The mouse set conjecture for sets of reals[*][†]

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Abstract

We show that the *Mouse Set Conjecture* for sets of reals is true in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular". As a consequence, we get that below $AD_{\mathbb{R}}$ + "Θ is regular", models of $AD^+ + \neg AD_{\mathbb{R}}$ are hybrid mice over $\mathbb{R}$. Such a representation of models of AD^+ is important in core model induction applications.

One of the central open problems in descriptive inner model theory is the conjecture known as the *Mouse Set Conjecture* (MSC). It conjectures that under AD^+

[*]2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.
[†]Keywords: Mouse, inner model theory, descriptive set theory, hod mouse.
[‡]First author's work is partially based upon work supported by the National Science Foundation under Grant No DMS-0902628.

ordinal definable reals are exactly those that appear in ω_1-iterable mice. The counterpart of this conjecture for sets of reals conjectures that under AD^+, the sets of reals which are ordinal definable from a real are exactly those that appear in countably iterable mice over $\mathbb{R}$. In [3], the first author proved that MSC holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular", but MSC for sets of reals was left open. The goal of this paper is to establish that MSC for sets of reals holds in the minimal model of $AD_{\mathbb{R}}$ + "Θ is regular".

We will establish a stronger form of MSC known as the *Strong Mouse Set Conjecture* ($SMSC$). We say $\mathcal{M}$ is countably κ-iterable if all of its sufficiently elementary countable substructures are κ-iterable. We say $\mathcal{M}$ is countably iterable if $\mathcal{M}$ is countably ω_1-iterable. Thus, under AD, if $\mathcal{M}$ is countably iterable then $\mathcal{M}$ is countably $\omega_1 + 1$-iterable.

In what follows, we will let "hod pair" stand for a hod pair below $AD_{\mathbb{R}}$ + "Θ is regular", i.e., the corresponding hod mouse cannot have inaccessible limit of Woodin cardinals (see Definition 1.34 of [3]). Given an iteration strategy Σ for a countable structure, we let $Code(\Sigma)$ be the set of reals coding Σ for trees of length ω_1. Given a hod pair $(\mathcal{P}, \Sigma)$ we let

$$Lp^{\Sigma}(\mathbb{R}) = \cup\{\mathcal{M} : \mathcal{M} \text{ is a sound countably iterable } \Sigma\text{-mouse over } \mathbb{R} \text{ projecting to } \mathbb{R}\}.$$

The following is the statement of $SMSC$ for sets of reals. Recall the notions of branch condensation and fullness preservation from [3] (see Definition 2.14 and Definition 2.27 of [3]). Recall that OD_X stands for the class of sets ordinal definable from a finite sequence consisting of members of X.

The Strong Mouse Set Conjecture for sets of reals, $SMSC(\mathbb{R})$: Assume AD^+. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is fullness preserving. Then

$$\{A \subseteq \mathbb{R} : \exists x \in \mathbb{R}(A \text{ is } OD_{\{\Sigma, x\}})\} = Lp^{\Sigma}(\mathbb{R}).$$

To following is the main theorem of this paper.

Theorem 0.1 *Assume $AD^+ + V = L(\wp(\mathbb{R}))$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that the following holds.*

1. *$\mathcal{P}$ does not have inaccessible limit of Woodin cardinals.*

2. *Σ has branch condensation and is fullness preserving.*

3. *MSC for Σ holds, i.e., for every $x, y \in \mathbb{R}$, $x \in OD(\Sigma, y)$ iff x is in a Σ-mouse over y.*

4. *Every set of reals A is $OD(\Sigma, x)$ for some real x.*

Then

$$\wp(\mathbb{R}) = \wp(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R}).$$

In particular, $V = L(Lp^{\Sigma}(\mathbb{R}))$.

Corollary 0.2 *Suppose $V = L(\wp(\mathbb{R}))$ and AD^+ holds. Suppose further that for any α such that $\theta_\alpha < \Theta$, letting $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$, $L(\Gamma, \mathbb{R}) \vDash \neg AD_{\mathbb{R}}$. Then $SMSC(\mathbb{R})$ holds. In particular, $SMSC(\mathbb{R})$ is true in the minimal model of $AD_{\mathbb{R}}$+"Θ is regular".*

Proof. It is shown in [3] that if $(\mathcal{P}, \Sigma)$ is as in the hypothesis of Theorem 0.1 then clause 3 holds in $L(\Gamma_{\alpha+1})$ where α is such that $\theta_\alpha = w(Code(\Sigma))$ and $\Gamma_{\alpha+1} = \{A \subseteq \mathbb{R} : w(A) < \theta_{\alpha+1}\}$. It then follows from Theorem 0.1 that $\Gamma_\alpha = \wp(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})$ implying that $SMSC(\mathbb{R})$ holds. □

All the background material that we will need in this paper is spelled out in [3]. We assume that our reader is familiar with some aspects of it. One important comment is that in general hybrid mice over $\mathbb{R}$ or any non-self-wellordered set are not defined (recall that a set X is self-wellordered if there is a wellordering of it in $\mathcal{J}_\omega(X)$). Given an iteration strategy Σ with hull condensation, the Σ-mice over self-wellordered sets are defined according to the following principle. At a typical stage where we would like to add more of Σ to the model, we choose the least tree $\mathcal{T}$ for which $\Sigma(\mathcal{T})$ hasn't been defined. However, $\mathbb{R}$ isn't self-wellordered and hence, we cannot choose the least such $\mathcal{T}$.

In [3], the first author gave a definition of premice over any non-self-wellordered sets under the hypothesis that $\mathcal{M}_1^{\#,\Sigma}$ exists, i.e., there is a minimal active Σ-mouse with one Woodin cardinal (see Definition 3.37 of [3]). This extra assumption is benign as under AD^+ whenever $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is fullness preserving, $\mathcal{M}_1^{\#,\Sigma}$ exists and is Θ-iterable. The proof is the same as the proof that shows that $AD^{L(\mathbb{R})}$ implies that $\mathcal{M}_1^{\#}$ exists and is Θ-iterable in $L(\mathbb{R})$ (see [11]). One consequence of the indexing of the strategy introduced in Definition 3.37 of [3] is that it allows us to perform S-constructions, which we will use in this paper (see Chapter 3 of [3]).

Corollary 0.2 has been used in core model induction applications. See, for instance, [2], [4], [5] or Chapter 7 of [6]. Before we begin the proof of Theorem 0.1, we introduce Prikry tree forcing associated with Martin's measure on degrees.

Acknowledgments: The author's would like to express their gratitude to the referee for a very helpful list of comments. These comments made the paper significantly better.

1 Prikry tree forcing on degrees

We develop the notion of Prikry forcing that we need in a general context. Assume $ZF - Replacement + AD$. Let $\mathcal{D}$ be the set of Turing degrees. Let $f : \mathcal{D}^{<\omega} \to HC$ be some function. We would like to define *Prikry tree forcing* on degrees associated to f. Let μ be Martin's measure. We let $(p, A) \in \mathbb{P}^f$ if

1. $p \in \mathcal{D}^{<\omega}$,
2. for any $n < lh(p)$, $(f(p \restriction n), p \restriction n) \in L[p(n)]$,
3. $A \subseteq \cup_{n<\omega}\mathcal{D}^{<n}$ is a tree with stem p such that for every $q \in A$ (in particular, $p \subseteq q$),

$$\{d : q^\frown d \in A\} \in \mu.$$

Given $(p, A), (q, B) \in \mathbb{P}^f$ we let

$$(p, A) \preccurlyeq (q, B) \text{ iff } p \text{ end-extends } q,\ A \subseteq B \text{ and } p \in B.$$

We say $p \in \mathcal{D}^{<\omega}$ is a *precondition* if it satisfies 1 and 2 above. Given a precondition p and $d \in \mathcal{D}$, we say d is *valid* at p if $p^\frown d$ is a precondition.

Given a $\mathbb{P}^f$-generic G we let $g = \cup\{p : \exists X(p, X) \in G\}$. We then let

$$G^i =_{def} f(g \restriction i + 1) \text{ and } f(G) =_{def} \cup_{i<\omega} G^i$$

The following is proved by a standard fusion argument.

Lemma 1.1 *$\mathbb{P}^f$ has the Prikry property. More precisely, suppose Z is a countable set of $\mathbb{P}^f$-terms, ϕ is a formula, and $(p, A) \in \mathbb{P}^f$. Then there is a condition $(p, W) \in \mathbb{P}^f$ deciding $\phi[\tau]$ for all $\tau \in Z$ such that $W \in OD_{Z,\{f,p,A\}}$.*

Proof. We will show that there is a condition (p, T_τ) deciding $\phi[\tau]$ such that $\langle T_\tau : \tau \in Z\rangle \in OD_{Z,\{f,p,A\}}$. It then follows that $(p, \cap_{\tau\in Z} T_\tau)$ is as desired. We say q is *positive* if $(\exists Y)\ ((q, Y) \Vdash \phi[\tau])$, *negative* if $(\exists Y)((q, Y) \Vdash \neg\phi[\tau])$, and *ambiguous* if it is neither positive nor negative. Notice that q cannot be both positive and negative. Fixing τ, we shrink A to some tree T such that given any $r \in T$ and any one step extensions $q_1, 1_2 \in T$ of r, both q_1 and q_2 are simultaneously ambiguous, positive or negative.

We define a sequence of functions $\langle H^i : i < \omega\rangle$ such that

$$dom(H^i) = \{q : p \trianglelefteq q \text{ and } q \text{ is a precondition}\}$$

and $rng(H^i) \subseteq \{0, 1, 2\}$. First define H on $\{(q, d) : q^\frown d \text{ is a precondition}\}$ by

$$H(q, d) = \begin{cases} 0: & q^\frown d \text{ is positive} \\ 1: & q^\frown d \text{ is negative} \\ 2: & q^\frown d \text{ is ambiguous.} \end{cases}$$

Now, let $H^0(q) = i$ if for μ-a.e. d is such that $H(q, d) = i$. Given $\langle H^i : i \le k\rangle$ define H^{k+1} by setting $H^{k+1}(q) = i$ if for μ-a.e. d is such that $H^k(q^\frown d) = i$.

We then define a decreasing sequence of conditions (p, T^i) by induction as follows. We will have that $(p, T^0) \preceq (p, A)$. We define T^0 by induction on the length of conditions. We let $T^0 \restriction m$ be T^0 restricted to sequences of length m. Suppose we have defined $T^0 \restriction m + 1$ for $m + 1 \ge lh(p)$. Given $q \in T^0 \restriction m$ such that $lh(q) = m$ we let

$$\{q^\frown d \in A : H(q, d) = H^0(q)\}$$

be the one step extensions of q in T^0. This finishes our description of T^0.

Suppose now we have defined $\langle (p, T^i) : i \le k\rangle$ and $T^{k+1} \restriction m + 1$. Given $q \in T^{k+1} \restriction m$ such that $lh(q) = m$, we let

$$\{q^\frown d \in T^k : H^k(q^\frown d) = H^{k+1}(q)\}$$

be the one step extensions of q in T^{k+1}. This finishes our description of $\langle (p, T^i) : i \le \omega\rangle$. Let $T_\tau = \cap_{i<\omega} T^i$.

We claim that (p, T_τ) decides τ. Suppose not. We then have two conditions (q, X) and (r, Y) such that both are below (p, T_τ) and

1. $lh(q) = lh(r)$,

2. $(q, X) \Vdash \phi[\tau]$,

3. $(r, Y) \Vdash \neg\phi[\tau]$.

Let now s be the common initial segment of q and r. Let $s = (d_i : i \leq m)$, $q = s^\frown(q_i : i < n)$ and $r = s^\frown(r_i : i < n)$. It follows from our construction that

$$H(s^\frown(q_i : i < n-1), q_{n-1}) = H^0(s^\frown(q_i : i \leq n-1)) = H^1(s^\frown(q_i : i < n-2)) = \\ ... = H^{n-1}(s)$$
$$H(s^\frown(r_i : i < n-1), r_{n-1}) = H^0(s^\frown(r_i : i \leq n-1)) = H^1(s^\frown(r_i : i < n-2)) = \\ ... = H^{n-1}(s).$$

It then follows that $H(s^\frown(q_i : i < n-1), q_{n-1}) = H(s^\frown(r_i : i < n-1), r_{n-1})$, which is a contradiction. □

We now turn to proving Theorem 0.1.

2 The proof

We assume $AD^+ + V = L(\wp(\mathbb{R}))$ and let $(\mathcal{P}, \Sigma)$ be as in the hypothesis of Theorem 0.1. Given a good pointclass Γ[1] and $a \in HC$, we let $Lp^{\Gamma,\Sigma}(a)$ be the union of sound Σ-mice over a projecting to a whose iteration strategy is coded by a set in Γ. Our first lemma is an easy lemma. Below, $MC(\Sigma)$ (mouse capturing relative to Σ) is the statement that for every $x, y \in \mathbb{R}$, $x \in OD_{\Sigma,y}$ if and only if there is an ω_1-iterable sound Σ-mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$.

Lemma 2.1 *For any good pointclass $\Gamma \neq \Sigma^2_1(Code(\Sigma))$ there is a good pointclass $\Gamma_1 \neq \Sigma^2_1(Code(\Sigma))$ such that $\Gamma \cup \{Code(\Sigma)\} \subseteq \Gamma_1$ and for any $a \in HC$,*

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1,\Sigma}(a).$$

Proof. Fix a good pointclass $\Gamma \neq \Sigma^2_1(Code(\Sigma))$. Because $MC(\Sigma)$ holds, using $\Sigma_1(Code(\Sigma))$-reflection, we can find Γ_1 and α such that $\Gamma_1 \neq \Sigma^2_1(Code(\Sigma))$, $\Gamma \cup \{Code(\Sigma)\} \subseteq \Delta_{\Gamma_1}$, $\mathcal{J}_\alpha(\Gamma_1, \mathbb{R}) \vDash ZF - Replacement$, $\Gamma_1 = (\Sigma^2_1(Code(\Sigma)))^{\mathcal{J}_\alpha(\Gamma_1,\mathbb{R})}$ and

$$\mathcal{J}_\alpha(\Gamma_1, \mathbb{R}) \vDash MC(\Sigma).$$

□

Suppose now that $\wp(\mathbb{R}) \neq Lp^\Sigma(\mathbb{R})$. Using Σ_1-reflection we get $\Gamma \subset \undertilde{\Delta}^2_1(Code(\Sigma))$ and $\alpha < \undertilde{\delta}^2_1(Code(\Sigma))$ such that

[1] i.e., a point class closed under $\exists^{\mathbb{R}}$, continuous preimages and images, and having the scale property

1. $\Gamma = \wp(\mathbb{R}) \cap \mathcal{J}_\alpha(\Gamma, \mathbb{R})$ and α ends a Σ_1-gap,

2. $\mathcal{J}_\alpha(\Gamma, \mathbb{R}) \vDash \phi$ where ϕ is the conjunction of the following statements:

   (a) $ZF - Replacement + DC_{\mathbb{R}} + MC(\Sigma)$,

   (b) there is an OD set of reals A such that $A \notin Lp^\Sigma(\mathbb{R})$.

We let $N = \mathcal{J}_\alpha(\Gamma, \mathbb{R})$. Let U be the set of pairs $(x, y) \in \mathbb{R}^2$ such that y codes a sound Σ-mouse $\mathcal{M}$ over x that projects to x and has an ω_1-iteration strategy in N.

Since $MC(\Sigma)$ holds in N, U is a universal $(\Sigma^2_1(Code(\Sigma)))^N$-set. Let $A \in N$ be an OD set of reals witnessing clause (b) of ϕ. We assume that A has the minimal Wadge rank among the sets witnessing clause b of ϕ. Using the results of Chapter 3 of [1], we can get $\vec{B} = \langle B_i : i < \omega \rangle$ which is a semiscale on U^c such that each $B_i \in (OD_\Sigma)^N$. The following fact is a well known consequence of $MC(\Sigma)$.

Proposition 2.2 *The following statements are true.*

1. *There is a cone of x such that there is $\mathcal{M} \trianglelefteq Lp^\Sigma(x)$ such that $\rho_\omega(\mathcal{M}) = x$ and $\mathcal{M}$ doesn't have an iteration strategy in N.*

2. *Let x be a base of the above cone. Then for every $a \in HC$ such that $x \in \mathcal{J}_\omega(a)$, there is $\mathcal{M} \trianglelefteq Lp^\Sigma(a)$ such that $\rho(\mathcal{M}) = a$ and $\mathcal{M}$ doesn't have an iteration strategy in N.*

Proof. Clause 2 follows from clause 1. To see this, fix a real x such that it is base for the cone of clause 1. Then whenever $a \in HC$ is such that $x \in a$ and y is a real coding a generically over $Lp^\Sigma(a)$, then $Lp^\Sigma(a)[y] = Lp^\Sigma(y)$. Indeed, this follows from S-constructions (see Section 2.11 of [3]).

Clause 1 is an easy consequence of $MC(\Sigma)$. Indeed, suppose clause 1 fails. Then

(1) there is an $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$ such that $x \leq_T y$, $Lp^\Sigma(y) = (Lp^\Sigma(y))^N$.

Because $MC(\Sigma)$ holds, letting C be the set of pairs $(y, z) \in \mathbb{R}^2$ such that $x \leq_T y$ and z codes an ω_1-iterable sound Σ-mouse $\mathcal{M}$ over y projecting to y, C is a universal $\Sigma^2_1(Code(\Sigma))$ set. Because $\Gamma \subset \utilde{\Delta}^2_1(Code(\Sigma))$, we cannot have that C is the universal $(\Sigma^2_1(Code(\Sigma)))^N$ set. It follows from (1), however, that $C \in N$ and $N \vDash$ "C is the universal $\Sigma^2_1(Code(\Sigma))$-set", contradiction. □

Let now x be a base of the cone from clause 1 of Proposition 2.2. We say a is *good* if $a \in HC$ and $x \in \mathcal{J}_\omega(a)$. For each good a let $\mathcal{M}(a)$ be the least Σ-mouse with no iteration strategy in N. Let F be the set of pairs $(a, \mathcal{M}(a))$. It follows that if F^* is the set of reals coding F then $F^* \in \undertilde{\Delta}^2_1(Code(\Sigma))$. Furthermore, there is a set $C \in \undertilde{\Delta}^2_1(Code(\Sigma))$ such that for every good a, the set of reals coding the unique iteration strategy of $\mathcal{M}(a)$ is Wadge reducible to C. Let then

$$D = \{(y, \sigma) \in \mathbb{R}^2 : y \text{ codes a good } a \text{ and } \sigma \text{ codes a continuous function } f \text{ such that } f^{-1}[C] \text{ is the iteration strategy of } \mathcal{M}(a)\}.$$

We have that $D \in \undertilde{\Delta}^2_1(Code(\Sigma))$.

Let Γ_1 be a good pointclass such that $F^*, Code(\Sigma), \vec{B}, U, C, D \in \undertilde{\Delta}_{\Gamma_1}$. Moreover, it follows from Lemma 2.1 that we can require that for any $a \in HC$

$$C_{\Gamma_1}(a) = Lp^{\Gamma_1, \Sigma}(a)$$

Let now $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ be as in Theorem 1.2.9 of [3] with the property that $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Code(\Sigma), \vec{B}, U, C, D$ (where Suslin capturing is defined on page 36 of [3], also see the next paragraph) [2]. We have that for any $\eta < \delta_z$, $C_{\Gamma_1}(\mathcal{N}^*_z|\eta) \in \mathcal{N}^*_z$.

Let $\Phi = (\undertilde{\Sigma}^2_1)^N$. We have that Φ is a good pointclass. Because $\vec{B}$ is Suslin captured by $\mathcal{N}^*_z$, we have $(\delta^+_z)^{\mathcal{N}^*_z}$-complementing trees $T, S \in \mathcal{N}^*_z$ which capture $\vec{B}$

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[2]For convenience, we restate Theorem 1.2.9 of [3].

Theorem 2.3 (Woodin, Theorem 10.3 of [9]) *Assume AD^+. Suppose Γ is a good pointclasses and there is a good pointclass Γ^* such that $\Gamma \subseteq \Delta_{\Gamma^*}$. Suppose (N, Ψ) Suslin, co-Suslin capture Γ. There is then a function F defined on $\mathbb{R}$ such that for a Turing cone of x, $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$ such that*

1. *$N \in L_1[x]$,*
2. *$\mathcal{N}^*_x|\delta_x = \mathcal{M}_x|\delta_x$,*
3. *$\mathcal{M}_x$ is a Ψ-mouse: in fact, $\mathcal{M}_x = \mathcal{M}^{\Psi,\#}_1(x)|\kappa_x$ where κ_x is the least inaccessible cardinal of $\mathcal{M}^{\Psi,\#}_1$.[3],*
4. *$\mathcal{N}^*_x \vDash$ "δ_x is the only Woodin cardinal",*
5. *Σ_x is the unique iteration strategy of $\mathcal{M}_x$,*
6. *$\mathcal{N}^*_x = L(\mathcal{M}_x, \Lambda)$ where Λ is the restriction of Σ_x to stacks $\vec{\mathcal{T}} \in \mathcal{M}_x$ that have finite length and are based on $\mathcal{M}_x \restriction \delta_x$,*
7. *$(\mathcal{N}^*_x, \Sigma_x)$ Suslin, co-Suslin captures $Code(\Psi)$ and hence, $(\mathcal{N}^*_x, \Sigma_x)$ Suslin, co-Suslin captures Γ,*
8. *$(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ is a self-capturing background triple.*

in the sense that whenever $i : \mathcal{N}_z^* \to \mathcal{N}$ is an iteration embedding according to Σ_z and g is a generic over $\mathcal{N}$ for a poset of size $\leq i(\delta_z)$ then

$$p[i(T)] \cap \mathcal{N}[g] = B \cap \mathcal{N}[g].$$

Let κ be the least cardinal of $\mathcal{N}_z^*$ which is $< \delta_z$-strong in $\mathcal{N}_z^*$.

Next, we fix a notation. For each $a \in HC$, we let $\mathcal{W}(a) = Lp^{\Gamma,\Sigma}(a)$. Using the results of Section 2.11 of [3], we have that if $g \subseteq Coll(\omega, a)$ is $\mathcal{W}(a)$-generic then

$$\mathcal{W}(a)[g] = \mathcal{W}(a,g) = \mathcal{W}(x_g) \qquad (1).$$

where x_g is the generic real coding a. The following claim is standard.

Lemma 2.4 $\mathcal{N}_z^* \vDash$ "*κ is a limit of cardinals η such that η is a Woodin cardinal in $\mathcal{W}(\mathcal{N}_z^*|\eta)$*".

Proof. Working in $\mathcal{N}_z^*$, let $\lambda = \delta_z^{++}$ and let $\pi : M \to \mathcal{N}_z^*|\lambda$ be an elementary substructure such that

1. $T, S \in ran(\pi)$ and
2. letting $crit(\pi) = \eta$, $V_\eta^{\mathcal{N}_z^*} \subseteq M$, $\pi(\eta) = \delta_z$ and $\eta > \kappa$.

By elementarity, we have that $M \vDash$ "η is Woodin". Letting $\pi^{-1}(\langle T, S\rangle) = \langle \bar{T}, \bar{S}\rangle$, we have that $(\bar{T}, \bar{S})$ Suslin captures $\vec{B}$ over M at $(\eta^+)^M$. This implies that whenever $a \in M|(\eta^+)^M$, $\mathcal{W}(a) \in M$. To see this, first note that we have that whenever $g \subseteq Coll(\omega, a)$ is M-generic and x_g is the generic real then $\mathcal{W}(x_g) \in M$. But using (1) above, we have that $\mathcal{W}(x_g) = \mathcal{W}(a)[g]$. Therefore, $\mathcal{W}(a) \in M[g]$. Since g was arbitrary, we have that $\mathcal{W}(a) \in M$.

We now have that $\mathcal{W}(\mathcal{N}^*|\eta) \in M$ and since $M \vDash$ "η is Woodin", we have that $\mathcal{W}(\mathcal{N}^*|\eta) \vDash$ "η is Woodin". Because κ is $< \delta_z$-strong in $\mathcal{N}_z^*$ and because $a \to \mathcal{W}(a)$ is definable over $\mathcal{N}_z^*$, we have that for unboundedly many $\nu < \eta$, $\mathcal{W}(\mathcal{N}_z^*|\nu) \vDash$ "ν is Woodin". □

2.1 A Σ-mouse beyond N

In this section, we prove the following important lemma.

Lemma 2.5 *There is a Σ-mouse $\mathcal{N}$ such that there is a sequence $(\gamma_i : i < \omega)$ with the property that*

1. *$(\gamma_i : i < \omega)$ is the sequence of Woodin cardinals of $\mathcal{N}$,*

2. *letting $\gamma = \sup_{i<\omega} \gamma_i$, $\rho_\omega(\mathcal{N}) < \gamma$,*

3. *for some $k < \omega$, $\mathcal{N}$ is a sound Σ-mouse over $\mathcal{N}|\gamma_k$,*

4. *for every cutpoint cardinal $\eta < \gamma$, $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathcal{W}(\mathcal{N}|\eta)$, and*

5. *letting Λ be the $(\omega, \omega_1, \omega_1)$-strategy of $\mathcal{N}$, $Code(\Lambda) \notin \Gamma$ and Λ is Γ-fullness preserving (i.e., clause 4 holds for any Λ-iterate of $\mathcal{N}$).*

We now begin the proof of Lemma 2.5. We continue with previous subsection's notation and start working in $\mathcal{N}_z^*$. Our aim is to use fully backgrounded constructions of $\mathcal{N}_z^*$ to produce a mouse $\mathcal{N}^*$ such that for some l, $\mathcal{N} = \mathcal{C}_l(\mathcal{N}^*)$ has the desired properties. Let $\langle \eta_i : i < \omega \rangle$ be the first ω cardinals below κ such that for every $i < \omega$, $\mathcal{W}(\mathcal{N}_z^*|\eta_i) \vDash$ "η_i is a Woodin cardinal" (it follows from Lemma 2.4 that there are such cardinals). Let now $\langle \mathcal{N}_i : i < \omega \rangle$ be a sequence constructed according to the following rules:

1. $\mathcal{N}_0 = (\mathcal{J}^{\vec{E},\Sigma})^{\mathcal{N}_z^*|\eta_0}$,

2. $\mathcal{N}_{i+1} = (\mathcal{J}^{\vec{E},\Sigma}[\mathcal{N}_i])^{\mathcal{N}_z^*|\eta_{i+1}}$.

Let $\mathcal{N}_\omega = \cup_{i<\omega} \mathcal{N}_i$.

Claim 1. For every $i < \omega$, $\mathcal{N}_\omega \vDash$ "η_i is a Woodin cardinal" and $\mathcal{N}_\omega|(\eta_i^+)^{\mathcal{N}_\omega} = \mathcal{W}(\mathcal{N}_i)$.

Proof. It is enough to show that

1. $\mathcal{N}_{i+1} \vDash$ "η_i is a Woodin cardinal",

2. no level of $\mathcal{N}_{i+1}$ projects across η_i, and

3. $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = \mathcal{W}(\mathcal{N}_i)$.

To show 1-3, it is enough to show that if $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(\mathcal{Q}) \leq \eta_i$ then the fragment of the iteration strategy of $\mathcal{Q}$ that acts on trees above η_i is coded by a set in Γ (this is simply because $\mathcal{N}_{i+1}$ is Γ-full). Fix then i and let $\mathcal{Q} \trianglelefteq \mathcal{N}_{i+1}$ be such that $\rho_\omega(\mathcal{Q}) \leq \eta_i$. Let ξ be such that if $\mathcal{S}$ is the ξth model of the fully backgrounded construction producing $\mathcal{N}_{i+1}$ then $\mathcal{Q}$ is the core of $\mathcal{S}$. Let $\pi : \mathcal{Q} \to \mathcal{S}$ be the uncollapse map. It is a fine structural map but that is irrelevant and we suppress this point.

Let $\nu < \eta_{i+1}$ be a cardinal such that $\mathcal{S}$ is the ξth model of the full background construction of $\mathcal{N}_z^*|\nu$. Let Ψ be the fragment of Σ_z that acts on non-dropping trees that are based on $\mathcal{N}_z^*|(\nu^+)^{\mathcal{N}_z^*}$ and are above η_i. We have that Ψ induces an iteration strategy Ψ^* for $\mathcal{S}$ and that π-pullback of Ψ^* is an iteration strategy for $\mathcal{Q}$. It is then enough to show that $Code(\Psi) \in \Gamma$.

Notice that whenever $\mathcal{T}$ is a tree on $\mathcal{N}_z^*|(\nu^+)^{\mathcal{N}_z^*}$ according to Ψ and $b = \Psi(\mathcal{T})$ then $\mathcal{Q}(b, \mathcal{T})$ is defined. Also, notice that because of our choice of η_{i+1}, for any such $\mathcal{T}$ and b, $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$. Because the function $a \to \mathcal{W}(a)$ is coded by a set in Γ, we have that $Code(\Psi) \in \Gamma$.

$\square$

Claim 2. There is $\mathcal{Q} \trianglelefteq (\mathcal{J}^{\vec{E},\Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$ such that $\rho_\omega(\mathcal{Q}) < \eta_\omega$.

Proof. To see this suppose not. Let $\mathcal{R} = (\mathcal{J}^{\vec{E},\Sigma}(\mathcal{N}_\omega))^{\mathcal{N}_z^*}$. It follows from universality of $\mathcal{R}$ (with respect to Σ-mice that have iteration strategies in Γ), we have that

$$Lp^{\Gamma_1,\Sigma}(\mathcal{N}_\omega) \trianglelefteq \mathcal{R}.$$

It follows from our choice of Γ_1 and from our hypothesis that $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$. Notice that if $\mathcal{M}(a)$ is defined for some a then because of our choice of Γ_1, $\mathcal{M}(a) \trianglelefteq Lp^{\Gamma_1,\Sigma}(a)$.

We claim that $\mathcal{M}(\mathcal{N}_0)$ is defined. To see this, notice that x is generic over $\mathcal{J}[\mathcal{N}_0]$ for the extender algebra at η_0. Hence, if $g \subseteq Coll(\omega, \eta_0)$ is $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$-generic such that $x \in Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)[g]$, then by the results of Section 2.11 of [3], we have that

$$Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)[g] = Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0[g]).$$

But now, because $x \in \mathcal{J}[\mathcal{N}_0][g]$, we have that $\mathcal{M}(\mathcal{N}_0[g])$ is defined, and by our choice of Γ_1, we have that $\mathcal{M}(\mathcal{N}_0[g]) \trianglelefteq Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0[g])$. Again using the results of Section 2.11 of [3], we have that some initial segment of $Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$ has an iteration strategy which is not coded by a set of reals in Γ. Hence, $\mathcal{M}(\mathcal{N}_0)$ is defined.

Because $\mathcal{M}(\mathcal{N}_0)$ is defined, we have that $\mathcal{M}(\mathcal{N}_0) \trianglelefteq Lp^{\Gamma_1,\Sigma}(\mathcal{N}_0)$ and therefore, $\mathcal{M}(\mathcal{N}_0) \trianglelefteq \mathcal{N}_\omega$. However, it follows from the proof of Claim 1 that all initial segments of $\mathcal{N}_\omega$ projecting to η_0 have an iteration strategy coded by a set in Γ. This implies that $\mathcal{M}(\mathcal{N}_0)$ has an iteration strategy coded by a set in Γ, contradiction! $\square$

Let now $\mathcal{N}^* \trianglelefteq Lp(\mathcal{N}_\omega)$ be least such that $\rho_\omega(\mathcal{N}^*) < \eta_\omega$. Let l be least such that $\rho_l(\mathcal{N}^*) < \eta_\omega$ and let k be least such that $\rho_l(\mathcal{N}^*) < \eta_k$. In what follows, we will regard $\mathcal{N}^*$ as a Σ-mouse over $\mathcal{N}^*|\eta_k$. We let $\mathcal{N} = \mathbb{C}_l(\mathcal{N}^*)$. Thus, $\mathcal{N}$ is sound (as a Σ-mouse over $\mathcal{N}|\eta_k$). We let $\langle \gamma_i : i < \omega \rangle$ be the Woodin cardinals of $\mathcal{N}$ and $\gamma = \sup_{i<\omega} \gamma_i$. Let Λ be the $(\omega, \omega_1, \omega_1)$-strategy of $\mathcal{N}$ induced by Σ_z. Notice that $Code(\Lambda) \notin \Gamma$ because otherwise, since $\mathcal{N}_\omega$ is Γ-full, $\mathcal{N} \trianglelefteq \mathcal{N}_\omega \trianglelefteq \mathcal{N}^*$.

Claim 3. Λ is Γ-fullness preserving.

Proof. To see this fix $\mathcal{N}_1$ which is a Λ-iterate of $\mathcal{N}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $i : \mathcal{N} \to \mathcal{N}_1$ exists. If $\mathcal{N}_1$ isn't Γ-full then there is a cutpoint ν of $\mathcal{N}_1$ and a sound Σ-mouse $\mathcal{Q}$ over $\mathcal{N}_1|\nu$ with (ω, ω_1)-iteration strategy Ψ such that $Code(\Psi) \in \Gamma$, $\rho_\omega(\mathcal{Q}) = \nu$ and $\mathcal{Q} \ntrianglelefteq \mathcal{N}_1$.

Subclaim. Ψ can be extended to an $(\omega, \omega_1, \omega_1)$-iteration strategy.

Proof. We can find a good pointclass Γ^* such that $Code(\Psi) \in \underset{\sim}{\Delta}_{\Gamma^*}$. Using Theorem 1.2.9 of [3], we can find $(\mathcal{N}_y^*, \Sigma_y, \delta_y)$ that Suslin captures $Code(\Psi)$. Notice that Σ_y is an $(\omega, \omega_1, \omega_1)$-iteration strategy. It follows from universality that $\mathcal{Q} \trianglelefteq (\mathcal{J}^{\vec{E},\Sigma}[\mathcal{N}_1|\nu])^{\mathcal{N}_y^*|\delta_y}$. Hence, $\mathcal{Q}$ has an $(\omega, \omega_1, \omega_1)$-iteration strategy Ψ^+. Because Ψ is the unique (ω, ω_1)-iteration strategy of $\mathcal{Q}$, we have that Ψ^+ extends Ψ. □

We now compare $\mathcal{Q}$ with $\mathcal{N}_1$. Let $\mathcal{S}$ be the comparison tree on the $\mathcal{Q}$ side with last model $\mathcal{Q}^*$ and $\mathcal{T}$ be the comparison tree on the $\mathcal{N}_1$ side with last model $\mathcal{N}_1^*$. Because $\mathcal{Q} \ntrianglelefteq \mathcal{N}_1$, we must have that $\mathcal{N}_1^* \trianglelefteq \mathcal{Q}^*$ and $\pi^{\mathcal{T}} : \mathcal{N}_1 \to \mathcal{N}_1^*$ exists. Because the (ω, ω_1)-fragment of Λ is the unique (ω, ω_1)-iteration strategy of $\mathcal{N}$, we must have that it is the $\pi^{\mathcal{T}} \circ i$-pullback of $\Psi_{\mathcal{N}_1^*, \vec{\mathcal{T}}^\frown \mathcal{S}}$ (recall that this is the strategy of $\mathcal{N}_1^*$ induced by Ψ). This implies that $\Lambda \in \Gamma$, contradiction. □

It is now clear that $(\mathcal{N}, \Lambda)$ is as desired. This completes the proof of Lemma 2.5.

2.2 A Prikry generic

In this subsection, while working in N, we define a Prikry forcing with the property that the generic object produces a sound countably iterable Σ-mouse $\mathcal{R}$ over $\mathbb{R}$ such that $\mathcal{R} \in N$ and extends $(Lp^\Sigma(\mathbb{R}))^N$. Clearly this is a contradiction.

We now start working in N. We now describe a function $f : \mathcal{D}^{<\omega} \to HC$ such that if $G \subseteq \mathbb{P}^f$ is N-generic then $f(G)$ is a Σ-premouse such that certain $\mathcal{J}^{\vec{E},\Sigma}$-construction of it is an initial segment of some Λ-iterate of $\mathcal{N}$.

Following [3], we say $\mathcal{Q}$ is Σ-*suitable* (in N) if for some ordinal δ

1. δ is the unique Woodin cardinal of $\mathcal{Q}$,
2. $o(\mathcal{Q}) = \sup_{n<\omega}(\delta^{+n})^{\mathcal{Q}}$,
3. $\mathcal{Q}$ is full with respect to Σ-mice, i.e., for any cutpoint η, $Lp^\Sigma(\mathcal{Q}|\eta) \trianglelefteq \mathcal{Q}$.

We let $\delta^{\mathcal{Q}}$ be the Woodin cardinal of $\mathcal{Q}$. Similarly we can define the notion of a Σ-suitable $\mathcal{Q}$ over any set a. In particular, if $\mathcal{Q}$ is Σ-suitable and $\mathcal{R}$ is Σ-suitable over $\mathcal{Q}$ then $\mathcal{R} \models$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". Because we will only deal with Σ-suitable structures, we omit Σ and just say suitable instead of Σ-suitable.

A normal iteration tree $\mathcal{U}$ on a suitable $\mathcal{P}$ is *short* if for all limit $\xi \leq lh(\mathcal{U})$,

$$Lp^{\Sigma}(\mathcal{M}(\mathcal{U}|\xi)) \models \text{"}\delta(\mathcal{U}|\xi) \text{ is not Woodin"}.$$

Otherwise, we say that $\mathcal{U}$ is *maximal*. We say that a suitable $\mathcal{P}$ is *short tree iterable* if for any short tree $\mathcal{T}$ on $\mathcal{P}$, there is a cofinal wellfounded branch b such that $\mathcal{Q}(b, \mathcal{T})$ exists and if $\pi_b^{\mathcal{T}} : \mathcal{P} \to \mathcal{M}_b^{\mathcal{T}}$ exists then $\mathcal{M}_b^{\mathcal{T}}$ is suitable.

Write $\mathcal{P}_y$ for the premouse coded by the real y. Let a be countable transitive and $d \in \mathcal{D}$ be such that a is coded by a real recursive in d. Put

$$\mathcal{F}_a^d = \{\mathcal{P}_z : \; z \leq_T d, \mathcal{P}_z \text{ is a short-tree iterable suitable premouse over } a\}$$

Lemma 2.6 *For any fixed a, there is a cone of d such that $\mathcal{F}_a^d \neq \emptyset$.*

Proof. If not, the failure of the statement in the claim is a Σ_1 statement. Call this statement $\phi[a]$. Using $\undertilde{\Sigma}_1^2(Code(\Sigma))$-reflection, we get a transitive model

$$H \models ZF^- + \Theta = \Theta_{Code(\Sigma)} + \phi[a],$$

$\mathbb{R} \subseteq H$ and $\wp(\mathbb{R}) \cap H \subsetneq \undertilde{\Delta}_1^2(Code(\Sigma))$.

Let Γ^* be a good pointclass beyond H. Such a Γ^* exists by our assumption on H. We use Theorem 1.2.9 of [3] to get a triple $\langle N_w^*, \delta_w, \Sigma_w \rangle$ (for some real w) that Suslin captures the universal Γ^* set. Using universality of fully backgrounded constructions and the proofs of the claims from the proof of Lemma 2.5 (or the results of Section 3.2.2 of [3]), we conclude that the $(\mathcal{J}^{\vec{E},\Sigma}[a])^{\mathcal{N}_w^*|\delta_w}$ reaches a premouse $\mathcal{Q}_a$ such that in H, $\mathcal{Q}_a$ is short tree iterable and suitable (with respect to H). This contradicts our assumptions on H. □

For each a and for each Turing degree d from the cone of Lemma 2.6, we can simultaneously compare all $\mathcal{Q} \in \mathcal{F}_a^d$ while doing the generic genericity iteration to make d generic over the common part of the final model $\mathcal{Q}_a^{d,-}$. This process (hence $\mathcal{Q}_a^{d,-}$) depends only on d. Set

$$\mathcal{Q}_a^d = Lp_\omega^{\Sigma}(\mathcal{Q}_a^{d,-}) \text{ and } \delta_a^d = o(\mathcal{Q}_a^{d,-}).$$

Recall that we are working in N (thus, we really have that $\mathcal{Q}_a^d = Lp_\omega^{\Gamma,\Sigma}(\mathcal{Q}_a^{d,-})$).

Lemma 2.7 *The following statements are true (in N).*

1. $\mathcal{Q}_a^d$ and δ_a^d depend only on d.

2. $\mathcal{Q}_a^{d,-}$ is Σ-full (no levels of $\mathcal{Q}_a^d$ project strictly below δ_a^d).

3. $\mathcal{Q}_a^d \models \delta_a^d$ is Woodin.

4. $\wp(a) \cap \mathcal{Q}_a^d = \wp(a) \cap OD_\Sigma(a \cup \{a\})$ and $\wp(\delta_a^d) \cap \mathcal{Q}_a^d = \wp(\delta_a^d) \cap OD_\Sigma(Q_a^{d,-} \cup \{Q_a^{d,-}\})$.

5. $\delta_a^d = \omega_1^{L[S,d]}$.

Proof. 1-4 just follow from our definitions. We consider 5. Let S be the tree of a $(\Sigma_1^2(Code(\Sigma)))^N$ scale on a universal $(\Sigma_1^2(Code(\Sigma)))^N$ set U. Suppose that in $L[S,d]$, the process producing $\mathcal{Q}_a^d$ stops at stage $\alpha < \omega_1^{L[S,d]}$. We then have that $\mathcal{Q}_a^d$ is countable in $L[S,d]$. The suitability of $\mathcal{Q}_a^d$ then implies that $\mathbb{R} \cap L[S,d] \subseteq \mathcal{Q}_a^d[d]$. It then follows that δ_a^d, the Woodin of $\mathcal{Q}_a^d$, is countable in $L[S,d]$ while it is a cardinal in $\mathcal{Q}_a^d[d]$ (because the extender algebra of $\mathcal{Q}_a^d$ at δ_a^d is δ_a^d-cc). Hence, $\omega_1^{L[S,d]}$ is countable in $L[S,d]$, contradiction! $\square$

We now define $f : \mathcal{D}^{<\omega} \to HC$ by induction on $\mathcal{D}^n$. Fix $(\mathcal{N}, \Lambda)$ as in Lemma 2.5 and let k be as in clause 3. Below we use the notation of Lemma 2.5. We let $f(\emptyset) = \mathcal{N}|\gamma_k$. Suppose we have defined $f \restriction \mathcal{D}^{n+1}$. Given $p \in \mathcal{D}^{n+2}$, we let

$$f(d) = \begin{cases} \mathcal{Q}^d_{f(p \restriction n+1)} & : f(p \restriction n+1) \text{ is countable in } L[d] \\ \emptyset & : \text{otherwise} \end{cases}$$

Suppose now that $G \subseteq \mathbb{P}^f$ is N-generic. Let $\mathcal{Q}_i = G^i$ and let $\mathcal{Q}_\omega = f(G)$. We let δ_i be the largest Woodin cardinal of $\mathcal{Q}_i$. Without loss of generality, we assume that if $(\langle d \rangle, X) \in G$ then $\mathcal{N}$ is countable in $L[d]$.

Given an increasing function $h : \omega \to \omega$, we define $\langle \mathcal{Q}_i^h, \mathcal{Q}_i^{h,*} : i < \omega \rangle$ according to the following procedure:

1. $\mathcal{Q}_0^{h,*}$ is the output of $\mathcal{J}^{\vec{E},\Sigma}[a]$ construction done in $\mathcal{Q}_{h(0)+1}$ using extenders with critical point $> \delta_{h(0)}$.

2. $\mathcal{Q}_0^h = (Lp^\Sigma(\mathcal{Q}_0^{h,*}))^{\mathcal{Q}_{h(0)+2}}$.

3. $\mathcal{Q}_{i+1}^{h,*}$ is the output of $\mathcal{J}^{\vec{E},\Sigma}[\mathcal{Q}_i^h]$ construction done in $\mathcal{Q}_{h(i+1)+1}$ using extenders with critical point $> \delta_{h(i+1)}$.

4. $\mathcal{Q}_{i+1}^h = (Lp^\Sigma(\mathcal{Q}_{i+1}^{h,*}))^{\mathcal{Q}_{h(i+1)+2}}$.

We let $\mathcal{Q}^h_\omega = \cup_{i<\omega}\mathcal{Q}^h_i$.

Lemma 2.8 *For some increasing function $h : \omega \to \omega$ such that $h \in V$, $\mathcal{Q}^h_\omega$ is an initial segment of a Λ-iterate of $\mathcal{N}$.*

Proof. Let $\vec{d} = \langle d_i : i < \omega\rangle$ be the generic sequence of degrees given by G. We define h recursively. It will have the property that $\mathcal{Q}^h_i$ is a Λ-iterate of $\mathcal{N}|(\gamma^+_{k+i+1})^{\mathcal{N}}$. While defining h, we also define a sequence $\vec{H} = \langle \mathcal{N}_i, \mathcal{U}_i, b_i : i \in [-1, \omega)\rangle$ such that

1. $\mathcal{N}_{-1} = \mathcal{N}$,
2. for each i, $\mathcal{U}_i$ is an iteration tree on $\mathcal{N}_i$ and $b_i = \Lambda(\oplus_{m<i+1}\mathcal{U}_m)$,
3. for each i, $\mathcal{N}_{i+1} = \mathcal{M}^{\mathcal{U}_i}_{b_i}$,
4. for each i, $\pi^{\mathcal{U}_i}_{b_i}$-exists and letting $\pi_{i,j} : \mathcal{N}_i \to \mathcal{N}_j$ be the composition of iteration embeddings, $\mathcal{U}_i$ is a tree based on $\mathcal{N}_i|[\pi_{-1,i}(\gamma_{k+i}), \pi_{-1,i}(\gamma_{k+i+1}))$,
5. $\mathcal{Q}^h_i = \mathcal{N}_i|(\pi_{-1,i}(\gamma^+_{k+i+1})^{\mathcal{N}})$
6. for each i, $h(i) = m + 1$ where m is the least integer such that $\vec{H} \restriction i + 1$ is countable in $L[d_m]$.

1-6 above tell us how to define the sequence. To see that we can always arrange 6, recall that $\vec{d}$ is cofinal in the set of degrees. To see that $h \in V$, recall that Prikry property implies that $\mathbb{P}^f$ doesn't add new reals. To see 5, notice that by our construction, $\vec{H} \restriction i$ is generic over $\mathcal{Q}_i$ for the extender algebra at δ_i. □

We let h be as in Lemma 2.8. We let $\mathcal{S}_i = \mathcal{Q}^h_i$ and $\mathcal{S}_\omega = \mathcal{Q}^h_\omega$. Also, let $\mathcal{S}$ be the Λ-iterate of $\mathcal{N}$ such that $\mathcal{S}_\omega \trianglelefteq \mathcal{S}$. Because $\rho(\mathcal{N}) \leq \gamma_k$, we have that $\rho(\mathcal{S}) \leq \gamma_k$. Let $\langle \eta_n : n < \omega\rangle$ be the Woodin cardinals of $\mathcal{S}$. Let $\eta_\omega = \sup_{n<\omega} \eta_n$. Notice that in $V[G]$, $\mathcal{S}_\omega$ is (ω, ω_1)-iterable for short trees.

We now have that there is $g \subseteq Col(\omega, < \eta_\omega)$-generic over $\mathcal{S}$ such that

$$\cup_{n<\omega}\mathbb{R}^{\mathcal{S}[g\cap\eta_n]} = \mathbb{R}.$$

Next we perform an S-construction (see Section 2.11 of [3], [7], [8] or [10]) to translate $\mathcal{S}$ to a Σ-mouse over $\mathbb{R}$. To see that the translation procedure works, let $\lambda = \Theta^{\mathcal{J}_\alpha(U,\mathbb{R})}$. Notice that $\mathbb{P}^f \in \mathcal{J}_\lambda(\mathbb{R})$ and that all extenders of $\mathcal{S}$ above η_ω have critical point $> \lambda$. Thus, we can translate Σ-premice over $\mathcal{J}_\lambda[\mathcal{S}_\omega]$ to Σ-premice over $\mathcal{J}_\lambda(\mathbb{R})$. Let then $\mathcal{W}$ be the Σ-premouse over $\mathbb{R}$ that is the result of translating $\mathcal{S}$ into a Σ-premouse over $\mathcal{R}$.

Lemma 2.9 $(Lp^{\Sigma}(\mathbb{R}))^N \lhd \mathcal{W}$.

Proof. Suppose $\mathcal{W} \trianglelefteq (Lp^{\Sigma}(\mathbb{R}))^N$. Notice that $\mathcal{W}$ is OD^N_{Σ}. Notice that $\mathcal{N}$ is the δ_k-core of $\mathcal{S}$. Let then τ be a name for a sound Σ-premouse over $\mathcal{N}|\delta_k$ projecting to δ_k such that it is always realized as the δ_k-core of the translation of $\mathcal{W}$ into an extension of $\mathcal{Q}^h_{\omega}$. Then τ is OD^N_{Σ} and hence, there is OD^N_{Σ} condition $(\emptyset, X)$ that decides τ. It then follows that if $\mathcal{N}^*$ is the premouse given by τ and $(\emptyset, X)$ then $\mathcal{N}^* = \mathcal{N}$. But this implies that $\mathcal{N} \in OD^N_{\mathcal{N}|\delta_k,\Sigma}$ and hence, by N-fullnes of $\mathcal{N}$, $\mathcal{N} \in \mathcal{N}$, contradiction. □

Let then $\mathcal{R} \trianglelefteq \mathcal{W}$ be the first level of $\mathcal{W}$ such that

$$(Lp^{\Sigma}(\mathbb{R}))^N \trianglelefteq \mathcal{R} \text{ and } \rho_{\omega}(\mathcal{R}) = \mathbb{R}.$$

The next two lemmas finishes the prove of Theorem 0.1.

Lemma 2.10 $\mathcal{R} \in V$.

Proof. Suppose not. Using DC we can find $\pi : H \to \mathcal{J}_{\mu}(\wp(\mathbb{R}))$ such that $\mu > \Theta$ and H is countable. We can further assume that $\Sigma, \Lambda, N \in rng(\pi)$. Let then $\bar{N} = \pi^{-1}(N)$. Let $g \subseteq Coll(\omega, \bar{N})$ be H-generic and let $g_1, g_2 \subseteq \pi^{-1}(\mathbb{P}^f) \in H[g]$ be two different $\bar{N}$-generics. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the versions of $\mathcal{R}$ defined for $\bar{N}$ using g_1 and g_2 respectively. Because both are (ω, ω_1)-iterable, we have that $\mathcal{R}_1 = \mathcal{R}_2$. Hence, the version of $\mathcal{R}$ for $\bar{N}$ is $OD^{H[g]}_{\Sigma^H}$ and hence, it is in H. □

It remains to show that $\mathcal{R}$ is in N and countably iterable in N. Granted this, we obtain the desired contradiction, hence complete the proof of Theorem 0.1.

Lemma 2.11 *$\mathcal{R} \in N$ and $\mathcal{R}$ is countably iterable in N.*

Proof. First, we show $\mathcal{R} \in N$. We can assume $o(\mathcal{R})$ is limit and $\rho_1(\mathcal{R}) = \mathbb{R}$ (if not, look at the mastercode structure of $\mathcal{R}$). In V, we can write $\mathcal{R} = \cup_{\xi < o(\mathcal{R})} Th^{\mathcal{R}|\xi}(\mathbb{R})$. Notice that for all $\xi < o(\mathcal{R})$, $|Th^{\mathcal{R}|\xi}(\mathbb{R})|_w < |A|_w$ (where A is the least OD^N_{Σ} set of reals such that $A \notin Lp^{\Sigma}(\mathbb{R}))^N$). Since $\mathcal{R}$ is a well-ordered union of sets Wadge reducible to A, it follows from a theorem of Kechris that $\mathcal{R}$ is projective in A. This implies that $\mathcal{R} \in N$.

It remains to show $\mathcal{R}$ is countably iterable in N. Working in N, given $\sigma \in \wp_{\omega_1}(\mathbb{R})$ we say σ is bad if there is a non-iterable sound Σ-premouse $\mathcal{W}$ over σ projecting to σ and an embedding $\pi : \mathcal{W} \to \mathcal{R}$. Notice that (in V) $\mathcal{R}$ is countably (ω, ω_1)-iterable and hence, for each $\sigma \in \wp_{\omega_1}(\mathbb{R})$ there is at most one such $\mathcal{W}$. We denote it by $\mathcal{W}(\sigma)$.

To show that $\mathcal{R}$ is countably iterable in N, it is enough to show that for stationary many σ, $\mathcal{W}(\sigma)$ is undefined. Towards a contradiction assume that for a club C of σ, $\mathcal{W}(\sigma)$ is defined. Then the set

$$B = \{(\sigma, \mathcal{W}(\sigma)) : \sigma \in C\}.$$

is $OD^N_{\Sigma,u}$ for some real u. It follows that for every $\sigma \in C$ such that $u \in \sigma$, $\mathcal{W}(\sigma) \in (Lp^\Sigma(\sigma))^N$. Because for every σ, $\mathcal{W}(\sigma)$ has an (ω, ω_1)-iteration strategy in V, we get that $\mathcal{W}(\sigma) \trianglelefteq (Lp^\Sigma(\sigma))^N$, which is a contradiction. □

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