mathcal{T}$ is on $\mathcal{P}$, $\mathcal{U}$ is on $\mathcal{Q}$ and $\mathcal{T}$ and $\mathcal{U}$ have a common last model.

Using this observation and the results of Section 3.2 one can internalize the directed system associated to $\mathcal{W}_z^f$. More precisely, suppose x is a real coding $\mathcal{W}_z^f$ and κ is an inaccessible strong cutpoint of $\mathcal{M}_{2k}(x)$ such that κ is below the first Woodin of $\mathcal{M}_{2k}(x)$ and κ is a limit of strong cutpoints, then one can form the direct limit of all correct iterates of $\mathcal{W}_z^f$ that are in $\mathcal{M}_{2k}(x)$. Notice that in Section 3.2, our internalization process didn't use $\mathcal{W}_z$ as a parameter in the definition. Here too we could make do without $\mathcal{W}_z^f$ but we don't it. Before we move on, let us then lay down the notation that is slowly evolving and becoming rather cumbersome.

1. We let $\mathcal{F}_z^{+,f} = \{\mathcal{P} : \mathcal{P}$ is a correct iterate of $\mathcal{W}_z^f\}$, $\mathcal{J}_z^{+,f} = \{(\mathcal{P}, \alpha) : \mathcal{P} \in \mathcal{F}_z^{+,f} \wedge \alpha < \delta^{\mathcal{P}}\}$, and $\mathcal{R}_z^{+,f}$ is the prewellordering defined on $\mathcal{J}_z^{+,f}$ by:

   $$(\mathcal{P}, \alpha) R_z^{+,f} (\mathcal{Q}, \beta) \text{ iff } \mathcal{Q} \text{ is a correct iterate of } \mathcal{P} \text{ and } i^f_{\mathcal{P},\mathcal{Q}}(\alpha) \leq \beta.$$

   We let $\leq_z^{+,f}$ be the prewellordering of $\mathcal{F}_z^{+,f}$ given by:

   $$\mathcal{P} \leq_z^{+,f} \mathcal{Q} \text{ iff } \mathcal{Q} \text{ is a correct iterate of } \mathcal{P}.$$

2. We let $\mathcal{I}_z^f = \{(\mathcal{P}, s) : \mathcal{P} \in \mathcal{F}_z^f \wedge s \in Ord^{<\omega} \wedge \mathcal{P}$ is strongly s-iterable $\}$, $\mathcal{F}_z^f = \{H_s^{\mathcal{P}} : (\mathcal{P}, s) \in \mathcal{I}_z^f\}$ and $\mathcal{J}_{z,s}^f = \{(\mathcal{P}, \alpha) : \mathcal{P} \in \mathcal{F}_z^{+,f} \wedge \alpha < \gamma_s^{\mathcal{P}}\}$. We let R_z^f be the prewellordering of $\mathcal{J}_z^f$ given by:

   $$(\mathcal{P}, \alpha) R_{z,s}^f (\mathcal{Q}, \beta) \text{ iff } \mathcal{Q} \text{ is a correct iterate of } \mathcal{P} \text{ and } \pi_{\mathcal{P},\mathcal{Q},s}(\alpha) \leq \beta.$$

   We let $\leq_z^f$ be the prewellordering of $\mathcal{I}_z^f$ given by:

   $(\mathcal{P}, s) \leq_z^f (\mathcal{Q}, t)$ iff $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$ and $s \subseteq t$. We have that $\leq_z^f$ is directed.

3. Given $\mathcal{P}$ and $s \in Ord^{<\omega}$ such that $(\mathcal{P}, s) \in \mathcal{I}_z^f$, if $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$ then we let $\pi^f_{\mathcal{P},\mathcal{Q},s} : H_s^{\mathcal{P}} \to H_s^{\mathcal{Q}}$ be the s-iterability embedding. z will be clear from the context and hence, we omit it. Recall that we let $\pi_{\mathcal{P},\mathcal{Q},s} : H_s^{\mathcal{P}} \to H_s^{\mathcal{Q}}$ be the s-iterability embedding where $\mathcal{P}, \mathcal{Q}$ are suitable $\mathcal{P}$ is s-iterable and $\mathcal{Q}$ is a correct iterate of $\mathcal{P}$.

4. We let $\mathcal{M}^f_{\infty,z}$ be the direct limit of $(\mathcal{F}^f_z, \leq^f_z)$ under the maps $\pi^f_{\mathcal{P},\mathcal{Q},s}$ and $\mathcal{M}^{+,f}_{\infty,z}$ be the direct limit of $(\mathcal{F}^{+,f}_z, \leq^{+,f}_z)$ under the iteration maps $i^f_{\mathcal{P},\mathcal{Q}}$. By the proof of Lemma 3.10, $\mathcal{M}^{+,f}_{\infty,z} = \mathcal{M}^f_{\infty,z}$.

5. We let $\pi^f_{\mathcal{P},\infty,s} : H^{\mathcal{P}}_s \to_{\Sigma_1} \mathcal{M}^f_{\infty,z}$ and $\pi_{\mathcal{P},\infty,s} : H^{\mathcal{P}}_s \to_{\Sigma_1} \mathcal{M}_{\infty,z}$ be the corresponding iteration embeddings.

6. Recall that $\delta_{\infty,z} = \delta^{\mathcal{M}_{\infty,z}}$. We change the above introduced notation and write $\delta^f_{\infty,z} = \delta^{\mathcal{M}^f_{\infty,z}}$. Thus, $\delta^f_{\infty,z} = \nu_z$.

7. We let $\gamma^f_{\infty,s,z} = \pi^f_{\mathcal{P},\infty,s}$"$\gamma^{\mathcal{P}}_s$ for some $\mathcal{P}$ such that $(\mathcal{P}, s) \in \mathcal{I}^f_z$. Recall that $\gamma_{\infty,s,z} = \pi_{\mathcal{P},\infty,s}$"$\gamma^{\mathcal{P}}_s$ for some $\mathcal{P}$ such that $(\mathcal{P}, s) \in \mathcal{I}_z$.

8. We let $\mathcal{M}^f_{\infty,\kappa,z,x}$ be the direct limit of $\mathcal{W}^f_z$ constructed inside $\mathcal{M}_{2k}(x)$ at κ. Here x codes $\mathcal{W}^f_z$ and κ is an inaccessible strong cutpoint of $\mathcal{M}_{2k}(x)$ which is less than the first Woodin of $\mathcal{M}_{2k}(x)$ and is a limit of strong cutpoints of $\mathcal{M}_{2k}(x)$.

9. We let $\mathcal{M}_{\infty,\kappa,z,x}$ be the direct limit of $\mathcal{W}_{2k+1,z}$ constructed inside $\mathcal{M}_{2k}(x)$. Here x codes $\mathcal{W}_{2k+1,z}$ and κ is an inaccessible strong cutpoint of $\mathcal{M}_{2k}(x)$ which is less than the first Woodin of $\mathcal{M}_{2k}(x)$ and is a limit of strong cutpoints of $\mathcal{M}_{2k}(x)$.

10. We let $\mathcal{M}^f_{\infty,z,x} = \mathcal{M}^f_{\infty,\kappa,z,x}$ and $\mathcal{M}_{\infty,z,x} = \mathcal{M}_{\infty,\kappa,z,x}$ where κ is the least inaccessible of $\mathcal{M}_{2k}(x)$.

11. $\pi^f_{\mathcal{P},\infty,s,x} : \mathcal{P} \to \mathcal{M}^f_{\infty,z,x}$ and $\pi_{\mathcal{P},\infty,s,x} : \mathcal{P} \to \mathcal{M}_{\infty,z,x}$ be the corresponding iteration embeddings.

12. If a is a countable transitive set such that $\mathcal{W}_z \in a$ or $\mathcal{W}^f_z \in a$ then we let $\mathcal{M}^f_{\infty,\kappa,z,a}$, $\mathcal{M}_{\infty,\kappa,z,a}$, $\mathcal{M}^f_{\infty,z,a}$, $\mathcal{M}_{\infty,z,a}$, $\pi^f_{\mathcal{P},\infty,s,a}$, and $\pi_{\mathcal{P},\infty,s,a}$ be the corresponding objects.

Our first lemma is that R_z's dominate R^f_z's.

Lemma 3.19 *For every z if w is a real coding $\mathcal{W}^f_z$ then for every m, $\left|R^f_{z,s_m}\right| \leq |R_{w,s_m}|$.*

Proof. Fix z, w and m as in the hypothesis. We now construct an order preserving embedding $f : \left|R^f_{z,s_m}\right| \to |R_{z,s_m}|$.

Suppose $\mathcal{P}$ is such that $(\mathcal{P}, s_m) \in \mathcal{I}_w$. By iterating if necessary, we get that there are conditions in the extender algebra of $\mathcal{P}$ that force that the generic object is a pair $(\mathcal{Q}, \alpha) \in \mathcal{J}^f_{z,s_m}$. The formula expressing this has $\mathcal{W}^f_z$ as a parameter and essentially says that $\mathcal{Q}$ is a correct iterate of $\mathcal{W}^f_z$ and $\alpha < \gamma^{\mathcal{Q}}_m$. Because if $G \subseteq Coll(\omega, \delta^{\mathcal{P}})$ is $\mathcal{M}_{2k}(\mathcal{P})$-generic and $x_g \in \mathcal{M}_{2k}(\mathcal{P})[g]$ is the real coding $\mathcal{P}|\delta^{\mathcal{P}}$ then we can form $\mathcal{M}^f_{\infty,z,x_g}$ [3], there are conditions p in the extender algebra of $\mathcal{P}$ that decide values for $\pi^f_{\dot{\mathcal{Q}},\infty,\check{s}_m,x_g}(\check{\alpha})$ where $(\dot{\mathcal{Q}}, \alpha)$ is the generic object containing p. Notice that the value of $\pi^f_{\dot{\mathcal{Q}},\infty,\check{s}_m,x_g}(\check{\alpha})$ is independent of g. We then let $\mathcal{A}^{\mathcal{P}}$ be a maximal antichain of conditions p such that

1. p forces that the generic object is a pair $(\mathcal{Q}, \alpha) \in \mathcal{I}^f_{z,s_m}$,

2. for some β, $\mathcal{M}_{2k}(\mathcal{P}) \vDash$ "$p \Vdash\!\Vdash_{Coll(\omega,\delta^{\mathcal{P}})} \pi^f_{\dot{\mathcal{Q}},\infty,\check{s}_m,x_g}(\check{\alpha}) = \check{\beta}$".

Notice that we can assume that $\mathcal{A}^{\mathcal{P}} \in H^{\mathcal{P}}_{s_m}$. For each $p \in \mathcal{A}^{\mathcal{P}}$ let β_p be the witness for 2. We can then define $\leq^{\mathcal{P}}$ on $\mathcal{A}^{\mathcal{P}}$ by: $p \leq^{\mathcal{P}} q \leftrightarrow \beta_p \leq \beta_q$. Notice that $\left|\leq^{\mathcal{P}}\right| < \gamma^{\mathcal{P}}_{s_m}$. We have that $p \leq^{\mathcal{P}} q$ iff $\mathcal{M}_{2k}(\mathcal{P}) \vDash (p, q) \Vdash$ " if $\dot{G} = ((\dot{\mathcal{Q}}, \check{\alpha}), (\dot{R}, \check{\beta}))$ then $(\dot{Q}, \check{\alpha}) R^f_{\check{z},\check{s}_m}(\dot{\mathcal{R}}, \check{\beta})$".

Fix now $(\mathcal{Q}, \alpha) \in \mathcal{I}^f_{z,s_m}$. We say $(\mathcal{P}, p)$ is $(\mathcal{Q}, \alpha)$-stable if

1. $(\mathcal{Q}, \alpha)$ is generic for the extender algebra of $\mathcal{P}$ and $p \in G$ where $G \subseteq \mathbb{B}^{\mathcal{P}}$ is the generic object such that $x_G = (\mathcal{Q}, \alpha)$,

2. $p \in \mathcal{A}^{\mathcal{P}}$ and $\beta_p = \pi^f_{\mathcal{Q},\infty,s_m,\mathcal{P}[\mathcal{Q}]}(\alpha)$,

3. whenever $(\mathcal{R}, q)$ is such that $\mathcal{R}$ is a correct iterate of $\mathcal{P}$ such that $(\mathcal{Q}, \alpha)$ is generic over $\mathcal{R}$ for the extender algebra at $\delta^{\mathcal{R}}$ and letting $G \subseteq \mathbb{B}^{\mathcal{R}}$ be the generic such that $x_G = (\mathcal{Q}, \alpha)$, $q \in \mathcal{A}^{\mathcal{R}} \cap G$,

[3] Notice that one can show via $\mathcal{S}$-constructions that $\mathcal{M}_{2k}(\mathcal{P})[g] = \mathcal{M}_{2k}(x)$.

$$\beta_q = \pi_{\mathcal{P},\mathcal{R},s_m}(\beta_p).$$

Thus, $q =_{\leq^{\mathcal{R}}} \pi_{\mathcal{P},\mathcal{R},s_m}(p)$.

We claim that for every $(\mathcal{Q},\alpha) \in \mathcal{I}^f_{z,s_m}$ there is a $(\mathcal{Q},\alpha)$-stable $(\mathcal{P},p)$. To see this assume not and fix $(\mathcal{Q},\alpha) \in \mathcal{I}^f_{z,s_m}$ such that there is no $(\mathcal{Q},\alpha)$-stable pair $(\mathcal{P},p)$. Let $\mathcal{P}_0$ be such that $(\mathcal{Q},\alpha)$ is generic for the extender algebra of $\mathcal{P}_0$. Letting $G \subseteq \mathbb{B}^{\mathcal{P}}$ be the generic object such that $x_G = (\mathcal{Q},\alpha)$, we have a unique condition $p_0 \in \mathcal{A}^{\mathcal{P}} \cap G$. Because $(\mathcal{P}_0, p_0)$ isn't $(\mathcal{Q},\alpha)$-stable, there is $\mathcal{P}_1$ which is a correct iterate of $\mathcal{P}_0$ and is such that $(\mathcal{Q},\alpha)$ is generic over $\mathcal{P}_1$ for the extender algebra at $\delta^{\mathcal{P}_1}$ and if $p_1 \in \mathcal{A}^{\mathcal{P}_1} \cap H$ where $H \subseteq \mathbb{B}^{\mathcal{P}_1}$ is the $\mathcal{P}_1$-generic such that $x_H = (\mathcal{Q},\alpha)$ then

$$\beta_{p_1} \neq \pi_{\mathcal{P},\mathcal{R},s_m}(\beta_{p_0})$$

Let

$$i =_{def} i_{\mathcal{M}_{2k}(\mathcal{P}_0),\mathcal{M}_{2k}(\mathcal{P}_1)} \restriction \mathcal{M}^f_{\infty,z,\mathcal{P}_0} : \mathcal{M}^f_{\infty,z,\mathcal{P}_0} \to \mathcal{M}^f_{\infty,z,\mathcal{P}_1}.$$

Then by Dodd-Jensen we have that

$$i(\pi^f_{\mathcal{Q},\infty,s_m,\mathcal{P}_0}(\alpha)) \geq \pi^f_{\mathcal{Q},\infty,s_m,\mathcal{P}_1}(\alpha),$$

implying that

$$i(\beta_{p_0}) \geq \beta_{p_1}.$$

But because $i(\beta_{p_0}) = \pi_{\mathcal{P},\mathcal{R},s_m}(\beta_{p_0})$ and $\beta_{p_1} \neq \pi_{\mathcal{P},\mathcal{R},s_m}(\beta_{p_0})$, we get that

$$\beta_{p_1} < i(\beta_{p_0}).$$

Continuing this construction we get $\langle \mathcal{P}_k, p_k : k < \omega \rangle$ such that $\mathcal{P}_0$ is a correct iterate of $\mathcal{W}_w$, $\mathcal{P}_{k+1}$ is a correct iterate of $\mathcal{P}_k$ and $\beta_{p_{k+1}} < i_{\mathcal{P}_k,\mathcal{P}_{k+1}}(\beta_{p_k})$. Let then $\mathcal{P}$ be the direct limit of $\mathcal{P}_k$'s under the embeddings $i_{\mathcal{P}_k,\mathcal{P}_{k+1}}$ and let $\sigma_k : \mathcal{P}_k \to \mathcal{P}$ be the direct limit embedding. Then letting $\xi_k = \sigma_k(\beta_{p_k})$, we get that $\langle \xi_k : k \in \omega \rangle$ is a descending sequence of ordinals, contradiction.

For each $(\mathcal{Q}, \alpha) \in \mathcal{I}^f_{z,s_m}$ let $A_{\mathcal{Q},\alpha} = \{(\mathcal{P}, p) : (\mathcal{P}, p)$ is $(\mathcal{Q}, \alpha)$-stable $\}$. Let $B_{\mathcal{Q},\alpha} = \{(\mathcal{P}, \xi) : \exists p((\mathcal{P}, p) \in A_{\mathcal{Q},\alpha} \wedge |p|_{\mathcal{A}^{\mathcal{P}}} = \xi)\}$. Then notice that if $(\mathcal{P}_i, \xi_i) \in B_{\mathcal{Q}_i,\alpha_i}$ for $i = 0, 1$ then

$$(\mathcal{Q}_0, \alpha_0) R^f_{z,s_m} (\mathcal{Q}_1, \alpha_1) \leftrightarrow (\mathcal{P}_0, \xi_0) R_{w,s_m} (\mathcal{P}_1, \xi_1)$$

To see this, let $\mathcal{P}$ be a common correct iterate of $\mathcal{P}_0$ and $\mathcal{P}_1$ such that $(\mathcal{Q}_0, \alpha_0)$ and $(\mathcal{Q}_1, \alpha_1)$ are generic for the extender algebra of $\mathcal{P}$. Then let $G_i \subseteq \mathbb{B}^{\mathcal{P}}$ be the $\mathcal{P}$-generic such that $x_{G_i} = (\mathcal{Q}_i, \alpha_i)$ $(i = 0, 1)$. Let $p_i \in \mathcal{A}^{\mathcal{P}} \cap G_i$. Suppose now $i_{\mathcal{P}_0,\mathcal{P}}(p_0) \leq^{\mathcal{P}} i_{\mathcal{P}_1,\mathcal{P}}(p_1)$. Because of stability we have that

$$i_{\mathcal{P}_k,\mathcal{P}}(\beta_{p_k}) = \pi_{\mathcal{Q}_k,\infty,s_m,\mathcal{P}}(\alpha_\kappa) \quad k = 0, 1.$$

Because $i_{\mathcal{P}_k,\mathcal{P}}(\beta_{p_k}) = \beta_{i_{\mathcal{P}_k,\mathcal{P}(p_k)}}$ $(k = 0, 1)$ and $i_{\mathcal{P}_0,\mathcal{P}}(p_0) \leq^{\mathcal{P}} i_{\mathcal{P}_1,\mathcal{P}}(p_1)$, we get that

$$\pi_{\mathcal{Q}_0,\infty,s_m,\mathcal{P}}(\alpha_0) \leq \pi_{\mathcal{Q}_1,\infty,s_m,\mathcal{P}}(\alpha_1).$$

This then implies that

$$\mathcal{M}_{2k}(\mathcal{P})[(\mathcal{Q}_0, \alpha_0), (\mathcal{Q}_1, \alpha_1)] \vDash \text{“}(\mathcal{Q}_0, \alpha_0) R^f_{z,s_m} (\mathcal{Q}_1, \alpha_1)\text{”}.$$

Hence, $(\mathcal{Q}_0, \alpha_0) R^f_{z,s_m} (\mathcal{Q}_1, \alpha_1)$. The other direction is similar.

Let then $f : \left|R^f_{z,s_m}\right| \to |R_{w,s_m}|$ be given by $f(\nu) = \eta$ if whenever $(\mathcal{Q}, \alpha) \in \mathcal{I}^f_{z,s_m}$ is such that $|(\mathcal{Q}, \alpha)|_{R^f_{z,s_m}} = \nu$ then for any $(\mathcal{P}, \beta) \in B_{\mathcal{Q},\alpha}$, $|(\mathcal{P}, \beta)|_{R_{w,s_m}} = \eta$. The proof just used can be easily modified to show that f is order preserving and hence, $\left|R^f_{z,s_m}\right| \leq |R_{w,s_m}|$. $\square$

The proof of Lemma 3.19 can be used to prove the following.

Corollary 3.20 *For any $m \in \omega$ and $z, w \in \mathbb{R}$, if $z \leq_T w$ then $|R_{z,s_m}| \leq |R_{w,s_m}|$ and $|R|_{z,s_m} \leq \left|R^f_{z,s_m}\right|$.*

Next, we prove Woodin's result. As we said before, we do not know if the proof given here is Woodin's original proof but it is probably very similar to it.

Theorem 3.21 (Woodin, unpublished) *Assume $AD + V = L(\mathbb{R})$. For $k \in \omega$, κ^1_{2k+3} is the least cardinal δ of* HOD *such that*

$$\mathcal{M}_{2k}(\mathrm{HOD}|\delta) \vDash \text{“}\delta \textit{ is Woodin”}.$$

Proof. It easily follows from Lemma 3.3 and the remarks following it that for each $z \in \mathbb{R}$, $\left|R^{+,f}_z\right| < \delta^1_{2k+3}$. The proof is just that $R^{+,f}_z$ is $\Sigma^1_{2k+3}(\mathcal{W}^f_z)$, i.e., in any code of $\mathcal{W}^f_z$. To finish the proof of Theorem 3.21, we need then to show that $\delta^f_{\infty,z} \leq \kappa^1_{2k+3}$ and that $\delta^f_{\infty,z} \geq \kappa^1_{2k+3}$. We start with the first.

Suppose that for some z, $\delta^f_{\infty,z} > \delta^1_{2k+3}$. Let $U \subseteq \mathbb{R}$ be the set

$$\{(x, y) : y \text{ codes } \Pi^1_{2k+2}\text{-iterable, } 2k+1\text{-small premouse } \mathcal{M} \text{ over } x \text{ such that } \mathcal{M} \text{ has } 2k+1 \text{ Woodins and a last extender }\}.$$

Then U is Π^1_{2k+2} and we can let $T \subseteq \omega^{<\omega} \times (\kappa^1_{2k+3})^{<\omega}$ be a tree such that $p[T] = U$. It follows by Theorem 2.1 that for every w

$$\mathcal{M}^f_{\infty,w}|\delta^f_{\infty,w} = \mathrm{HOD}|\delta^f_{\infty,w}.$$

and therefore, there is w which codes $\mathcal{W}^f_z$ and is such that $T \in \mathcal{M}^f_{\infty,w}|\eta$ for some $\eta < \delta^f_{\infty,w}$ (because we are assuming that $\delta^f_{\infty,z} > \kappa^1_{2k+3}$ and by Lemma 3.19, we have that $\delta^f_{\infty,z} \leq \delta^f_{\infty,w}$). Let then $\mathcal{P} \in \mathcal{F}^f_w$ be such that there is $S \in \mathcal{P}|\delta^{\mathcal{P}}$ such that $i^f_{\mathcal{P},\infty}(S) = T$. We can fix l such that $S \in H^{\mathcal{P}}_l$. Let u be a real coding $(\mathcal{W}^f_w, \mathcal{P})$. Let $S^* = \pi^f_{\mathcal{P},\infty,s_l,\mathcal{W}_u}(S)$. We claim that $\mathcal{M}_{2k+1}(u) \vDash$ “$p[S^*_u] \neq \emptyset$”.

To see that $\mathcal{M}_{2k+1}(u) \vDash$ “$p[S^*_u] \neq \emptyset$”, fix a correct iterate $\mathcal{R}$ of $\mathcal{P}$ such that for some y there is $h \in (\gamma^{\mathcal{R}}_l)^\omega$ such that if $g = \pi^f_{\mathcal{R},\infty,s_l}$”$h$ then $(u, y, g) \in [T]$. Notice that $\mathcal{M}_{2k+1}(u) = \mathcal{M}_{2k}(\mathcal{W}_u)$. Iterate $\mathcal{W}_u$ to make $(\mathcal{R}, y)$ generic. Let $\mathcal{Q}$ be this iterate. Let $\bar{g} = \pi^f_{\mathcal{R},\infty,s,\mathcal{Q}[\mathcal{R},y]}$. Then for every k, we must have that

$$(y \restriction k, \bar{g} \restriction k) \in (\pi^f_{\mathcal{W}^f_w,\infty,s_l,\mathcal{Q}}(S))_u = (\pi^f_{\mathcal{R},\infty,s_l,\mathcal{Q}[\mathcal{R},y]}(S))_u.$$

This means that $[(\pi^f_{\mathcal{W}^f_w,\infty,s_l,\mathcal{Q}}(S))_u] \neq \emptyset$. By absoluteness we have that

$$\mathcal{M}_{2k}(\mathcal{Q}) \vDash [(\pi^f_{\mathcal{W}^f_w,\infty,s_l,\mathcal{Q}}(S))_u] \neq \emptyset.$$

It then follows by elementarity that

$$\mathcal{M}_{2k+1}(u) \vDash \text{“}p[S^*_u] \neq \emptyset\text{”}.$$

It is, however, a well-known fact that there cannot be $y \in \mathcal{M}_{2k+1}(u)$ which codes a Π^1_{2k+2}-iterable $2k+1$-small premouse $\mathcal{M}$ over u which has $2k+1$-Woodins and a last extender.[4] This contradiction shows that $\delta^f_{\infty,z} \leq \kappa^1_{2k+3}$.

To show that $\delta^f_{\infty,z} \geq \kappa^1_{2k+3}$, it is enough to show that $\delta_{\infty,0} \geq \kappa^1_{2k+3}$. For this, we show that every Π^1_{2k+2}-set is $\delta_{\infty,0}$-Suslin. Let $\delta = \delta_{\infty,0}$. To see that the universal Π^1_{2k+2}-set is δ-Suslin let $\mathcal{Q} = \mathcal{M}^\#_{2k}(\mathcal{M}_{\infty,0}|\delta)$. Notice that $\mathcal{Q}$ has size δ. Let U be the universal Pi^1_{2k+2}-set. Let ϕ be Π^1_{2k+2} such that $x \in U \leftrightarrow \phi(x)$. Let T be the tree of attempts to construct a triple (x, z, π) such that

1. z codes a premouse $\mathcal{M}_z$,

2. $\pi : M_z \to \mathcal{Q}$,

3. x is generic over M_z for the extender algebra at the least Woodin of $\mathcal{M}_z$,

4. $M_z[x] \vDash \phi[x]$.

Let then $S = \{(s, f) : s \in \omega^{<\omega},\ f \in [\delta]^{<\omega}$ and f codes f_0, f_1 such that $(s, f_0, f_1) \in T\}$. Then, because $\mathcal{M}_{2k}(z)$ is $\Pi^1_{2k+2}(z)$-correct, it is not hard to see that $p[S] = U$. This then completes the proof that $\delta^f_{\infty,z} = \kappa^1_{2k+3}$. □

As a corollary to Lemma 3.19, we get the following.

[4]One way to see this is to use a result from [18]. It is shown there that $x \in \mathcal{Q}_{2k+1}(u) \leftrightarrow x$ is in every Π^1_{2k+2}-iterable, $2k+1$-small premouse $\mathcal{M}$ which has $2k+1$ Woodins and a last extender. Thus, if there was such a premouse $\mathcal{M} \in \mathcal{M}_{2k+1}(u)$ then as $Q_{2k+1}(u) = \mathbb{R}^{\mathcal{M}_{2k+1}(u)}$, $\mathcal{M} \in \mathcal{M}$, contradiction!

Corollary 3.22 *For every $z \in \mathbb{R}$, $\delta_{\infty,z} = \kappa^1_{2k+3}$.*

3.4 The proof of the main theorem

In this subsection, we work towards the proof of the main theorem. Recall that

$$a_{2k+1,m} = \sup\{|\leq^*| : \leq^* \in \underset{\sim}{\Gamma}_{2k+1,m}\}$$

We let $\gamma_{\infty,m,x} = \gamma_{\infty,s_m,x}$ and $\gamma^f_{\infty,m,x} = \gamma^f_{\infty,s_m,x}$ and let

$$b_{2k+1,m} = \sup_{x\in\mathbb{R}} \gamma_{\infty,m,x}.$$

Notice that it follows from Lemma 3.19 that

$$b_{2k+1,m} = \sup_{x\in\mathbb{R}} \gamma^f_{\infty,m,x}.$$

It follows from Theorem 3.21 that

$$\kappa^1_{2k+3} = \sup_{m\in\omega} b_{2k+1,m}.$$

To make the notation as simple as possible, we fix an odd integer $2k+1$. We will omit it from various subscripts from now until the end of this subsection. Here is our first lemma.

Lemma 3.23 $a_{2k+1,m} \leq b_{2k+1,m+1}$.

Proof. Fix $m \in \omega$ and let $\leq^* \in \underset{\sim}{\Gamma}_{2k+1,m}$. Let z^*, ϕ be such that for all $x, y \in \mathbb{R}$,

$$x \leq^* y \leftrightarrow \mathcal{M}_{2k}(z^*, x, y) \vDash \phi[z^*, x, y, s_m].$$

Suppose towards a contradiction that $|\leq^*| = \sup_{x\in\mathbb{R}} \gamma_{\infty,x,m+1}$ (this may produce another real parameter, but we assume that it is already part of z^*).

First notice that for every l, $\sup_{x\in\mathbb{R}} \gamma_{\infty,l,x} < \kappa^1_{2k+3}$. This is because if $\sup_{x\in\mathbb{R}} \gamma_{\infty,l,x} = \kappa^1_{2k+3}$ then because $\mathrm{cf}(\kappa^1_{2k+3}) = \omega$, there must be x such that $\gamma_{\infty,l,x} = \kappa^1_{2k+3}$. But since $\delta_{\infty,x} > \gamma_{\infty,l,x}$, we get a contradiction. Thus, we can fix $z \in \mathbb{R}$ and $r \in \omega$ such that $z^* \leq_T z$ and $\gamma_{\infty,r,z} > \sup_{x\in\mathbb{R}} \gamma_{\infty,m,x}$.

Following Hjorth (see [3]), using coding lemma, we get $w \in \mathbb{R}$ and a $\Sigma^1_{2k+3}(w)$ set $B \subseteq \mathbb{R}^2$ such that $z \leq_T w$

1. if $(x, y) \in B$ then $x \in dom(\leq^*)$, $y \in dom(\leq_{z,r})$ and $|x|_{\leq^*} = |y|_{\leq_{z,r}}$,

2. for every $x \in dom(\leq^*)$ there is $y \in dom(\leq_{z,r})$ such that $(x, y) \in B$.

Let R be $\Pi^1_{2k+2}(w)$ such that $(x, y) \in B \leftrightarrow \exists u R(w, x, y, u)$. We now construct an embedding of $\leq^*$ into $R_{w,s_{m+1}}$. Let $A = \{(x, y, u) : R(w, x, y, u)\}$. Notice that whenever a is a transitive set, $\leq^* \cap \mathbb{R}^{\mathcal{M}_{2k}(a)} \in \mathcal{M}_{2k}(a)$. We will abuse our notation and write $\leq^*$ for $\leq^* \cap \mathbb{R}^{\mathcal{M}_{2k}(a)}$.

Given a suitable $\mathcal{P}$, there is a maximal antichain $\mathcal{A} \subseteq \mathbb{B}^{\mathcal{P}}$ such that if $p \in \mathcal{A}$ then for some α, $\mathcal{M}_{2k}(\mathcal{P}) \vDash$

1. $p \Vdash$ "$x_G = (x, y, u) \in A$",

2. $p \Vdash$ "$\Vdash_{Coll(\omega,\delta^{\mathcal{P}})} |x|_{\leq^*} = \alpha$".

Notice that we can take $\mathcal{A} \in H^{\mathcal{P}}_{m+1}$. Let then $\mathcal{A}^{\mathcal{P}}$ be the least such maximal antichain. We can define $\leq^{\mathcal{P}}$ on $\mathcal{A}^{\mathcal{P}}$ as follows. Given $p \in \mathcal{A}$, let α_p be the ordinal α as in 2. Then for $p, q \in \mathcal{A}$, we let $p \leq^{\mathcal{P}} q$ iff $\alpha_p \leq \alpha_q$. Notice that $\left|\leq^{\mathcal{P}}\right| < \gamma^{\mathcal{P}}_{m+1}$. The remaining part of the proof is similar to the proof of Lemma 3.19.

Given now an $x \in dom(\leq^*)$, a suitable $\mathcal{P}$ and $p \in \mathcal{A}^{\mathcal{P}}$ we say $(\mathcal{P}, p)$ is x-stable if there is $(x, y, u) \in A$ which is generic over $\mathcal{P}$ for $\mathbb{B}^{\mathcal{P}}$ and

1. if $G \subseteq \mathbb{B}^{\mathcal{P}}$ is such that $x_G = (x, y, u)$ then $p \in G$,

2. whenever $(\mathcal{R}, q)$ is such that $\mathcal{R}$ is a correct iterate of $\mathcal{P}$ such that some $(x, y^*, u^*) \in A$ is generic over $\mathcal{R}$ for $\mathbb{B}^{\mathcal{R}}$, and $q \in \mathcal{A}^{\mathcal{R}} \cap H$ where $H \subseteq \mathbb{B}^{\mathcal{R}}$ is the $\mathcal{R}$-generic such that $x_H = (x, y^*, u^*)$, then

$$|q| =_{\leq^{\mathcal{R}}} \left|\pi_{\mathcal{P},\mathcal{R},s_{m+1}}(p)\right|.$$

We claim that for every $x \in dom(\leq^*)$ there is x-stable $(\mathcal{P}, p)$. To see this, suppose not. First let y, u be such that $(x, y, u) \in A$. Then let $\mathcal{P}$ be suitable such that (x, y, u) is generic for $\mathbb{B}^{\mathcal{P}}$. There is then $p \in \mathcal{A}^{\mathcal{P}}$ such that if $G \subseteq \mathbb{B}^{\mathcal{P}}$ is $\mathcal{P}$-generic such that $x_G = (x, y, u)$ then $p \in G$. Let $\alpha = \alpha_{\mathcal{P},p}$. Because $(\mathcal{P}, p)$ isn't x-stable we must have that there is a correct iterate $\mathcal{R}$ of $\mathcal{P}$ such that some $(x, y^*, u^*) \in A$ is generic over $\mathcal{R}$ for $\mathbb{B}^{\mathcal{R}}$, and if H is the generic such that $x_H = (x, y^*, u^*)$ and $q \in H \cap \mathcal{A}^{\mathcal{R}}$ then

$$|q| \neq_{\leq^{\mathcal{R}}} \left|\pi_{\mathcal{P},\mathcal{R},s_{m+1}}(p)\right|.$$

Let y code $(\mathcal{Q}, \beta)$ and let y^* code $(\mathcal{Q}^*, \beta^*)$. Notice that $(\mathcal{Q}, \beta) =_{R_{z,r}} (\mathcal{Q}^*, \beta^*)$. Let also

$$i = i_{\mathcal{P},\mathcal{R}} \restriction \mathcal{M}_{\infty,z,\mathcal{P}} : \mathcal{M}_{\infty,z,\mathcal{P}} \to \mathcal{M}_{\infty,z,\mathcal{R}}.$$

We have that

$$i \circ \pi_{\mathcal{Q},\infty,r,z,\mathcal{P}} : H_r^{\mathcal{P}} \to H_r^{\mathcal{M}_{\infty,z,\mathcal{R}}}.$$

Because of Dodd-Jensen then we get that

$$i(\pi_{\mathcal{Q},\infty,r,z,\mathcal{P}}(\beta)) \geq \pi_{\mathcal{Q}^*,\infty,r,z\mathcal{R}}(\beta^*).$$

Notice that equality cannot hold. To see this, suppose $i(\pi_{\mathcal{Q},\infty,r,z,\mathcal{P}}(\beta)) = \pi_{\mathcal{Q}^*,\infty,r,z,\mathcal{R}}(\beta^*)$. We have that,

$$\mathcal{M}_{2k}(\mathcal{P}) \models p \Vdash \text{“if } (x_G)_2 = (\dot{\mathcal{Q}}, \dot{\beta}) \text{ then } \pi_{\dot{\mathcal{Q}},\infty,r,z,\mathcal{P}}(\dot{\beta}) = \check{\xi}\text{”}^5.$$

where $\check{\xi} = \pi_{\mathcal{Q},\infty,r,z,\mathcal{P}}(\beta)$. We then have by elementarity that there is $\mathcal{R}$-generic $H \subseteq \mathbb{B}^{\mathcal{R}}$ such that $i_{\mathcal{P},\mathcal{R}}(p) \in H$ and if $(x_H)_2 = (\mathcal{S}, \nu)$ then $\pi_{\mathcal{S},\infty,r,z,\mathcal{R}}(\nu) = i_{\mathcal{P},\mathcal{R}}(\xi)$. But since we are assuming that $i(\pi_{\mathcal{Q},\infty,r,z,\mathcal{P}}(\beta)) = \pi_{\mathcal{Q}^*,\infty,r,z,\mathcal{R}}(\beta^*)$, we must have that $(\mathcal{S}, \nu) =_{R_{z,r}} (\mathcal{Q}^*, \beta^*)$ and by the choice of B we must have that $(x_H)_1 =_{\leq^*} x$. This then implies that $i_{\mathcal{P},\mathcal{R}}(p) =_{\leq^{\mathcal{R}}} q$, contradiction. Thus we must have that

[5] Here we think of a real x as coding a triple (x_1, x_2, x_3).

$$i(\pi_{\mathcal{Q},\infty,z,w\oplus\mathcal{P},r}(\beta)) > \pi_{\mathcal{Q}^*,\infty,z,w\oplus\mathcal{R},r}(\beta^*).$$

Let then $\mathcal{P}_0 = \mathcal{P}$, $(x, y_0, u_0) = (x, y, u)$, $\mathcal{P}_1 = \mathcal{R}$ and $(x, y_1, u_1) = (x, y^*, u^*)$. Let $(\mathcal{Q}_0, \beta_0)$ be the pair coded by y_0 and let $(\mathcal{Q}_1, \beta_1)$ be the pair coded by y_1. Let $\xi_i = \pi_{\mathcal{Q}_i,\infty,z,r,\mathcal{P}_i}(\beta_i)$ for $i = 0, 1$. It then follows from our discussion that $i_{\mathcal{P}_0,\mathcal{P}_1}(\xi_0) > \xi_1$.

By a repeated application of the argument used in the previous paragraph, we can get $\langle \mathcal{P}_l, (\mathcal{Q}_l, \beta_l), \xi_l : l \in \omega\rangle$ such that

1. $\mathcal{P}_l \in \mathcal{F}_w$,
2. $\mathcal{P}_{l+1}$ is a correct iterate of $\mathcal{P}_l$,
3. $(\mathcal{Q}_l, \beta_l) \in \mathcal{I}_{w,r}$ and (Q_l, β_l) is generic over $\mathcal{P}_l$ for $\mathbb{B}^{\mathcal{P}_l}$,
4. $\pi_{\mathcal{Q}_l,\infty,r,z,\mathcal{P}_l}(\beta_l) = \xi_l$,
5. $i_{\mathcal{P}_l,\mathcal{P}_{l+1}}(\xi_l) > \xi_{l+1}$.

Letting $\sigma_{l,j} : \mathcal{P}_l \to \mathcal{P}_j$ be the iteration embedding, letting $\mathcal{Q}$ be the direct limit of $\langle \mathcal{P}_l, \sigma_{l,j} : l < j < \omega\rangle$ and letting $\sigma_l : \mathcal{P}_l \to \mathcal{Q}$ be the iteration embedding we get that $\langle \sigma_l(\xi_l) : l < \omega\rangle$ is a decreasing sequence of ordinals, contradiction. Thus, indeed, for every x there is an x-stable $(\mathcal{P}, p)$.

Let then for each x, S_x be the set of x-stable $(\mathcal{P}, p)$'s and let $\beta_{\mathcal{P},p} = |p|_{\leq^{\mathcal{P}}}$. Using uniformization, we can choose $(\mathcal{P}_x, p_x) \in S_x$. Notice now that

$$x \leq^* y \leftrightarrow (\mathcal{P}_x, p_x) \leq_{w,m} (\mathcal{P}_y, p_y).$$

(To see this, let $\mathcal{R}$ be a common iterate of $\mathcal{P}_x$ and $\mathcal{P}_y$ such that for some $u, v, u^*, v^* \in \mathbb{R}$, (x, u, v) and (y, u^*, v^*) are generic over $\mathcal{R}$ for $\mathbb{B}^{\mathcal{R}}$. Then by x and y stability, we must have that $x \leq^* y$ holds if and only if $i_{\mathcal{P}_x,\mathcal{R}}(p_x) \leq^{\mathcal{R}} i_{\mathcal{P}_y,\mathcal{R}}(p_y)$.) We then have that $x \to (\mathcal{P}_x, p_x)$ is an order preserving map of $\leq^*$ into $R_{w,m+1}$. Therefore, $|\leq^*| \leq |R_{w,m+1}| \leq \sup_{x\in\mathbb{R}} \gamma_{\infty,x,m+1}$, contradiction! $\square$

We thus have that $\sup_{m\in\omega} a_{2k+1,m} \leq \kappa^1_{2k+3}$. Notice that for each $m \in \omega$ and $w \in \mathbb{R}$, $R_{w,m} \in \Gamma_{2k+1,m+1}(w)$. Because we have that $\sup_{m\in\omega} b_{2k+1,m} = \kappa^1_{2k+3}$, we easily get that $\sup_{m\in\omega} a_{2k+1,m} = \kappa^1_{2k+3}$. This then finishes the proof of the Main Theorem.

4 Some remarks

First of all, it turns out that $b_{2k+1,m}$ is a cardinal for every m and moreover, $b_{2k+1,m} < \kappa^1_{2k+3}$. Here is the proof.

Lemma 4.1 *For every m, $b_{2k+1,m} < \kappa^1_{2k+3}$ and $b_{2k+1,m}$ is a cardinal.*

Proof. We have that $b_{2k+1,m} < \kappa^1_{2k+3}$ because if for some m, $b_{2k+1,m} = \kappa^1_{2k+3}$ then because $\mathrm{cf}(\kappa^1_{2k+3}) = \omega$, we can fix x such that $\gamma_{\infty,s_m,x} = \kappa^1_{2k+3}$. But this contradicts Theorem 3.21. Thus, we have that $b_{2k+1,m} < \kappa^1_{2k+3}$. Suppose no that for some m, $b_{2k+1,m}$ isn't a cardinal. Let $\kappa = |b_{2k+1,m}|$. Then $\kappa < b_{2k+1,m}$ and there is $A \subseteq \kappa$ such that A codes a well-ordering of κ of length $b_{2k+1,m}$. There is then a real z such that $A \in \mathrm{HOD}_z$. We then can get w such that $z \leq_T w$ and $\kappa < \gamma^f_{\infty,s_m,w}$. It follows that $A \in \mathrm{HOD}_w$ and in particular, $\gamma^f_{\infty,s_m,w}$ isn't a cardinal of HOD_w. But clearly $\gamma^f_{\infty,s_m,w}$ is a cardinal of HOD_w, contradiction. $\square$

We do not know if $a_{2k+1,m} = b_{2k+1,m}$. A more interesting question that comes up naturally is what is the exact place of $b_{2k+1,m}$ in the sequence of $\aleph$'s. We conjecture that $a_{2k+1,0} = \delta^1_{2k+2}$. One evidence for this is that by Hjorth's aforementioned result, $a_{1,0} = u_2 = \omega_2 = \delta^1_2$. More generally, Jackson showed that the sup of the lengths of Π^1_{2k} prewellorderings is δ^1_{2k} and Π^1_{2k} is a subclass of $\Gamma_{2k+1,0}$. The general question is open.

Where is this line of research going? Is there more one can do? It seems to be possible to use the directed system associated with $\mathcal{M}_{2n+1}$ to prove Kechris-Martin kind of results for Π^1_{2k+3}

(see [6]). In particular, one should be able to prove that Π^1_{2k+2} is closed under quantification over κ^1_{2k+3}. Another application should be the uniqueness of $L[T_{2k}]$, i.e., it should be possible to prove, using ideas from this paper, that $L[T_{2k}]$ is independent of the choice of the scale that produces T_{2k}. This would generalize Hjorth's theorem on the uniqueness of $L[T_2]$ (see [2]). It should also be possible to prove results like Solovay's Δ^1_3-coding result (see [17]) for higher levels of projective hierarchy. The author, however, has no intuition on whether it is possible to use directed systems to carry out Jackson's analysis of projective ordinals. From inner model theoretic point of view, Jackson's analysis remains a mystery.

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