$\mathsf{Sealing}$ from iterability[*†‡]

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Abstract

We show that if V has a proper class of Woodin cardinals, a strong cardinal, and a generically universally Baire iteration strategy (as defined in the paper) then $\mathsf{Sealing}$ holds after collapsing the successor of the least strong cardinal to be countable. This result is complementary to [8, Theorem 3.1] where it is shown that $\mathsf{Sealing}$ holds in a generic extension of a certain minimal universe. The current theorem is more general in that no minimal assumption is needed. A corollary of the main theorem is that $\mathsf{Sealing}$ is consistent relative to the existence of a Woodin cardinal which is a limit of Woodin cardinals. This improves significantly on the first consistency of $\mathsf{Sealing}$ obtained by W.H. Woodin.

The $\mathsf{Largest\ Suslin\ Axiom}$ (LSA) is a determinacy axiom isolated by Woodin. It asserts that the largest Suslin cardinal is inaccessible for ordinal definable bijections. Let $\mathsf{LSA}-\mathsf{over}-\mathsf{uB}$ be the statement that in all (set) generic extensions there is a model of LSA whose Suslin, co-Suslin sets are the universally Baire sets. The other main result of the paper shows that assuming V is as above, in the universe $V[g]$, where g is V-generic for the collapse of the successor of the least strong cardinal to be countable, the theory $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ fails; this implies that $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ is not equivalent to $\mathsf{Sealing}$ (over the base theory of $V[g]$).

We identify elements of the Baire space ω^ω with reals. Throughout the paper, by a "set of reals A", we mean $A \subseteq \omega^\omega$. A set of reals A is γ-*universally Baire* if there are trees T, U on $\omega \times \lambda$ for some λ such that $A = p[T] = \mathbb{R}\backslash p[U]$ and whenever g is a $<\gamma$-generic, in $V[g]$, $p[T] = \mathbb{R}\backslash p[U]$. We write A^g for $p[T]^{V[g]}$; this is the

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canonical interpretation of A in $V[g]$.[1] A is *universally Baire* if A is γ-universally Baire for all γ. Let Γ^∞ be the set of universally Baire sets. Given a generic g, we let $\Gamma^\infty_g = (\Gamma^\infty)^{V[g]}$ and $\mathbb{R}_g = \mathbb{R}^{V[g]}$. The next definition is due to Woodin.

Definition 0.1 $\mathsf{Sealing}$ *is the conjunction of the following statements.*

1. *For every set generic g, $L(\Gamma^\infty_g, \mathbb{R}_g) \vDash \mathsf{AD}^+$ and $\wp(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.*
2. *For every set generic g over V, for every set generic h over $V[g]$, there is an elementary embedding*

$$j : L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_h, \mathbb{R}_h).$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A^h$.

$\mathsf{Sealing}$ is a form of Shoenfield-type generic absoluteness for the theory of universally Baire sets. In this paper, we will avoid motivational discussion as [8] has a lengthy introduction to the subject. We should say, however, that $\mathsf{Sealing}$ is an important hypothesis in set theory and particularly in inner model theory for several reasons. If a large cardinal theory ϕ implies $\mathsf{Sealing}$ then the Inner Model Program for building canonical inner models of ϕ cannot succeed (at least with the criteria for defining "canonical inner models" as is done to date), cf [8, Sealing Dichotomy]. $\mathsf{Sealing}$ signifies a place beyond which new methodologies are needed in order to advance the Core Model Induction techniques. In particular, to obtain consistency strength beyond $\mathsf{Sealing}$ from strong theories such as the Proper Forcing Axiom, one needs to construct canonical subsets of Γ^∞ (third-order objects), instead of elements of Γ^∞ like what has been done before (see [8, Section 1] for a more detailed discussion). The consistency of $\mathsf{Sealing}$ was first demonstrated by Woodin, who showed that if there is a proper class of Woodin cardinals and a supercompact cardinal κ then $\mathsf{Sealing}$ holds after collapsing 2^{2^κ} to be countable. Woodin's proof can be found in [2].

One of the main corollaries of the Theorem 0.4 is that the set theoretic strength of $\mathsf{Sealing}$ is below a Woodin cardinal that is a limit of Woodin cardinals; this improves significantly the aforementioned result of Woodin. Another proof of this fact was presented in [8], where the authors establish an actual equiconsistency for $\mathsf{Sealing}$. One advantage of this proof is that no smallness assumption is made (unlike [8]). Another, perhaps more important, advantage of the current proof over the one presented in [8] is that this proof is more accessible. Our proof of $\mathsf{Sealing}$ is based on

[1]One can show A^g does not depend on the choice of T, U.

iterability and uses recent ideas from descriptive inner model theory. However, in this paper, our aim is to present the proof of our main theorem, Theorem 0.4, without using any fine structure theory or heavy machinery from inner model theory, so that the paper is accessible to the widest possible audience. We will only assume general knowledge of iterations, iteration strategies and Woodin's extender algebra, both of which are topics that can be presented without any fine structure theory. For instance, the reader can consult [1] or [3]. The fact that the hypothesis of Theorem 0.4 is weaker than a Woodin cardinal that is a limit of Woodin cardinals follows from a very recent work of Steel ([12]) and the first author ([5]), and this fact will not be proven here, as it is well beyond the scope of this paper.

Given a transitive model Q of set theory and a Q-cardinal κ, we let $Q|\kappa = H^Q_\kappa$. We say E is a (κ, λ)-short extender over Q if there is a Σ_1-elementary embedding $j : Q \to M$ such that

1. M is transitive,
2. $M = \{j(f)(a) : f : \kappa^{<\omega} \to Q, f \in Q \text{ and } a \in \lambda^{<\omega}\}$,
3. $j(\kappa) \geq \lambda$, and
4. $E = \{(a, A) : a \in \lambda^\omega, A \subseteq [\kappa]^{|a|} \text{ and } a \in j(A)\}$.

κ is called the *critical point of* E and λ the *length of* E. We write $\kappa = \mathrm{crit}(E)$ and $\lambda = lh(E)$. M is then called the *ultrapower of* Q *by* E and is uniquely determined by Q and E. We write $M = Ult(Q, E)$. Given a set X and an extender E, we say E *coheres* X if $X \cap V_{lh(E)} = j(X) \cap V_{lh(E)}$. For more on extenders, the reader can consult [3] or [1].

Suppose P is a transitive model of set theory. We let $\mathsf{ile}(P)$ be the set of *inaccessible-length extenders* of P. More precisely $\mathsf{ile}(\mathsf{P})$ consists of extenders $E \in P$ such that $P \vDash$ "$lh(E)$ is inaccessible and $V_{lh(E)} = V^{Ult(V,E)}_{lh(E)}$."

Definition 0.2 *We say that $\mathcal{P}$ is a pre-iterable structure if $\mathcal{P} = (P, \mathsf{ile}(P))$ where P is a transitive model of* ZFC.

When we talk about iterability for $\mathcal{P}$, we mean iterability with respect to extenders in $\vec{E}^{\mathcal{P}} =_{def} \mathsf{ile}(P)$ (and its images). Thus, the relevant iterations are those that are built by using extenders in $\vec{E}$ and its images.

Recall from [3] that an iteration $\mathcal{T}$ is *normal* if the extenders used in it have increasing lengths and each extender E used along $\mathcal{T}$ is applied to the least possible model, i.e. E is applied to the first model $\mathcal{M}^{\mathcal{T}}_\alpha$ where the ultrapower $Ult(\mathcal{M}^{\mathcal{T}}_\alpha, E)$

makes sense. Following Jensen, we will say that $\mathcal{T}$ is a *smooth iteration* (of its base model) if it can be represented as a *stack* of normal iterations. More precisely, $\mathcal{T} = (\mathcal{T}_i : i < \eta)$ where $\mathcal{T}_0$ is a normal iteration of the base model of $\mathcal{P}$ and for $i \in (0, \eta)$, $\mathcal{T}_i$ is a normal iteration of the last model of $\mathcal{T}_{i-1}$ if i is a successor ordinal and on the direct limit of $(\mathcal{T}_j : j < i)$ under the iteration embeddings if i is limit. We say that a pre-iterable structure $\mathcal{P}$ is *smoothly iterable* if player II has a wining strategy in the iteration game of arbitrary length that produces smooth iterations. Recall that in iteration games, player I picks the extenders while player II plays branches at limit steps. We say that Σ is an iteration strategy for $\mathcal{P}$ if it is a winning strategy for $\mathcal{P}$ in the iteration game that produces arbitrary length smooth iterations.

Finally we state self-iterability. The Unique Branch Hypothesis (UBH) is the statement that every normal iteration tree $\mathcal{T}$ on V has at most one cofinal well-founded branch. The Generic Unique Branch Hypothesis (gUBH) says that UBH holds in all set generic extensions. The notion of generically universally Baire (guB) strategy appears in the next section as Definition 1.5.

Definition 0.3 *We say that self-iterability holds if the following holds in V.*

1. gUBH.
2. $\mathcal{V} = (V, \mathsf{ile(V)})$ *is a pre-iterable structure that has a guB-iteration strategy.*

Notice that because of clause 1, the iteration strategy in clause 2 is unique.

Theorem 0.4 *Assume self-iterability holds, and suppose there is a class of Woodin cardinals and a strong cardinal. Let κ be the least strong cardinal of V and let $g \subseteq Coll(\omega, \kappa^+)$ be V-generic. Then $V[g] \vDash \mathsf{Sealing}$.*

As mentioned above, a corollary of Theorem 0.4, via a non-trivial amount of work in [12] and [5], is

Corollary 0.5 *Con*(ZFC + *there is a Woodin cardinal which is a limit of Woodin cardinals*) *implies Con*($\mathsf{Sealing}$).

The main idea behind the proof of Theorem 0.4 originates in [8]. The most relevant portion of that paper is [8, Theorem 3.1]. We should note that the hypothesis of Theorem 0.4 cannot be weakened to just gUBH for plus-2 iterations as this form of UBH holds in a minimal mouse with a strong cardinal, a class of Woodin cardinals and a stationary class of measurable cardinals[2], but this theory is weaker than $\mathsf{Sealing}$ as shown by [8, Theorem 3.1].

[2] This fact is due to Steel, see [10, Theorem 3.3].

The Largest Suslin Axiom was introduced by Woodin in [13, Remark 9.28]. The terminology is due to the first author. Here is the definition.

Definition 0.6 *The* Largest Suslin Axiom*, abbreviated as* LSA*, is the conjunction of the following statements:*

1. AD^+.
2. *There is a largest Suslin cardinal.*
3. *The largest Suslin cardinal is* OD*-inaccessible.*

In the hierarchy of determinacy axioms, which one may appropriately call the Solovay Hierarchy[3], LSA is an anomaly as it belongs to the successor stage of the Solovay Hierarchy but does not conform to the general norms of the successor stages of the Solovay Hierarchy. Prior to [7], LSA was not known to be consistent. In [7], the first author showed that it is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals. Nowadays, the axiom plays a key role in many aspects of inner model theory, and features prominently in Woodin's Ultimate L framework (see [14, Definition 7.14] and Axiom I and Axiom II on page 97 of [14][4]).

Definition 0.7 *Let* $\mathsf{LSA}-\mathsf{over}-\mathsf{uB}$ *be the statement: For all V-generic g, in $V[g]$, there is $A \subseteq \mathbb{R}_g$ such that $L(A, \mathbb{R}_g) \vDash \mathsf{LSA}$ and Γ^∞_g is the Suslin co-Suslin sets of $L(A, \mathbb{R}_g)$.*

[8] shows that Sealing is equiconsistent with $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ over the theory ZFC+ "there is a proper class of Woodin cardinals and the class of measurable cardinals is stationary". In this paper, we show that in general, one cannot replace "equiconsistent" with "equivalent". Recall from [12] the statement of *Hod Pair Capturing* (HPC): A is a Suslin co-Suslin set, there is a least-branch (lbr) hod pair $(\mathcal{P}, \Sigma)$ such that A is definable from parameters over $(HC, \in, \Sigma)$. *No Long Extender* (NLE) is the statement: there is no countable, ω_1+1-iterable pure extender premouse M such that there is a long extender on the M-sequence.The notion of least-branch hod mice (lbr hod mice) is defined precisely in [12, Section 5].

[3]Solovay defined what is now called the Solovay Sequence (see [13, Definition 9.23]). It is a closed sequence of ordinals with the largest element Θ, where Θ is the least ordinal that is not a surjective image of the reals. One then obtains a hierarchy of axioms by requiring that the Solovay Sequence has complex patterns. LSA is an axiom in this hierarchy. The reader may consult [6] or [13, Remark 9.28].

[4]The requirement in these axioms that there is a strong cardinal which is a limit of Woodin cardinals is only possible if $L(A, \mathbb{R}) \vDash \mathsf{LSA}$.

Theorem 0.8 *Suppose self-iterability holds and there is a proper class of Woodin cardinals. Suppose* HPC *and* NLE *hold. Then* $V \models \mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ *fails.*

Remark 0.9 *1. The hypotheses of Theorem 0.8 hold in the universe of lbr hod mice that have a proper class of Woodin cardinals (cf. [12]). So such hod mice satisfy "*$\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ *fails."*

2. Woodin has independently shown that $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ *can fail. More precisely,* $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ *fails assuming there is a proper class of Woodin cardinals, a proper class of strong cardinals, and there is an inaccessible cardinal which is a limit of Woodin cardinals and strong cardinals.*

Remark 0.9(1), Theorem 0.8, and the fact that self-iterability and HPC hold in any generic extension of an lbr hod mouse with a proper class of Woodin cardinals give us the following.

Corollary 0.10 *Let V be the universe of an lbr hod mouse with a proper class of Woodin cardinals and a strong cardinal. Assume* NLE. *Let κ be the least strong cardinal of $\mathcal{P}$ and $g \subseteq Coll(\omega, \kappa^+)$ be V-generic. Then $V[g] \models$* Sealing *holds and* $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ *fails.*

Throughout this paper, except in Section 1, we assume the hypothesis of Theorem 0.4. Throughout this paper, except in Section 1, κ will stand for the least strong cardinal. In this paper, especially in Section 2, we will make heavy use of Neeman's "realizable maps are generic" result that appears as [4, Corollary 4.9.2]. Sections 4 and 5 make heavy use of the results of Section 2 to show that for V-generic $g \subseteq Coll(\omega, \kappa^+)$, where κ is as in Theorem 0.4, for $V[g]$ generic h, one can realize Γ^∞_{g*h} as the derived model of an iterate of a countable substructure of $V_\gamma[g*h]$ for some large γ (Lemma 5.1). This is then used to prove Theorem 0.4 in Section 6. The last section proves Theorem 0.8.

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1 Generically universally Baire iteration strategies

In this paper we will need three properties of iteration strategies, namely *Skolem-hull condensation*, *pullback condensation* and *generically universal Bairness*. We now define these notions.

We say $(\mathcal{P}, \Psi)$ is an *iterable pair* if $\mathcal{P}$ is a pre-iterable structure and Ψ is a strategy for it. Suppose $(\mathcal{P}, \Psi)$ is a pre-iterable pair. If $\mathcal{T}$ is a smooth iteration of $\mathcal{P}$ according to Ψ with last model $\mathcal{Q}$ then we write $\Psi_{\mathcal{T},\mathcal{Q}}$ for the strategy of $\mathcal{Q}$ induced by Ψ. Namely, $\Psi_{\mathcal{T},\mathcal{Q}}(\mathcal{U}) = \Psi(\mathcal{T}^\frown\mathcal{U})$. When $\Psi_{\mathcal{T},\mathcal{Q}}$ is independent of $\mathcal{T}$ we will drop it from our notation. Given a $\mathcal{P}$-cardinal ξ, we write $\Psi_{\mathcal{P}|\xi}$ for the fragment of Ψ that acts on smooth iterations based on $\mathcal{P}|\xi$. Here $\mathcal{P}|\xi = H^{\mathcal{P}}_{\xi}$.

Continuing with $(\mathcal{P}, \Psi)$, suppose $\pi : \mathcal{N} \to \mathcal{P}$ is elementary. Given a smooth iteration $\mathcal{T}$ of $\mathcal{N}$ we can define the copy $\pi\mathcal{T}$ on $\mathcal{P}$ which may or may not have well-founded models. The construction of $\pi\mathcal{T}$ was introduced in [3] on page 17. Suppose now that $\mathcal{T}$ is such that $\pi\mathcal{T}$ is according to Ψ and $\mathcal{T}$ is of limit length. Let $b = \Psi(\pi\mathcal{T})$. It follows from the construction of $\pi\mathcal{T}$ that b is a well-founded branch of $\mathcal{T}$.

We then say Λ is the π-pullback of Ψ if for any smooth iteration $\mathcal{T}$ on $\mathcal{N}$ that is according to Λ, $\pi\mathcal{T}$ is according to Ψ. It is customary to let Λ be Ψ^π.

Definition 1.1 *Suppose $(\mathcal{P}, \Psi)$ is an iterable pair. We say Ψ has Skolem-hull condensation if whenever $\mathcal{T}$ is an iteration according to Ψ, ξ is such that $\mathcal{T} \in V_\xi$ and $\pi : M \to V_\xi$ is elementary such that $(\mathcal{P}|\xi, \Psi_{\mathcal{P}|\xi}, \mathcal{T}) \in rng(\pi)$ then $\pi^{-1}(\mathcal{T})$ is according to $\Psi^{\pi}_{\mathcal{P}|\xi}$.*

Definition 1.2 *Suppose $(\mathcal{P}, \Psi)$ is an iterable pair. We say Ψ has pullback condensation if whenever $\mathcal{T}$ is an iteration according to Ψ with last model $\mathcal{Q}$ and $\mathcal{U}$ is an iteration of $\mathcal{Q}$ according to $\Psi_{\mathcal{T},\mathcal{Q}}$ with last model $\mathcal{R}$ then $\Psi^{\pi^{\mathcal{U}}}_{\mathcal{T}^\frown\mathcal{U},\mathcal{R}} = \Psi_{\mathcal{T},\mathcal{Q}}$.*

The following theorems are easy consequences of UBH (gUBH), and are probably not due to the authors.

Theorem 1.3 *Assume UBH and suppose λ is inaccessible. Then $V_\lambda \vDash \mathsf{UBH}$.*

Theorem 1.4 *Assume self-iterability and suppose Ψ is the unique strategy of $\mathcal{V}$. Then Ψ has Skolem-hull condensation and pullback condensation.*

Suppose $(\mathcal{P}, \Psi)$ is an iterable pair. Given a strong limit cardinal κ and $F \subseteq Ord$, set

$$W^{\Psi,F}_{\kappa} = (H_\kappa, F \cap \kappa, \mathcal{P}|\kappa, \Psi_{\mathcal{P}|\kappa} \restriction H_\kappa, \in).$$

Given a structure Q in a language extending the language of set theory with a transitive universe, and an $X \prec Q$, we let M_X be the transitive collapse of X and $\pi_X : M_X \to Q$ be the inverse of the transitive collapse. In general, the preimages of objects in X will be denoted by using X as a subscript. Suppose in addition $Q = (R, ...\mathcal{P}...)$ where $\mathcal{P}$ is a pre-iterable structure and Φ is an iteration strategy of $\mathcal{P}$. We will then write $X \prec (Q|\Phi)$ to mean that $X \prec Q$ and the strategy of $\mathcal{P}_X$ that we are interested in is Φ^{π_X}. We set $\Lambda_X = \Phi^{\pi_X}$.

Motivated by the definition of universally Baire sets that involves club of generically correct hulls, we make the following definition.

Definition 1.5 *We say Ψ is a generically universally Baire (guB) strategy for a pre-iterable $\mathcal{P} = (P, \vec{E})$ if there is a formula $\phi(x)$ in the language of set theory augmented by three relation symbols and $F \subseteq Ord$ such that for every inaccessible cardinal κ and for every countable*

$$X \prec (W_\kappa^{\Psi,F}|\Psi_{\mathcal{P}|\kappa})$$

whenever

(a) $g \in V$ is M_X-generic for a poset of size $< \kappa_X$ and

(b) $\mathcal{T} \in M_X[g]$ is such that for some inaccessible $\eta < \kappa_X$, $\mathcal{T}$ is an iteration of $\mathcal{P}_X|\eta$,

the following conditions hold:

1. *if $lh(\mathcal{T})$ is a limit ordinal and $\mathcal{T} \in dom(\Lambda_X)$ then $\Lambda_X(\mathcal{T}) \in M_X[g]$,*
2. *$\mathcal{T}$ is according to Λ_X if and only if $\mathcal{M}_X[g] \vDash \phi[\mathcal{T}]$.*

We say that (ϕ, F) is a generic prescription of Ψ.

In Definition 1.5, we could demand that there is a club of X with the desired properties. However that would be equivalent to our definition as we can let F above code the desired club. In the next section our goal is to prove some basic facts about *guB*-strategies.

2 Generic interpretability of guB strategies

As we said in the introduction, from this point on we work under the hypothesis of Theorem 0.4. However, we will not use the existence of a strong cardinal until Section 5.

Let Ψ be the guB-strategy of $\mathcal{V} = (V, \mathsf{ile}(\mathsf{V}))$ and fix a generic prescription (ϕ, F) for Ψ. We will omit Ψ, F from our notation and just write W_κ instead of $W_\kappa^{\Psi,F}$. Given a cardinal α we will write Ψ_α for the fragment of Ψ that acts on iterations based on $\mathcal{V}|\alpha$. Often we will treat Ψ_α as a strategy for $\mathcal{V}|\alpha$ rather than a strategy for $\mathcal{V}$. Similarly, given an interval (α, β) we will write $\Psi_{\alpha,\beta}$ for the fragment of Ψ on iterations based on $\mathcal{V}|\beta$ above α. To make the notation simpler, often we will not specify the domain of Ψ_α that we have in mind (as in Lemma 2.1).

Let δ be a Woodin cardinal of $\mathcal{V}$. We first prove that Ψ_δ has canonical extensions in generic extensions of V. As a first step, we prove the following useful capturing result.

Lemma 2.1 *Suppose λ is an inaccessible cardinal and let $X \prec (W_\lambda|\Psi_\delta)$ be countable. Set $\Phi = \pi_X^{-1}(\Psi_\delta)$. Then $\Lambda_X \restriction M_X = \Phi$.*

Proof. Let $\mathcal{U} \in M_X$ be such that $\mathcal{U} \in dom(\Phi) \cap dom(\Lambda_X)$. Set $b = \Lambda_X(\mathcal{U})$. It follows that $b \in M_X$. Because $M_X \vDash \mathsf{gUBH}$, it follows that $\Phi(\mathcal{U}) = b$. □

Theorem 2.2 *Suppose δ is a Woodin cardinal and $\eta \geq \delta$ is an inaccessible cardinal. Let $g \subseteq Coll(\omega, \eta)$ be generic. Then, in $V[g]$, there is an Ord-strategy Σ for $\mathcal{V}|\delta$ such that the following hold.*

1. *$\Psi_\delta \subseteq \Sigma$,*
2. *Letting Λ be the ω_1-fragment of Σ, $V[g] \vDash$ "Λ is universally Baire".*
3. *For all $V[g]$-generic h, letting Λ^h be the canonical extension of Λ to $V[g * h]$, $\Lambda^h \restriction V[g] \subseteq \Sigma$.*

Proof. Let $\lambda > \eta$ be an inaccessible cardinal. Set $W = W_\lambda$, $\mathcal{P} = \mathcal{V}|\delta$ and given a iteration $\mathcal{T}$ of $\mathcal{P}$ of limit length and a cofinal well-founded branch b of $\mathcal{T}$, set $\psi[\mathcal{T}, b] = \phi[\mathcal{T}^\frown\{b\}] \wedge \forall\alpha < lh(\mathcal{T})\phi[\mathcal{T} \restriction \alpha + 1]$.

Working in $V_\lambda[g]$, let Σ be the strategy given by ψ. More precisely, let Σ be defined as follows.

1. $\mathcal{T} \in dom(\Sigma)$ if and only if $lh(\mathcal{T})$ is of limit length and for every limit $\alpha < lh(\mathcal{T})$ if $b = [0, \alpha)_\mathcal{T}$ then $V_\lambda[g] \vDash \psi[\mathcal{T}, b]$.
2. $\Sigma(\mathcal{T}) = b$ if and only if $V_\lambda[g] \vDash \psi[\mathcal{T}, b]$.

The following is an immediate consequence of our definitions.

Lemma 2.3 *Suppose $X \prec (W|\Psi_\delta)$ is a countable. Let $k \in V$ be M_X-generic. Suppose $(\mathcal{U}, b) \in M_X[k]$ is such that $M_X[k] \vDash \psi[\mathcal{U}, b]$. Then $\mathcal{U} \in dom(\Lambda_X)$ and $\Lambda_X(\mathcal{U}) = b$.*

We now work towards showing that Σ is a total strategy.

Lemma 2.4 *Suppose $\mathcal{T} \in dom(\Sigma)$. Then there is at most one branch b such that $V[g] \vDash \psi[\mathcal{T}, b]$.*

Proof. Towards a contradiction assume not. Let $X \prec (W|\Psi_\delta)$ be countable and $k \subseteq Coll(\omega, \eta_X)$ be M_X-generic with $k \in V$. Fix now $\mathcal{U}, b, c \in M_X[k]$ such that $M_X[k] \vDash \psi[\mathcal{U}, b] \wedge \psi[\mathcal{U}, c]$. It follows from Lemma 2.3 that $b = \Lambda_X(\mathcal{U}) = c$. Therefore, $b = c$. $\square$

Lemma 2.5 *Suppose $\mathcal{T} \in dom(\Sigma)$. Then there is a branch b such that $V_\lambda[g] \vDash \psi[\mathcal{T}, b]$.*

Proof. Towards a contradiction assume not. Let $X \prec (W|\Psi_\delta)$ be countable and $k \subseteq Coll(\omega, \eta_X)$ be M_X-generic. It follows that there is an iteration $\mathcal{U} \in M_X[k]$ of $\mathcal{P}_X$ such that

(a) for every $\alpha < lh(\mathcal{U})$, letting $b_\alpha = [0, lh(\mathcal{U}))_\mathcal{U}$, $M_X[k] \vDash \psi[\mathcal{U} \restriction \alpha, b_\alpha]$ but
(b) for no well-founded cofinal branch $b \in M_X[k]$ of $\mathcal{U}$, $M_X[k] \vDash \psi[\mathcal{U}, b]$.

It follows from (a) and Lemma 2.3 that $\mathcal{U} \in dom(\Lambda_X)$. Hence, setting $\Lambda_X(\mathcal{U}) = b$, $b \in M_X[k]$ and $M_X[k] \vDash \phi[\mathcal{U}^\frown\{b\}]$. Therefore, $M_X[k] \vDash \psi[\mathcal{U}, b]$. $\square$

Lemma 2.6 *Let $X \prec (W|\Psi_\delta)$ be countable and let $k \in V$ be M_X-generic for $Coll(\omega, \eta_X)$. Let Φ be the strategy of $\mathcal{P}_X$ defined by ψ in $M_X[k]$. Then $\Lambda_X \restriction HC^{M_X[k]} = \Phi$.*

Proof. Suppose that $\mathcal{T} \in M_X[k]$ is according to both Λ_X and Φ. Set $b = \Phi(\mathcal{T})$. Because $\Phi(\mathcal{T}) = b$ we have that $M_X[k] \vDash \phi[\mathcal{T}^\frown\{b\}]$. Hence, $\Lambda_X(\mathcal{T}) = b$. $\square$

Corollary 2.7 $V_\lambda[g] \vDash$ "*Σ is a total strategy extending $\Psi_\delta \restriction V_\lambda$*".

Proof. Lemma 2.4 and Lemma 2.5 imply that Σ is a total strategy. To show that it extends $\Psi_\delta \restriction V_\lambda$, we reflect. Let $X \prec (W|\Psi_\delta)$ be countable and let $k \subseteq Coll(\omega, \eta_X)$ be M_X-generic such that $k \in V$. Let Φ be the strategy of $\mathcal{P}_X$ defined by ψ over $M_X[k]$. It follows from Lemma 2.6 that $\Phi = \Lambda_X \restriction (M_X[k])$. It follows from Lemma 2.1 that $\Lambda_X \restriction M_X = \pi_X^{-1}(\Psi_\delta)$. Hence, $\pi_X^{-1}(\Psi_\delta) \subseteq \Phi$. $\square$

We now work towards showing that $\Lambda =_{def} \Sigma \restriction HC^{V[g]}$ is universally Baire. For this it is enough to show that ψ is generically correct. More precisely, it is enough to show that in $V[g]$, for a club of $X \prec (W, \Psi_\delta)$ such that $V_\eta \cup \{\eta\} \subseteq X$, whenever $k \in V[g]$ is $M_X[g]$-generic and $(\mathcal{T}, b) \in M_X[g][k]$,

$$M_X[g][h] \vDash \psi[\mathcal{T}, b] \leftrightarrow V[g] \vDash \psi[\mathcal{T}, b].$$

Working in V, fix $X \prec H_{\lambda^+}$ such that $W, \Psi_\delta \in X$. It is enough to show that our claim holds in M_X. Let $k \in V$ be $Coll(\omega, \eta_X)/M_X$-generic. Let Φ be the strategy defined by ψ over $M_X[k]$ and $\Psi = \pi_X^{-1}(\Psi_\delta)$. Let $Y \prec (W_X|\Psi)$ be any countable substructure in $M_X[k]$ such that $V_{\eta_X} \cup \{\eta_X\} \in X$ and let $h \in M_X[k]$ be $M_Y[k]$-generic. Fix $(\mathcal{T}, b) \in M_Y[k][h]$.

Suppose now that $M_Y[k][h] \vDash \psi[\mathcal{T}, b]$. Because $\pi_X[Y] \in V$ we have that $\mathcal{T}$ is according to Λ_Y and $\Lambda_Y(\mathcal{T}) = b$. But because $\pi_Y \restriction \eta_X = id$, we have that $\Lambda_Y = \Lambda_X$. Therefore, $\mathcal{T}$ is according to Λ_X and $\Lambda_X(\mathcal{T}) = b$. It follows from Lemma 2.6 that $\Phi(\mathcal{T}) = b$. The reader can easily verify that these implications are reversible, and so if $\Phi(\mathcal{T}) = b$ then $M_Y[k][h] \vDash \psi[\mathcal{T}, b]$.

Finally, we need to verify that if h is $V[g]$-generic for a poset of size $< \lambda$ then $\Lambda^h \restriction V_\lambda[g] = \Sigma$. This again can be verified by first reflecting in V. Indeed, working in V, fix $X \prec H_{\lambda^+}$ such that $W, \Psi_\delta \in X$. Let (k, Φ, Ψ) be as above. Let $\Gamma = \Phi \restriction HC^{M_X[k]}$. Let $h \in V$ be any $M_X[k]$-generic. We want to see that $\Gamma^h \restriction M_X[k] \subseteq \Phi$. To see this, let $\mathcal{T} \in \mathcal{M}_X[k]$ be according to both Γ^h and Φ. Let $b = \Gamma^h(\mathcal{T})$. It follows that $M_X[k][h] \vDash \psi[\mathcal{T}, b]$. Hence, $\mathcal{T} \in dom(\Lambda_X)$ and $\Lambda_X(\mathcal{T}) = b$. It follows from Lemma 2.6 that $\Phi(\mathcal{T}) = b$.

Thus far we have shown that Theorem 2.2 holds in $V_\lambda[g]$ for any inaccessible $\lambda > \eta$. Let Σ_λ be the strategy defined above. To finish the proof of Theorem 2.2 it is enough to show that if $\lambda_0 < \lambda_1$ are two inaccessible cardinals bigger than η then $\Sigma_{\lambda_1} \restriction V_{\lambda_0}[g] = \Sigma_{\lambda_0}$. This can be verified by a reflection argument similar to the ones given above.

Indeed, let $X \prec H_{\lambda_1^+}$ be such that $\{\lambda_0, W_{\lambda_0}, W_{\lambda_1}\} \in X$. Let $k \subseteq Coll(\omega, \eta_X)$ be M_X-generic such that $k \in V$. Let Φ_0 and Φ_1 be the versions of Σ_{λ_0} and Σ_{λ_1} in $M_X[k]$. It follows from Lemma 2.6 that for $i \in 2$, $\Phi_i = \Lambda_X \restriction M_{X \cap W_{\lambda_i}}[k]$. Therefore, $\Phi_0 \subseteq \Phi_1$. □

We record a useful corollary to the proof of Theorem 2.2. We let ψ be the formula used in the proof of Theorem 2.2. If Σ is as in Theorem 2.2 and k is $V[g]$-generic then we let Σ^k be the extension of Σ to $V[k]$.

Corollary 2.8 *Suppose λ is an inaccessible cardinal and k is $V[g]$-generic for a poset in $V_\lambda[g]$. Then $\Sigma^k \restriction V_\lambda[g][k]$ is defined via ψ. More precisely, the following conditions hold.*

1. $\mathcal{T} \in dom(\Sigma^k) \cap V_\lambda[g*k]$ *if and only if for every limit* $\alpha < lh(\mathcal{T})$*, setting* $b_\alpha = [0,\alpha)_\mathcal{T}$*,* $V_\lambda[g*k] \vDash \psi[\mathcal{T} \upharpoonright \alpha, b_\alpha]$*.*

2. *For* $\mathcal{T} \in dom(\Sigma^k) \cap V_\lambda[g*k]$*,* $\Sigma^k(\mathcal{T}) = b$ *if and only if* $V_\lambda[g*k] \vDash \psi[\mathcal{T}, b]$*.*

As the definition of Σ uses only parameters from V, it follows that in all generic extensions $V[h]$ of V, Ψ_δ has an extension Ψ_δ^h. For instance, we can define $\Psi_\delta^h(\mathcal{U})$ by first selecting η such that h is generic for a poset in V_η and $\mathcal{U} \in V_\eta[h]$ then picking a generic $g \subseteq Coll(\omega, \eta)$ such that $V[h] \subseteq V[g]$ and then finally setting $\Psi_\delta^h(\mathcal{U}) = \Sigma(\mathcal{U})$ where Σ is as in Theorem 2.2.

3 Some correctness results

Say $u = (\eta, \delta, \lambda)$ is a *good triple* if it is increasing, δ is a Woodin cardinal, and λ is an inaccessible cardinal. Fix a good triple and set $\Phi = \Psi_\delta \upharpoonright H_\lambda$. The goal of this section is to show that many Skolem hulls of Φ are computed correctly. We start by showing that a stronger form of Lemma 2.1 hold.

Lemma 3.1 *Suppose* $X \prec ((W_\lambda, u)|\Phi)$ *is countable and* $k \in V$ *is* M_X*-generic. Then*

$$\Phi_X^k \upharpoonright (M_X|\nu_X[k]) = \Lambda_X \upharpoonright (M_X|\nu_X[k])^{5}.$$

Proof. Fix $\mathcal{T} \in dom(\Phi_X^k) \cap dom(\Lambda_X)$ and set $\Phi_X^k(\mathcal{T}) = b$. It follows from Corollary 2.8 that $\mathcal{M}_X[k] \vDash \psi[\mathcal{T}, b]$. Therefore, $\Lambda_X(\mathcal{T}) = b$. □

The following is a straightforward corollary of Lemma 3.1 and can be proven by a reflection.

Corollary 3.2 *Suppose* g *is generic for a poset in* V_η *and* $X \prec ((W_\lambda, u)|\Phi^g)$ *is countable in* $V[g]$*. Let* $k \in V[g]$ *be* M_X*-generic. Then*

$$\Phi_X^k \upharpoonright (M_X|\nu_X[k]) = \Lambda_X \upharpoonright (M_X|\nu_X[k]).$$

Corollary 3.3 *Suppose* g *is generic for a poset in* V_η *and* $i : \mathcal{V} \to \mathcal{P}$ *is an iteration embedding via a normal iteration* $\mathcal{T}$ *of length* $< \lambda$ *that is based on* $\mathcal{V}|\delta$ *and is according to* Φ*. Then* $i(\Phi) = \Phi^g_{\mathcal{P}|i(\delta)} \upharpoonright \mathcal{P}$*.*

[5] Here Φ_X^k is the generic interpretation of Φ_X in $M_X[k]$ using the definition of Φ given in Theorem 2.2.

Proof. It is enough to prove the claim in some M_Z where $Z \prec ((H_{\lambda^{++}}, W_\lambda, u, \Phi)|\Phi)$ is countable. Let $h \in V$ be M_Z-generic for a poset in $M_Z|\eta_Z$, and let $\mathcal{U} \in M_Z|\lambda_Z[h]$ be a normal iteration of M_Z according to Φ_Z with last model $\mathcal{Q}$. We want to see that $\pi^{\mathcal{U}}(\Phi_Z) = (\Phi_Z)^h_{\mathcal{Q}|\pi^{\mathcal{U}}(\delta_Z)} \restriction \mathcal{Q}$.

Let $\mathcal{R}$ be the last model of $\pi_Z\mathcal{U}$ and $\sigma : \mathcal{Q} \to \mathcal{R}$ come from the copying construction. It follows from [4, Theorem 4.9.1] that σ is generic over $\mathcal{R}$ and $\mathcal{R}[\sigma] \in V$. It then follows from Corollary 3.2 that $\pi^{\mathcal{U}}(\Phi_Z) = (\pi^{\pi_Z\mathcal{U}}(\Phi))^\sigma$. It again follows from Corollary 3.2 that $\Phi^h_Z = \Lambda_Z \restriction M_Z[h]$, and hence

$$(\Phi_Z)^h_{\mathcal{Q}|\pi^{\mathcal{U}}(\delta_Z)} = (\Lambda_Z)_{\mathcal{Q}|\pi^{\mathcal{U}}(\delta_Z)} \restriction M_Z[h] = (\pi^{\pi_Z\mathcal{U}}(\Phi))^\sigma = \pi^{\mathcal{U}}(\Phi_Z).$$

□

Corollary 3.4 *Suppose g is generic for a poset in V_η and $i : \mathcal{V} \to \mathcal{P}$ is an iteration embedding via a normal iteration $\mathcal{T}$ of length $< \lambda$ that is based on $\mathcal{V}|\delta$ and is according to Φ. Let $X \prec ((W_\lambda, u)|\Phi^g)$ be countable in $V[g]$ and let $\mathcal{Q} \in HC^{V[g]}$ be such that there are embeddings $\sigma : M_X \to \mathcal{Q}$ and $\tau : \mathcal{Q} \to \mathcal{P}$ with the property that $i \circ \pi_X = \tau \circ \sigma$. Then for any $\mathcal{Q}$-generic $k \in V[g]$,*

$$(\sigma(\Phi_X))^k = (\tau\text{-}\mathit{pullback\ of\ } \Phi^g_{\mathcal{P}|i(\delta)}) \restriction \mathcal{Q}[k].$$

Proof. It is enough to prove the claim assuming g is trivial. The more general claim then will follow by using the proof of Corollary 3.3. It follows from [4, Corollary 4.9.2] that τ is generic over $\mathcal{P}$ and that $\mathcal{P}[\tau]$ is a definable class of V. Applying Corollary 3.2 and Corollary 3.3 in $\mathcal{P}$, we get that

$$(\sigma(\Phi_X))^k = (\tau\text{-pullback of } \Phi^g_{\mathcal{P}|i(\delta)}) \restriction \mathcal{Q}[k].$$

□

Corollary 3.5 *Suppose $i : \mathcal{V} \to \mathcal{P}$ is an iteration embedding via a normal iteration $\mathcal{T}$ of length $< \lambda$ that is based on $\mathcal{V}|\delta$ and is according to Φ. Let $h \in V$ be $\mathcal{P}$-generic for a poset in $\mathcal{P}|\lambda$. Then $i(\Phi)^h = \Phi_{\mathcal{P}|i(\delta)} \restriction \mathcal{P}[h]$.*

Proof. It is enough to prove the claim in some M_Z where $Z \prec ((H_{\lambda^{++}}, W_\lambda, u, \Phi)|\Phi)$. Let $\mathcal{U}$ and $j : M_Z \to \mathcal{Q}$ play the role of $(\mathcal{T}, i, \mathcal{P})$. Let $G \in M_Z$ be $\mathcal{Q}$-generic for a poset in $\mathcal{Q}|\lambda_X$. We want to see that $j(\Phi_Z)^G = (\Phi_Z)_{\mathcal{Q}|j(\delta_Z)} \restriction \mathcal{Q}[G]$. Let $\mathcal{T} = \pi_Z\mathcal{U}$, $\mathcal{P}$ the last model of $\mathcal{T}$, $j = \pi^{\mathcal{T}}$ and $\tau : \mathcal{Q} \to \mathcal{P}$ be the copy map.

It follows from Corollary 3.4 that $j(\Phi_Z)^G = (\tau\text{-pullback of } \Phi_{\mathcal{P}|i(\delta)}) \restriction \mathcal{Q}[G]$. But because $\Phi_Z = \Lambda_Z \restriction M_Z$ (see Lemma 3.1),

$$(\tau\text{-pullback of } \Phi_{\mathcal{P}|i(\delta)}) \restriction \mathcal{Q}[G] = (\Phi_Z)_{\mathcal{Q}|j(\delta_Z)} \restriction \mathcal{Q}[G].$$

□

Suppose M is a transitive model of set theory and ν is its least strong cardinal. Suppose $M \vDash$ "$u = (\eta, \delta, \lambda)$ is a good triple" and suppose $\mathcal{T}$ is a normal iteration of M. We say $\mathcal{T}$ is a *sealed* iteration if $\mathcal{T} = \mathcal{T}_0^\frown\{E_0\}$ is such that

1. $\mathcal{T}_0$ is a normal iteration of M of successor length based on $M|\delta$ with last model N,
2. $\mathcal{T}_0$ is above ν,
3. $E_0 \in N$ is an extender such that $\text{crit}(E_0) = \nu$, $lh(E_0) > \pi^{\mathcal{T}_0}(\delta)$,
4. N has an inaccessible cardinal in the interval $(\pi^{\mathcal{T}_0}(\delta), lh(E_0))$.

Clearly the last model of $\mathcal{T}$ is $Ult(M, E_0)$. We say that a normal iteration $\mathcal{T}$ is a stack of sealed iterations if for some $n < \omega$, $\mathcal{T} = \oplus_{i \leq n}\mathcal{T}_i$ such that $\mathcal{T}_i$ is a sealed iteration of its first model.

Corollary 3.6 *Suppose $u = (\eta, \delta, \lambda)$ is a good triple, g is generic for a poset in V_η and $\mathcal{T} \in V_\lambda[g]$ is a normal iteration of $\mathcal{V}$ that is a stack of sealed iterations and is according to Φ^g where $\Phi = \Psi_\delta$. Set $\mathcal{T} = \oplus_{i\leq n}\mathcal{T}_i$ and let $\mathcal{P}$ be the last model of $\mathcal{T}_{n-1}$ if $n > 0$ and $\mathcal{V}$ otherwise. Let $\mathcal{T}_n = (\mathcal{U}, E)$ and let $\mathcal{Q}$ be the last model of $\mathcal{U}$. Set $\nu = \pi^{\mathcal{U}}(\pi^{\oplus_{i<n}\mathcal{T}_i}(\delta))$. Then $\Phi^g_{Ult(\mathcal{P},E)|\nu} = \Phi^g_{\mathcal{Q}|\nu}$.*

Proof. We prove the claim in some M_Z where $Z \prec ((H_{\lambda^+}, W_\lambda, u, \Phi)|\Phi)$ is countable. Let h be M_Z-generic for a poset in $M_Z|\eta_Z$ and let $(\mathcal{W}, \mathcal{R}, \mathcal{W}_n, \mathcal{S}, \mathcal{X}, F, \xi) \in M_Z[h]$ play the role of $(\mathcal{T}, \mathcal{P}, \mathcal{T}_n, \mathcal{Q}, \mathcal{U}, E, \nu)$.

Changing our notation, let $\mathcal{P}$ be the last model of the π_Z-copy of $\oplus_{i<n}\mathcal{W}_i$ and let $\sigma : \mathcal{R} \to \mathcal{P}$ be the copy map. We have that σ is generic over $\mathcal{P}$ (see [4, Corollary 4.9.2]) and $\mathcal{P}[\sigma]$ is a definable classof V. Let $\mathcal{Q}$ be the last model of $\sigma\mathcal{X}$ and let $\tau_0 : \mathcal{S} \to \mathcal{Q}$ and $\tau_1 : Ult(\mathcal{R}, F) \to Ult(\mathcal{P}, \tau_0(F))$ come from the copying construction. Notice that

$$\tau_0 \restriction (\mathcal{S}|lh(F)) = \tau_1 \restriction (\mathcal{S}|lh(F)).$$

We then let τ be this common embedding. Set $\tau_0(F) = E$ and $\nu = \tau_0(\xi)$. We have that τ_0 and τ_1 are generic over $\mathcal{Q}$ and $Ult(\mathcal{P}, E)$ respectively.

We now want to see that in $M_Z[h]$, $(\Phi^h_Z)_{Ult(\mathcal{R},F)|\xi} = (\Phi^h_Z)_{\mathcal{S}|\xi}$. Notice that it follows from Lemma 3.1 that $\Phi^h_Z = \Lambda_Z \restriction M_Z[h]$. Let $\Gamma_0 = (\tau\text{-pullback of } \Phi_{\mathcal{Q}|\nu})$ and

$\Gamma_1 = (\tau\text{-pullback of } \Phi_{Ult(\mathcal{P},E)|\nu})$. It follows that

(0) $\Gamma_0 \restriction M_Z[h] = (\Phi_Z^h)_{\mathcal{S}|\xi}$ and $\Gamma_1 \restriction M_Z[h] = (\Phi_Z^h)_{Ult(\mathcal{R},F)|\xi}$.

Let $i : \mathcal{V} \to \mathcal{Q}$ and $j : \mathcal{V} \to Ult(\mathcal{P}, E)$ be the iteration maps. It follows from Corollary 3.3 that

(1) $\Phi_{\mathcal{Q}|\nu} \restriction \mathcal{Q} = i(\Phi)_\nu$ and $\Phi_{Ult(\mathcal{P},E)|\nu} = j(\Phi)_\nu$.

Because $Ult(\mathcal{P}, E)|lh(E) = \mathcal{Q}|lh(E)$, $lh(E) > \nu$ is an inaccessible cardinal in $\mathcal{Q}$, and $\mathcal{Q}|lh(E) \vDash \mathsf{gUBH}$, we have that

(2) $i(\Phi)_\nu \restriction (\mathcal{Q}|lh(E)) = j(\Phi)_\nu \restriction (\mathcal{Q}|lh(E)) =_{def} \Sigma$

implying by the way of (1) that

(3) $\Phi_{\mathcal{Q}|\nu} \restriction (\mathcal{Q}|lh(E)) = \Phi_{Ult(\mathcal{P},E)|\nu} \restriction (\mathcal{Q}|lh(E))$.

Using [4, Corollary 4.9.2] we can find $H \in V$ that is $\mathcal{Q}$-generic for a poset in $\mathcal{Q}|\nu$ and is such that $\tau_0 \in \mathcal{Q}[H]$. It now follows that $\tau \in Ult(\mathcal{P}, E)[H]$ as $\tau \in \mathcal{Q}|lh(E)[H]$. We now have that

(4) $(\Sigma^H)^{Ult(\mathcal{P},E)[H]} \restriction (\mathcal{Q}|lh(E)[H]) = (\Sigma^H)^{\mathcal{Q}[H]} \restriction (\mathcal{Q}|lh(E)[H])$.

Applying Corollary 3.5 to (4) we get that

(5) $\Phi_{\mathcal{Q}|\nu} \restriction (\mathcal{Q}|lh(E)[H]) = \Phi_{Ult(\mathcal{P},E)|\nu} \restriction (\mathcal{Q}|lh(E)[H])$.

iI follows from (5) that

(6) the ω_1-fragments of Γ_0 and Γ_1 are equal.

(6) then implies, by the way of (0), that $(\Phi_Z^h)_{Ult(\mathcal{R},F)|\xi} = (\Phi_Z^h)_{\mathcal{S}|\xi}$. $\square$

4 Capturing universally Baire sets

The following is a useful corollary of Theorem 2.2. Given a limit of Woodin cardinals ν and $g \subseteq Coll(\omega, <\nu)$, let

1. $\mathbb{R}^*_g = \cup_{\alpha<\nu} \mathbb{R}^{V[g \cap Coll(\omega,\alpha)]}$,
2. Δ_g be the set of reals $A \in V(\mathbb{R}^*)$ such that for some $\alpha < \nu$, there is a pair $(T, S) \in V[g \cap Coll(\omega, \alpha)]$ such that $V[g \cap Coll(\omega, \alpha)] \models$ "(T, S) are $< \nu$-complementing trees" and $p[T]^{V(\mathbb{R}^*)} = A$, and
3. $DM(g) = L(\Delta_g, \mathbb{R}^*_g)$.

Corollary 4.1 *Suppose ν is a limit of Woodin cardinals. Let $\delta < \nu$ be a Woodin cardinal, and let $g \subseteq Coll(\omega, <\nu)$ be V-generic. Then $\Psi^g_\delta \in DM(g)$.*

We next need a characterization of universally Baire sets via strategies. We show this in Lemma 4.4. The lemma is standard.

If ν is a Woodin cardinal we let EA_ν be the ω-generator version of the extender algebra associated with ν. Recall that we say the triple (M, δ, Φ) *Suslin, co-Suslin captures*[6] the set of reals B if there is a pair $(T, S) \in M$ such that $M \models$ "(T, S) are δ-complementing[7]" and

1. M is a countable transitive model of some fragment of ZFC,
2. Φ is an ω_1-strategy for M,
3. $M \models$ "δ is a Woodin cardinal",
4. for $x \in \mathbb{R}$, $x \in B$ if and only if there is an iteration $\mathcal{T}$ of M according to Φ with last model N such that x is generic over N for $\mathsf{EA}^N_{\pi^{\mathcal{T}}(\delta)}$ and $x \in p[\pi^{\mathcal{T}}(T)]$.

The next lemma is standard and originates in [3].

Lemma 4.2 *Suppose $u = (\eta, \delta, \lambda)$ is a good triple and g is V-generic for a poset in V_η. Suppose $X \prec (W_\lambda[g] | \Psi^g_{\eta,\delta})$ is countable in $V[g]$. Then whenever $\mathcal{T}$ is a countable iteration of M_X according to Λ_X with last model N, there is $\sigma : N \to W_\lambda[g]$ such that $\pi_X = \sigma \circ \pi^{\mathcal{T}}$.*

[6]This notion is probably due to Steel, see [11].

[7]For any poset $\mathbb{P}$ of size δ, for any generic $g \subseteq \mathbb{P}$, $V[g] \models$ "$p[T] = \mathbb{R} - p[S]$".

Proof. Let $\mathcal{P} = W_\lambda[g]$. Let $\pi_X\mathcal{T}$ be the copy of $\mathcal{T}$ on $V[g]$. Let W be the last model of $\pi_X\mathcal{T}$. There is then $\tau : N \to \pi^{\mathcal{U}}(\mathcal{P})$ such that $\pi^{\mathcal{U}} \circ \pi_X = \tau \circ \pi^{\mathcal{T}}$. It follows by absoluteness that there is $m : N \to \pi^{\mathcal{U}}(\mathcal{P})$ with $m \in W$ such that $\pi^{\mathcal{U}}(\pi_X) = m \circ \pi^{\mathcal{T}}$. The existence of σ follows from elementarity. $\square$

The next lemma is also standard, but we do not know its origin. To state it we need the following diversion.

A diversion.

Suppose M is a countable transitive model of set theory and Φ is a strategy of M. Let (η, g) be such that g is M-generic for a poset in $M|\eta$. Let Φ' be the fragment of Φ that acts on iterations that are above η. Then Φ' can be viewed as an iteration strategy of $M[g]$. This is because if $\mathcal{T}$ is an iteration of $M[g]$ above η, there is an iteration $\mathcal{U}$ of M that is above η and such that

1. $lh(\mathcal{T}) = lh(\mathcal{U})$,
2. $\mathcal{T}$ and $\mathcal{U}$ have the same tree structure,
3. for each $\alpha < lh(\mathcal{T})$, $M_\alpha^{\mathcal{T}} = \mathcal{M}_\alpha^{\mathcal{U}}[g]$,
4. for each $\alpha < lh(\mathcal{T})$, $E_\alpha^{\mathcal{T}}$ is the extension of $E_\alpha^{\mathcal{U}}$ onto $M_\alpha^{\mathcal{U}}[g]$.

Let Φ'' be the strategy of $M[g]$ with the above properties. We then say that Φ'' is induced by Φ'. We will often confuse Φ'' with Φ'.

Corollary 4.3 *Suppose (η, δ, λ) is a goof triple, g is generic for a poset of size $< \eta$ and $h \subseteq Coll(\omega, \lambda)$ is generic over V such that $V[g] \subseteq V[h]$. Let Σ be as in Theorem 2.2 applied to h and Ψ_δ, and let Φ be the fragment of $\Sigma \restriction V[g]$ that acts on iterations that are above η. Then Φ induced a strategy Φ' for $\mathcal{V}|\delta[g]$, and Φ' is projective in Φ.*

This ends our diversion, and we can now state our lemma.

Lemma 4.4 *Suppose $u = (\eta, \delta, \lambda)$ is a good triple and g is V-generic for a poset in V_η. Let $A \in \Gamma_g^\infty$. Then, in $V[g]$, there is a club of countable $X \prec (W_\lambda[g]|\Psi_{\eta,\delta}^g)$ such that $(M_X, \delta_X, \Lambda_X^g)$ Suslin, co-Suslin captures A. For each such X, let $X' = X \cap W_\lambda \prec W_\lambda$, and $(M_{X'}, \Lambda_{X'})$ be the transitive collapse of X' and its strategy. Then A is projective in $\Lambda_{X'}$. Moreover, these facts remain true in any further generic extension by a poset in $V_\eta[g]$.*

Proof. Let $\mathcal{P} = W_\lambda[g]$. Work in $V[g]$. Let (T, S) be λ-complementing trees such that $A = p[T]$. Let $X \prec W_\lambda[g]$ be countable such that $(T, S) \in X$. We claim that $(M_X, \delta_X, \Lambda^g_X)$ Suslin co-Suslin captures A. To see this fix a real x. Let $\mathcal{T}$ be any countable normal iteration of M_X such that

1. $\mathcal{T}$ is according to Λ^g_X,
2. $\mathcal{T}$ has a last model N,
3. x is generic for $\mathsf{EA}^N_{\pi^{\mathcal{T}}(\delta)}$.

Using Lemma 4.2, we can find $\sigma : N \to \mathcal{P}$ such that $\pi_X = \sigma \circ \pi^{\mathcal{T}}$.

Assume first $x \in A$. Then $x \in p[T]$. If now $x \notin p[\pi^{\mathcal{T}}(T_X)]$ then $x \in p[\pi^{\mathcal{T}}(S_X)]$ (this uses the fact that T_X, S_X are λ_X-complementing in M_X) and hence, $x \in p[S]$ (this follows from the fact that $\sigma[\pi^{\mathcal{T}}(S_X)] \subseteq S$). Thus, $x \in p[\pi^{\mathcal{T}}(T_X)]$.

Next suppose $x \in p[\pi^{\mathcal{T}}(T_X)]$. Then because $\sigma[\pi^{\mathcal{T}}(T_X)] \subseteq T$, $x \in p[T]$ implying that $x \in A$.

That Λ^g_X is projective in $\Lambda_{X'}$ follows from Corollary 4.3; hence A is projective in $\Lambda_{X'}$. We leave it to the reader to verify that these facts remain true in a further generic extension by a poset in $V_\eta[g]$. $\square$

5 A derived model representation of Γ^∞

In this section our goal is to establish a derived model representation of Γ^∞. We set $\iota = \kappa^+$ and fix $g \subseteq Coll(\omega, \iota)$.

We say $u = (\eta, \delta, \delta', \lambda)$ is a *good quadruple* if (η, δ, λ) and (η, δ', λ) are good triples with $\delta < \delta'$. Suppose $u = (\eta, \delta, \delta', \lambda)$ is a good quadruple and h is a $V[g]$-generic such that $g * h$ is generic for a poset in V_η. Working in $V[g * h]$, let $D(h, \eta, \delta, \lambda)$ be the club of countable

$$X \prec ((W_\lambda[g * h], u)|\Psi^g_{\eta,\delta})$$

such that $H^V_\iota \cup \{g\} \subseteq X$.

Suppose $A \in \Gamma^\infty_{g*h}$. Then for a club of $X \in D(h, \eta, \delta, \lambda)$, A is Suslin, co-Sulsin captured by $(M_X, \delta_X, \Lambda_X)$ and A is projective in Λ_X (see Lemma 4.4). Given such an X, we say X *captures* A.

Let $k \subseteq Coll(\omega, \Gamma^\infty_{g*h})$ be generic, and let $(A_i : i < \omega) = \Gamma^\infty_{g*h}$ and $(w_i : i < \omega) = \mathbb{R}_{g*h}$ be generic enumerations in $V[g * h * k]$. Let $(X_i : i < \omega) \in V[g * h * k]$ be such that for each i

1. $X_i \in D(h, \eta, \delta, \lambda)$, and

2. X_i captures A_i.

In particular, A_i is projective in $\Lambda_{X_i'}$, where $X_i' = X_i \cap W_\lambda$. We set $M_n^0 = M'_{X_n}$, $\pi_n^0 = \pi_{X_0}$, $\kappa_0 = \kappa_{X_0}$, $\nu_0 = \delta_{X_0}$, $\nu_0' = \delta'_{X_0}$, $\eta_0 = \eta_{X_0}$, $\delta_0 = \delta$, $\mathcal{P}_0 = \mathcal{V}$.

Next we inductively define sequences $(M_n^i : i, n < \omega)$, $(\pi_n^i : i, n < \omega)$, $(\Lambda_i : i \leq \omega)$, $(\tau_n^{i,i+1} : i, n < \omega)$, $(\delta_i : i < \omega)$, $(\nu_n : i < \omega)$, $(\nu_n' : i < \omega)$, $(\eta_n : n < \omega)$, $(\kappa_i : i < \omega)$, $(\theta_i : i < \omega)$, $(\mathcal{T}_i, E_i : i < \omega)$, $(M_i' : i < \omega)$, $(\mathcal{U}_i, F_i : i < \omega)$, $(\mathcal{P}_i : i \leq n)$, $(\mathcal{P}_i' : i < \omega)$, and $(\sigma_i : i < \omega)$ satisfying the following conditions (see Diagram 5.1).

(a) For all $i, n < \omega$, $\pi_n^i : \mathcal{M}_n^i \to \mathcal{P}_i$ and $rng(\pi_n^i) \subseteq rng(\pi_{n+1}^i)$.

(b) $\tau_n^{i,i+1} : M_n^i \to M_n^{i+1}$. Let $\tau_n : M_n^0 \to M_n^n$ be the composition of $\tau_n^{j,j+1}$'s for $j < n$.

(c) For all $i, n < \omega$, $\kappa_n = \tau_n(\kappa_0)$, $\eta_n = \tau_n(\eta_0)$, $\nu_n = \tau_n(\nu_0)$ and $\nu_n' = \tau_n(\nu_0')$.

(d) For all $n < \omega$, $\mathcal{T}_n$ is an iteration of $M_n^n|\nu_n'$ above ν_n that makes w_n generic and M_n' is its last model.

(e) $\theta_n = \pi^{\mathcal{T}_n}(\nu_n')$ and $E_n \in \vec{E}^{M_n'}$ is such that $lh(E_n) > \theta_n$ and $\mathrm{crit}(E_n) = \kappa_n$.

(f) for all m, n, $M_m^{n+1} = Ult(M_m^n, E_n)$ and $\tau_m^{n,n+1} = \pi_{E_n}^{M_m^n}$.

(g) $\mathcal{U}_n = \pi_n^n \mathcal{T}_n$, $\mathcal{P}_n'$ is the last model of $\mathcal{U}_n$, $\sigma_n : M_n' \to \mathcal{P}_n'$ is the copy map and $F_n = \sigma_n(E_n)$.[8]

(h) $\mathcal{P}_{n+1} = Ult(\mathcal{P}_n, F_n)$ and $\psi_m^{n+1} : \mathcal{M}_m^{n+1} \to \mathcal{P}_{n+1}$ is given by $\pi_m^{n+1}(\pi_{E_n}^{M_m^n}(f)(a)) = \pi_{F_n}^{\mathcal{P}_n}(\pi_m^n(f))(\sigma_n(a))$.

(i) $\Lambda_n = (\pi_n^n\text{-pullback of } (\Psi_\lambda^{g*h})_{\mathcal{P}_n|\psi_n(\nu_n)})_{\eta_n,\nu_n} = (\sigma_n\text{-pullback of } (\Psi_\lambda^{g*h})_{\mathcal{P}_n'|\sigma_n(\nu_n)})_{\eta_n,\nu_n}$ (see Corollary 3.6).

Let M_n^ω be the direct limit of $(M_n^m : m < \omega)$ under the maps $\tau_n^{m,m+1}$. Letting $\mathcal{P}_\omega$ be the direct limit of $(\mathcal{P}_n : n < \omega)$ and the compositions of $\pi_{F_n}^{\mathcal{P}_n}$, we have $\pi_n^\omega : M_n^\omega \to \mathcal{P}_\omega$ be the natural maps. Notice that

(1) for each $n < \omega$, $\kappa_n < \omega_1^{V[g*h]}$ and $sup_n \kappa_n = \omega_1^{V[g*h]}$.

It follows that if $\tau_n^m : M_n^m \to M_n^\omega$ is the direct limit embedding then

[8]So $\oplus_{i \leq n} \mathcal{T}_i$ and $\oplus_{i \leq n} \mathcal{U}_i$ are sealed iterations based on κ.

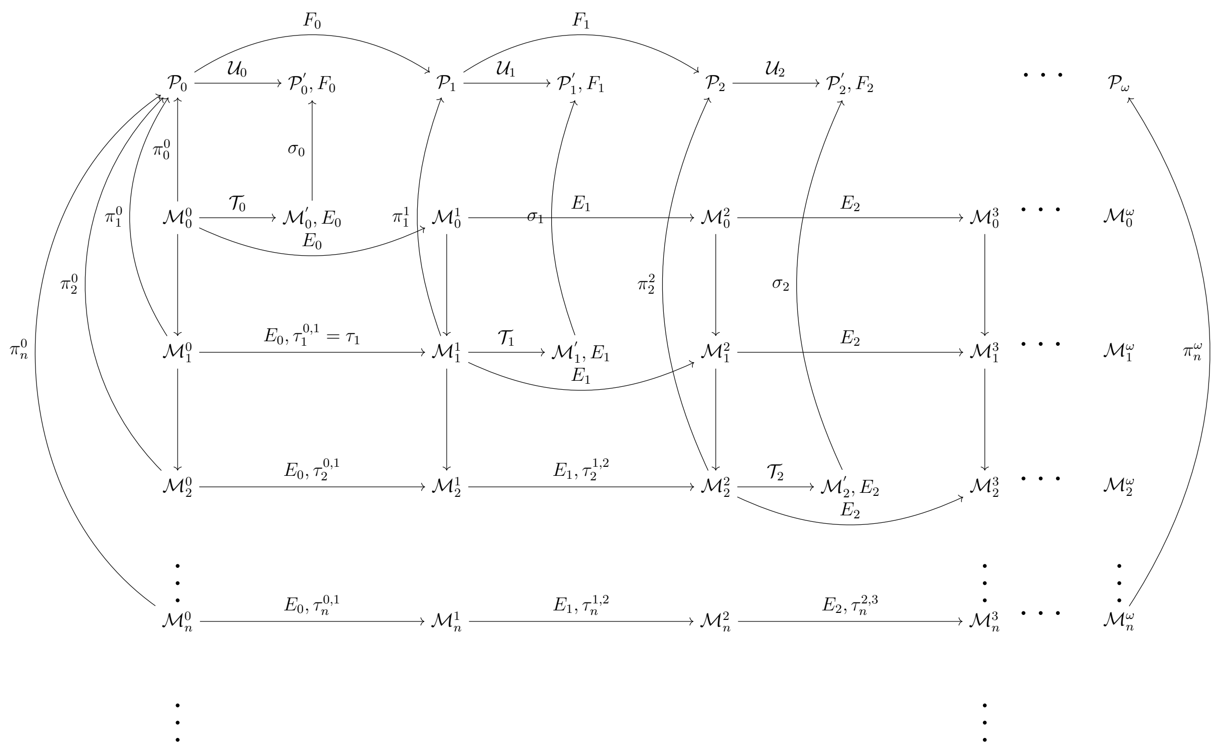

Figure 5.1: Diagram of the main argument

(2) $\tau_n^m(\kappa_n) = \omega_1^{V[g*h]}$.

Next, notice that

(3) for each m, n, p, letting $\iota_n = \tau_n(\iota_{X_0}) = \tau_n(\iota)$, $M_m^n|\iota_n = M_p^n|\iota_n$ and $\iota_n = (\kappa_n^+)^{M_m^n}$.
(4) for each m, n, p, $\pi_m^n \restriction (M_m^n|\iota_n) = \pi_p^n \restriction (M_p^n|\iota_n)$
(5) for each m, $n > 1$ and $p > n$, $M_m^n|\theta_{n-1} = M_m^p|\theta_{n-1}$.
(6) for each m, $n > 1$ and p with $p > n$, $\pi_m^n \restriction (M_m^n|\theta_{n-1}) = \pi_m^p \restriction (M_m^p|\theta_{n-1})$.

Because of condition (d) above we can find $G \subseteq Coll(\omega, <\omega_1^{V[g*h]})$ generic over M_0^ω such that $\mathbb{R}^{M_0^\omega[G]} = \mathbb{R}_{g*h}$ and $G \in V[g*h*k]$. It then follows from the results of Section 2 and Section 4 that

Lemma 5.1 $DM(G)^{M_0^\omega[G]} = L(\Gamma_{g*h}^\infty, \mathbb{R}_{g*h})$.

Proof. It follows from Corollary 3.6 and Lemma 4.4 that A_n is projective in Λ_n. It follows from Theorem 3.4 that $\Lambda_n \restriction HC^{V[g*h]} \in M_0^\omega[G]$ and it follows from Corollary 4.1 that $\Lambda_n \restriction HC^{V[g*h]} \in DM(G)^{M_0^\omega[G]}$. It follows that $\Gamma_{g*h}^\infty \subseteq DM(G)^{M_0^\omega[G]}$.

Moreover, it follows from Corollary 4.4 that any set in $DM(G)^{M_0^\omega[G]}$ is projective in some $\Lambda_n \restriction HC^{V[g*h]}$ and it follows from Theorem 2.2 that $\Lambda_n \restriction HC^{V[g*h]} \in \Gamma_{g*h}^\infty$. Thus, $DM(G)^{M_0^\omega[G]} \subseteq L(\Gamma_{g*h}^\infty, \mathbb{R}_{g*h})$. $\square$

We can also show variations of the above lemma for $\mathcal{M}_n^\omega$ for each $n < \omega$. Lemma 5.1 implies that in order to prove that Sealing holds, it is enough to establish clause 2 of Sealing as clause 1 immediately follows from Lemma 5.1.

To continue, it will be easier to introduce some terminology. We say that the sequence $(X_i : i < \omega)$ is cofinal in Γ_{g*h}^∞ as witnessed by $(A_i : i \in \omega)$ and $(w_i : i < \omega)$. We also say that $(M_0^n, \Lambda_n, \theta_n, \tau_{n,m} : n < m < \omega)$ is a Γ_{g*h}^∞-genericity iteration induced by $(X_i : i < \omega)$ where $\tau_{n,m} : M_0^n \to M_0^m$ is the composition of $\tau_0^{i,i+1}$ for $i \in [n, m)$.

6 A proof of Theorem 0.4

We now put together the results of the previous sections to obtain a proof of Theorem 0.4. Fix h and h' such that h is $V[g]$-generic and h' is $V[g*h]$-generic. We want to show that there is an embedding

$$j : L(\Gamma_{g*h}, \mathbb{R}_{g*h}) \to L(\Gamma_{g*h*h'}, \mathbb{R}_{g*h*h'})$$

such that for $A \in \Gamma_{g*h}$, $j(A) = A^{h'}$. Let $(\xi_i : i < \omega)$ be an increasing sequence of cardinals such that $g * h * h'$ is generic for a poset in V_{ξ_0}. Let $u_n = (\xi_i : i < n)$. Set $W = L(\Gamma_{g*h}, \mathbb{R}_{g*h})$ and $W' = L(\Gamma_{g*h*h'}, \mathbb{R}_{g*h*h'})$.

Because $(\Gamma_{g*h})^{\#}$ exists, there is only possibility for j as above. Namely, given a term τ, $n \in \omega$, $x \in \mathbb{R}_{g*h}$ and $A \in \Gamma^{\infty}_{g*h}$, we must have that

$$j(\tau^W(u_n, A, x)) = \tau^{W'}(u_n, A^{h'}, x).$$

What we must show is that j is elementary. The next lemma finishes the proof.

Lemma 6.1 *j is elementary.*

Proof. Let $u = (\eta, \delta, \delta', \lambda)$ be a good quadruple such that $\sup_{i<\omega} \xi_i < \eta$. Let $k \subseteq Coll(\omega, \Gamma^{\infty}_{g*h})$ be $V[g*h]$-generic and $k' \subseteq Coll(\omega, \Gamma^{\infty}_{g*h*h'})$ be $V[g*h*h']$-generic.

We have that Γ^{∞}_{g*h} is the Wadge closure of strategies of the countable substructures of $\mathcal{W}_\lambda$. More precisely, given $A \in \Gamma^{\infty}_{g*h}$, there is an $X \prec (W_\lambda | \Psi^{g*h}_{\eta,\delta})$ such that A is Wadge reducible to Λ_X. It follows that to show that j is elementary it is enough to show that given a formula ϕ, $m \in \omega$, $X \prec ((W_\lambda, u) | \Psi^{g*h}_{\eta,\delta})$ and a real $x \in \mathbb{R}_{g*h}$,

$$W \vDash \phi[u_m, \Lambda_X, x] \Rightarrow W' \vDash \phi[u_m, \Lambda^{h'}_X, x].$$

Fix then a tuple (ϕ, n, X, x) as above.

Working inside $V[g*h*k]$, let $(Y_i : i < \omega)$ be a cofinal sequence in Γ^{∞}_{g*h} as witnessed by some $\vec{A}$ and $\vec{w}$ such that $A_0 = \emptyset$, $w_0 = x$ and $Y_0 = X$.

Working inside $V[g*h'*k']$, let $(Z_i : i < \omega)$ be a cofinal sequence in $\Gamma^{\infty}_{g*h*h'}$ as witnessed by some $\vec{B}$ and $\vec{v}$ such that $B_0 = \emptyset$, $v_0 = x$ and $Z_0 = X$.

Let $(M_n, \Lambda_n, \theta_n, \tau_{n,l} : n < l < \omega)$ be a Γ^{∞}_{g*h}-genericity iteration induced by $(Y_i : i < \omega)$ and $(N_n, \Phi_n, \nu_n, \sigma_{n,l} : n < l < \omega)$ be a $\Gamma^{\infty}_{g*h*h'}$-genericity iteration induced by $(Z_i : i < \omega)$. It is not hard to see that we can make sure that $M_1 = N_1$ by simply selecting the same extender E_0 after $\mathcal{T}_0$.

Let $\zeta = \eta_X$ and $\Gamma = (\Psi_{\eta,\delta})_X$. Let M_ω be the direct limit along $(M_n : n < \omega)$ and N_ω the direct limit along $(N_n : n < \omega)$. For $n < \omega$, let κ_n be the least strong cardinal of M_n and κ'_n be the least strong cardinal of N_n. Let s^n_m be the first m indiscernibles of $(M_n | \kappa_n)$ and t^n_m be the first m indiscernibles of $(N_n | \kappa'_n)$. Notice that $(M_n|\kappa_n)^{\#} \in M_n$ and $(N_n|\kappa'_n)^{\#} \in N_n$. It follows that $\tau_{n,l}(s^n_m) = s^l_m$ and $\sigma_{n,l}(t^n_m) = t^l_m$ for $n < l \leq \omega$.

We then have the following sequence of implications. Below we let Γ^* be the name for the generic extension of Γ in the relevant model and $\dot{DM}$ be the name for the derived model.

$$
\begin{aligned}
W \vDash \phi[u_m, \Lambda_X, x] &\Rightarrow M_\omega[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa_\omega)} \dot{DM} \vDash \phi[s^\omega_m, \Gamma^*, x] \\
&\Rightarrow M_1[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa_1)} \dot{DM} \vDash \phi[s^1_m, \Gamma^*, x] \\
&\Rightarrow N_\omega[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa'_\omega)} \dot{DM} \vDash \phi[t^\omega_m, \Gamma^*, x] \\
&\Rightarrow W' \vDash \phi[u_m, \Lambda^{h'}_X, x].
\end{aligned}
$$

□

7 Tower Sealing

We recycle the notations from Section 5. We addtionally assume V is the universe of an lbr hod mouse. As before, we set $\iota = \kappa^+$ and fix $g \subseteq Coll(\omega, \iota)$ be V-generic. Let h be a $V[g]$-generic. In $V[g*h]$, let γ be a Woodin cardinal. Let $G \subseteq \mathbb{Q}_{<\gamma}$ be $V[g*h]$-generic and $j : V[g*h] \to M$ be the generic embedding induced by G (the proof for $\mathbb{P}_{<\delta}$ is similar). Since γ is Woodin, M is closed under ω-sequences in $V[g*h*G]$.

Let $D(h*G, \eta, \delta, \lambda)$ be defined as in Section 5 in $V[g*h*G]$. Here we may take η, δ, λ to be fixed points of j. We may also assume for any $A \in V[g*h*G]$ (M), if A is λ-Woodin in $V[g*h*G]$ (M, respectively), then A is uB in $V[g*h*G]$ (M, respectively). Let $\Gamma_0 = \Gamma^\infty_{g*h*G}$ and $\Gamma_1 = (\Gamma^\infty)^M$. We want to show that $\Gamma_0 = \Gamma_1$.

For $X \in D(h*G, \eta, \delta, \lambda)$, let $X' = X \cap W_\lambda$. Suppose X is closed under j, j^{-1}. Let $\pi : M_{X'} \to (W_\lambda, u)|\Psi_{\eta,\delta}$ be the uncollapse map, $j \circ \pi$ embeds $M_{X'}$ into $j((W_\lambda, u)|\Psi_{\eta,\delta})$. So letting $Y = X \cap j((W_\lambda, u)|\Psi_{\eta,\delta})$, then

$$\Lambda_{X'} \leq_w \Lambda^M_Y \tag{1}$$

where Λ^M_Y is defined in M as the strategy of M_Y, the transitive collapse of Y, obtained by pulling back M's strategy. Using the fact that X is closed under j, j^{-1}, $M_{X'}$ embeds elementarily into M_Y.

Claim 7.1 *Suppose $A \in \Gamma_0 \cup \Gamma_1$. There is a stationary set of $X \in D(h*G, \eta, \delta, \lambda)$ such that A is projective in $\Lambda_{X'}$.*

Proof. If $A \in \Gamma_0$, this follows from Lemma 4.4. If $A \in \Gamma_1$, then by applying Lemma 4.4 in M, we see that $A \leq_w \Lambda^M_Y$. It suffices then to show that for any club C, there is some $X \in C \cap D(h*G, \eta, \delta, \lambda)$ such that

$$\Lambda_{X'} \text{ is projective in } \Lambda^M_Y. \tag{2}$$

Let $X \in C \cap D(h * G, \eta, \delta, \lambda)$ be such that $sup(X \cap \delta)$ be minimal. Let Y be defined from X as above, then Y captures A in M. We need to see (2). Now both $(M_0 = M_{X'}[g * h]|\delta^{M_{X'}}, \Lambda_{X'})$ and $(M_1 = M_Y[g * h]|\delta^{M_Y}, \Lambda_Y^M)$ are both lbr hod pairs over the same base set. So in M, we can co-iterate them[9]. Let $\mathcal{T}$ be the iteration tree on the M_0-side with last model P and $\mathcal{U}$ be on the M_1-side with last model Q. By 1, $\Lambda_{X'} \leq_w \Lambda_Y^M$, so it is clear that $P \trianglelefteq Q$ (and the strategy of Q agrees with that of P on P, i.e. $\Lambda_Q \restriction P = \Lambda_P$).

We claim that $P = Q$. Let π_0 be the natural map from M_0 into M_1 (recall j maps $X'[g * h]$ into $Y[g * h]$ so π_0 is just the transitive collapse of that map) and π_1 be the natural map of M_1 into V_δ^M. Otherwise, $P \lhd Q$ and $\mathcal{T}$ does not drop on its main branch. Let $\sigma_0 : P \to V_\delta^M$ be such that $\sigma_0 \circ i^{\mathcal{T}} = \pi_1 \circ \pi_0$. Now note that π_0 is cofinal in δ^{M_Y}, so $\pi_1 \circ \pi_0$ is cofinal in the range of π_1. But $P \lhd Q$, so we get $\sup(\sigma_0 \circ i^{\mathcal{T}}[\delta^{M_{Y'}}]) < \sup(\pi_1 \circ \pi_0[\delta^{M_{Y'}}])$. Contradiction.

Since $A \leq_w \Lambda_P = \Lambda_Q$ and $\Lambda_{X'}$ is the $i^{\mathcal{T}}$-pullback of Λ_P, the conclusion of the lemma follows. $\square$

For X as in the claim, (1) tells us that

$$\Lambda_{X'} \in (\Gamma^\infty)^M. \tag{3}$$

Working in $V[g * h * G * k]$, where $k \subseteq Coll(\omega, \Gamma_0 \cup \Gamma_1)$ is $V[g * h * G]$-generic, let $(w_i : i < \omega)$ enumerate $\mathbb{R}^{V[g*h*G]}$, $(A_i : i < \omega)$ enumerate Γ_0, and $(B_i : i < \omega)$ enumerate Γ_1. Let $(X_i : i < \omega) \in V[g * h * k]$ be such that for each i

1. $X_i \in D(h * G, \eta, \delta, \lambda)$,
2. X_i is closed under j, j^{-1}, and
3. A_i and B_i are both projective in $\Lambda_{X_i'}$.

Note that by (3), for each i,

$$\Lambda_{X_i'} \in (\Gamma^\infty)^M \cap \Gamma^\infty_{g*h*G}. \tag{4}$$

3. follows from Claim 7.1. Now, using the sequence $(X_i' : i < \omega)$, we construct in $V[g * h * G * k]$, as in Section 5, the sequences $(M_n^i : i, n < \omega)$, $(\pi_n^i : i, n < \omega)$, $(\Lambda_i : i \leq \omega)$, $(\tau_n^{i,i+1} : i, n < \omega)$, $(\delta_i : i < \omega)$, $(\nu_n : i < \omega)$, $(\nu_n' : i < \omega)$, $(\eta_n : n < \omega)$, $(\kappa_i : i < \omega)$, $(\theta_i : i < \omega)$, $(\mathcal{T}_i, E_i : i < \omega)$, $(M_i' : i < \omega)$, $(\mathcal{U}_i, F_i : i < \omega)$, $(\mathcal{P}_i : i \leq n)$, $(\mathcal{P}_i' : i < \omega)$, and $(\sigma_i : i < \omega)$.

[9]The coiteration is into a common full backgrounded construction.

Let M_n^ω be the direct limit of $(M_n^m : m < \omega)$ under the maps $\tau_n^{m,m+1}$. Let $k \subseteq Coll(\omega, < \omega_1^{V[g*h*G]})$ be M_0^ω-generic; here recall from the construction in Section 5, $\omega_1^{V[g*h*G]}$ is a strong cardinal in M_0^ω and is the image of κ under the direct limit map. As before,

$$DM(k)^{M_0^\omega[k]} = L(\Gamma_0, \mathbb{R}_{g*h*G}). \tag{5}$$

Note that any finite initial segments of the above sequences are in M, as $X_i \in M$ for each i and $\mathbb{R}^M = \mathbb{R}^{V[g*h*G]}$. By the construction, (4), and the fact that B_i is projective in $\Lambda_{X_i'}$ for each i, we get

$$DM(k)^{M_0^\omega[k]} = L(\Gamma_1, \mathbb{R}^M). \tag{6}$$

Equations (5) and (6) give us $L(\Gamma_0, \mathbb{R}_{g*h*G}) = L(\Gamma_1, \mathbb{R}^M)$ and $\Gamma_0 = \Gamma_1$, as both sets are the Suslin co-Suslin sets of $DM(k)^{M_0^\omega[k]}$.

The rest of the clauses of $\mathsf{Tower\ Sealing}$ are proved as in the previous section.

8 $\mathsf{LSA} - \mathsf{over} - \mathsf{UB}$ may fail

In this section, we prove Theorem 0.8. We assume the hypotheses of Theorem 0.8. Here are some consequences of the hypotheses that we need (for (b), see Lemma 4.4):

(a) self-iterability (cf. Definition 0.3),

(b) letting λ be a limit of Woodin cardinals, and $g \subseteq Coll(\omega, < \lambda)$ be V-generic, then for any set A which is Suslin co-Suslin in the derived model given by g, $DM(g)$ (see Section 4), then A is Wadge reducible to $\Psi_\delta^g \restriction HC^{V[g]}$, for some Woodin cardinal $\delta < \lambda$. Furthermore, $\Psi_\delta^g \restriction HC^{V[g]} \in DM(g)$, in fact, $\Psi_\delta^g \restriction HC^{V[g]} \in \Gamma_g^\infty$.

Suppose for contradiction that $\mathsf{LSA} - \mathsf{over} - \mathsf{UB}$ holds. Let λ be an inaccessible cardinal which is a limit of Woodin cardinals in V. Let $h \subseteq Coll(\omega, < \lambda)$ be V generic. By our assumption, in $V[h]$, there is some set A such that

- $A \in V(\mathbb{R}^{V[h]})$;
- $L(A, \mathbb{R}) \vDash \mathsf{LSA}$;
- Γ_h^∞ is the Suslin co-Suslin sets of $L(A, \mathbb{R})$.
- $\Gamma_h^\infty = \Delta_h$, where Δ_h is defined at the beginning of Section 4.

Recall the notion of lbr hod mice is defined in [12]. We will not need the precise definition of these objects. However, we need some notions related to short-tree strategies. Let $\mathcal{P}$ is a premouse (or hod premouse), τ a cut point cardinal of $\mathcal{P}$ (typically the τ we consider will be a Woodin cardinal or a limit of Woodin cardinals of $\mathcal{P}$), and Σ is an iteration strategy of $\mathcal{P}$ acting on trees based on $\mathcal{P}|\tau$. Suppose $\mathcal{T}$ according to Σ is of successor length $\xi+1$. Then we say $\mathcal{T}$ is *short* if either $[0,\xi]_{\mathcal{T}}$ drops in model or else, letting i be the branch embedding, $i(\tau) > \delta(\mathcal{T})$; otherwise, we say $\mathcal{T}$ is *maximal*. We let Σ^{sh} be the short part of Σ; so Σ^{sh} is a partial strategy. In the following, we may not have a (total) iteration strategy, but a partial strategy Λ such that whenever $\mathcal{T}$ is according to Λ, if $\Lambda(\mathcal{T})$ is defined, then either $\Lambda(\mathcal{T})$ drops in model or else the branch embedding $i_{\Lambda(\mathcal{T})}(\tau) > \delta(\mathcal{T})$. We call such a Λ a *short-tree strategy*. We may turn Λ into a total strategy by assigning $\Lambda(\mathcal{T})$ to be $\mathcal{M}(\mathcal{T})^\sharp$ whenever a branch of $\mathcal{T}$ is not defined by Λ. Short tree strategies may be defined on stacks of normal trees as usual.

The proof of [9, Theorem 0.5] gives us a pair $(\mathcal{P},\Sigma)$ such that the following hold in $V(\mathbb{R}^{V[h]})$ (here the hypothesis $\mathsf{HPC}+\mathsf{NLE}$ is used):

1. $\mathcal{P}$ is a least-branch hod premouse (lpm) (cf. [12, Section 5]);

2. $\mathcal{P}$ has a largest Woodin cardinal $\delta = \delta^{\mathcal{P}}$ and letting $\kappa^{\mathcal{P}}$ be the least $<\delta$-strong cardinal in $\mathcal{P}$, then $\kappa^{\mathcal{P}}$ is a limit of Woodin cardinals;

3. Σ is a short-tree strategy of $\mathcal{P}$ and $\Sigma \in L(A,\mathbb{R})\backslash\Gamma^\infty_h$; furthermore, Σ is Suslin in $L(A,\mathbb{R})$;

4. for every $A \in \Gamma^\infty_h$, there is an iteration map $i : \mathcal{P} \to \mathcal{Q}$ according to Σ such that $A <_w \Sigma_{Q|\kappa^{\mathcal{Q}}}$, where $\kappa^{\mathcal{Q}}$ is the least $\delta^{\mathcal{Q}} = i(\delta^{\mathcal{P}})$-strong cardinal in $\mathcal{Q}$;

5. whenever $\mathcal{T}$ is according to Σ and $\Sigma(\mathcal{T})$ is not a branch, letting $\mathcal{Q} = \Sigma(\mathcal{T})$, $\Sigma_{\mathcal{T},\mathcal{Q}}$ satisfies (iii) and (iv);

General properties of sets of reals in derived models give:

6. there is some $\gamma < \lambda$ such that $(\mathcal{P}, \Sigma \restriction V[h\restriction\gamma]) \in V[h\restriction\gamma]$.

Lemma 8.1 *Fix a γ as in (vi). There is a Woodin cardinal $\delta < \lambda$ such that $\delta > \gamma$ and there is a tree $\mathcal{T}$ such that letting $\mathcal{Q} = \Sigma(\mathcal{T})$, $\Sigma_{\mathcal{T},\mathcal{Q}}$ satisfies (iii) and (iv) and is Wadge reducible to $\Psi^{g*h\restriction\gamma}_\delta$.*

Proof. Let δ be the least Woodin cardinal $> \gamma$. Let $\Psi = \Psi^{g*h\restriction\gamma}_\delta$. Let $(\mathcal{N}_\xi, \Lambda_\xi : \xi \leq \delta)$ be the models and strategies of the fully backgrounded (lbr) hod mouse construction over W_δ^Ψ (cf. [12]), where backgrounded extenders used have critical points $> max(\gamma, |\mathcal{P}|)$. Let $\mathcal{T}$ be according to Σ be the comparison tree of $\mathcal{P}$ against the above construction. By universality, there is $\xi \leq \delta$ such that

(i) $\Sigma(\mathcal{T}) = b$ exists and there is an iteration map $i : \mathcal{P} \to \mathcal{M}_\xi$ and $\Sigma_{\mathcal{T},\mathcal{M}_\xi} = \Lambda_\xi^{sh}$.

(ii) $\Sigma(\mathcal{T})$ does not exist ($\mathcal{T}$ is Σ-maximal), $\mathcal{M}_\xi = \Sigma(\mathcal{T})$, and $\Sigma_{\mathcal{T},\mathcal{M}_\xi} = \Lambda_\xi^{sh}$.

In either case, we get that $\Sigma_{\mathcal{T},\mathcal{M}_\xi}$ satisfies (3) and (4) above and $\Sigma_{\mathcal{T},\mathcal{M}_\xi} = \Lambda_\xi^{sh}$ is Wadge reducible to Ψ. $\square$

Let $\delta, \mathcal{T}, \mathcal{Q}$ be as in Lemma 8.1. Applying (b) in $DM(h)$, we get that $\Psi_\delta^{h\restriction\gamma} \restriction HC^{V[h]} \in \Gamma_h^\infty$. Lemma 8.1 then implies that $\Sigma_{\mathcal{T},\mathcal{Q}} \in \Gamma_h^\infty$. This contradicts (3). This completes the proof of Theorem 0.8.

References

[1] Ilijas Farah. The extender algebra and Σ_1^2 absoluteness. 2011. to appear in The Cabal Seminar, vol. 4, available at http://www.math.yorku.ca/~ifarah/preprints.html.

[2] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.

[3] D. A. Martin and J. R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[4] Itay Neeman. Inner models in the region of a Woodin limit of Woodin cardinals. *Ann. Pure Appl. Logic*, 116(1-3):67–155, 2002.

[5] Grigor Sargsyan. Generic generators. *to appear.*

[6] Grigor Sargsyan. Descriptive inner model theory. *Bull. Symbolic Logic*, 19(1):1–55, 2013.

[7] Grigor Sargsyan and Nam Trang. *The largest Suslin axiom.* Submitted. Available at math.rutgers.edu/~gs481/lsa.pdf.

[8] Grigor Sargsyan and Nam Trang. The exact consistency strength of generic absoluteness for universally Baire sets. 2019. Available at math.rutgers.edu/~gs481/lsa.pdf.

[9] J. R. Steel. HOD pair capturing and short tree strategies. 2016. Hand written notes. Available at https://math.berkeley.edu/∼steel/papers/HPC.oct2016.pdf.

[10] John R. Steel. Local k^c constructions. 2003. Available at math.berkeley.edu/∼steel/papers/Publications.html.

[11] John R. Steel. Derived models associated to mice. In *Computational prospects of infinity. Part I. Tutorials*, volume 14 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 105–193. World Sci. Publ., Hackensack, NJ, 2008. Available at author's website.

[12] John R. Steel. Normalizing iteration trees and comparing iteration strategies. 2016. Available at math.berkeley.edu/∼steel/papers/Publications.html.

[13] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the non-stationary ideal*, volume 1 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010.

[14] W. Hugh Woodin. In search of Ultimate-L: the 19th Midrasha Mathematicae Lectures. *Bull. Symb. Log.*, 23(1):1–109, 2017.