Announcement of recent results in descriptive inner model theory*†‡

Grigor Sargsyan
Department of Mathematics
Rutgers University

December 15, 2020

The Inner Model Program (IMPr) in set theory is the program of building canonical inner models for large cardinals, such as measurable cardinals, Woodin cardinals, superstrong cardinals, supercompact cardinals and other large cardinals. The Inner Model Problem (IMP) for the large cardinal axiom $\phi(\kappa)$ is the problem of building canonical inner models that satisfy $\exists\kappa\phi(\kappa)$. In many ways, IMPr is the response to Scott's celebrated theorem that Gödel's constructible universe doesn't have measurable cardinals. It is then interesting to know whether there can be canonical universes like Gödel's constructible universe that do carry measurable cardinals.

Since then the methods developed to study canonical inner models have found many deep applications in set theory including in the proofs of determinacy and in lower bound calculations of set theoretic and combinatorial hypothesis. Several deep connections between inner model theoretic notions, —e.g. extender models, iteration strategies and etc,—and descriptive set theoretic notions,—e.g. Suslin reprsentations, scales, universally Baire sets and etc,—have been established.

The goal of this paper is to outline some recent results in descriptive inner model theory that collectively imply demonstrate that one of the central methods for constructing canonical inner models cannot succeed according to the current norms. The approach that we rule out is the approach via the convergence of backgrounded constructions. Backgrounded here refers to the completeness of the ultrafilters or extenders used in the construction. Here we deal with one of the most liberal back-

*2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.
†Keywords: Mouse, inner model theory, descriptive set theory, hod mouse.
‡The author's research was partially supported by the NSF Career Award DMS-1352034.

grounding conditions, where the extender is certified to be used in the construction just in case it is *certified by a collapse*. Many precursors of this certification method, such as *countable completeness*, have appeared in the literature, but we will use the one introduced in [3] (see [3, Definition 2.2]). Extenders that are certified by a collapse are essentially derived from Skolem hulls that have strong closure properties describe in [3, Definition 2.2]. We let $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$ be the backgrounded construction in which all extenders used are certified by a collapse. This construction is described in [3, Definition 2.1].

As is well-known to inner model theorists, the desired goal of such K^{c} constructions is to produce a model of certain desired ordinal height. Standing in the way of convergence are certain technical fine structural properties, —*solidity and universality*, —that are needed to describe the next model in the construction. All of these technical conditions are a consequence of a more intelligible property called *countable iterability*. If all countable submodels of models appearing in a K^{c} construction are $\omega_1 + 1$-iterable (or countably iterable) then the K^{c} construction converges.

The main technical contribution of this paper (see Theorem 1.2), when coupled with other results, will establish the following theorem, which is a consequence of many years of work done by the entire community starting from 1960s.

Theorem 0.1 *It is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals that there is a transitive model of* ZFC *in which* $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$ *construction does not converge.*

In what follows we will make some general comments intended for experts, then introduce the topic to non-experts and then present the proof of our main technical theorem, Theorem 1.2.

1 K^{c} constructions for experts

The K^{c}-construction that Theorem 0.1 rules out in this paper is the one used in [3], which is perhaps the most liberal K^{c}-construction defined in the literature, allowing many extenders with very weak certificates to be put on the sequence of K^{c}. We denote this K^{c} by $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$. We next describe the exact chain of implications that rule out $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$. The first step is the following theorem (see [, Theorem 0.3]) which is a consequence of a very powerful covering theorem, [3, Theorem 3.4].

Step 1: (Jensen-Schimmerling-Schindler-Steel) Assume $\omega_2^\omega = \omega_2 + \neg\Box(\omega_3) + \neg\Box_{\omega_3}$. Let $g \subseteq Coll(\omega_3, \omega_3)$ be generic. Then if $V[g] \vDash$ "$\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$ converges" then $(\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}})^{\mathsf{V[g]}} \vDash$

"there is a subcompact cardinal".

Of course, the above theorem uses Schimmerling-Zeman characterization of $\square$ in extender models (see [5]). The next steps need some terminology.

Let † stand for the following theory:

1. ZF.
2. $V = L(\wp(\mathbb{R}))$.
3. AD^+.
4. There is no countable ω_1-iterable model of ZFC that has a Woodin cardinal that is a limit of Woodin cardinals.

We let ‡ be the same as † except we change clause 3 to $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$. It follows from [4] that models of ‡ satisfy HPC. The reason that this is important is that it implies, via [9], that HOD of models of ‡ can be represented as a union of directed system of hod pairs.

Assume now that $V \vDash$ ‡. Let $\mathcal{H}$ be the hod premouse representation of V_Θ^{HOD} and suppose $\alpha < \Theta$. We then set

1. $\Delta_\alpha = \{A \subseteq \mathbb{R} : w(A) < \alpha\}$,
2. $\mathbb{C}_\alpha^- = L_\alpha((\mathcal{H}|\alpha)^\omega)$,
3. $\mathbb{C}_\alpha = L((\mathcal{H}|\alpha)^\omega)$, and
4. $\mathbb{C}_\alpha^+ = \mathrm{HOD}((\mathcal{H}|\alpha)^\omega)$.

Definition 1.1 *Assume $V \vDash$ ‡ and suppose $\alpha < \Theta$. We let $\square^\alpha$ be the following statement: for some $\kappa < \alpha$*

1. *κ is a regular member of the Solovay sequence,*
2. *$\mathbb{C}_\alpha \vDash \kappa = \Theta$,*
3. *$\mathbb{C}_\alpha^+ \vDash \alpha = \kappa^+ + \mathrm{cf}(\alpha) = \alpha$,*
4. *$\mathbb{C}_\alpha^- \cap \wp(\mathbb{R}) = \mathbb{C}_\alpha \cap \wp(\mathbb{R}) = \mathbb{C}_\alpha^+ \cap \wp(\mathbb{R}) = \Delta_\kappa$,*
5. *$\wp(\kappa^\omega) \cap \mathbb{C}_\alpha = \wp(\kappa^\omega) \cap \mathbb{C}_\alpha^+$,*
6. *$\mathrm{cf}^V(\kappa) = \kappa \leq \mathrm{cf}^V(\alpha)$.*

If κ witnesses that $\Box^\alpha$ holds then we express this fact by saying that $\Box^{\kappa,\alpha}$ holds.

The main contribution of this paper is the consistency of ‡ with $\exists\alpha\Box^\alpha$.

Step 2: (with Larson) Suppose $V \models ‡ + \exists\alpha,\kappa\Box^{\kappa,\alpha}$. Fix κ, α such that κ is a regular member of the Solovay sequence and $\Box^{\kappa,\alpha}$ holds. Let $M = \mathbb{C}^+_\alpha$, and set $\mathbb{P} = (\mathbb{P}_{max} * Add(1,\kappa) * Add(1,\alpha))^M$. Let $G \subseteq \mathbb{P}$ be generic. Then $M[G] \models \omega_2^\omega = \omega_2 + \neg\Box(\omega_3) + \neg\Box_{\omega_3}$.

The above theorem builds upon [1] and, of course, [10]. In particular, it uses the key argument of Larson used in the proof of [1, Theorem 7.3]. The third step involves establishing the consistency of the hypothesis of the above theorem, which is the main task of this paper.

Step 3:

Theorem 1.2 *Suppose $\mathcal{P}$ is an (lbr) hod premouse such that there is an increasing sequence $\kappa_0 < \lambda_0 < \delta_0 < \eta_0$ having the following properties in $\mathcal{P}$:*

1. *η_0 is an inaccessible limit of Woodin cardinals.*

2. *δ_0 is the least Woodin cardinal δ with the property that if ξ is the second least $< \delta$-strong cardinal then ξ is a limit of Woodin cardinals.*

3. *κ_0 is the least $< \delta_0$-strong cardinal and λ_0 is the second $< \delta_0$-strong cardinal.*

Let $g \subseteq Coll(\omega, < \eta_0)$ be generic and let $M = L(\Gamma^\infty_g, \mathbb{R}_g)$. Then $M \models ‡ + \exists\alpha,\kappa\Box^{\kappa,\alpha}$.

Moreover, letting $\mathcal{M}_\infty$ be the direct limit of countable iterates of $(\mathcal{P}|\eta_0, \Sigma)$ where Σ is the internal strategy of $\mathcal{P}|\eta_0$[1] and setting $\kappa = \pi^\Sigma_{\mathcal{P}|\eta_0,\infty}(\kappa_0)$ and $\lambda = \pi^\Sigma_{\mathcal{P}|\eta,\infty}(\lambda_0)$, $M \models \Box^{\kappa,\lambda}$.

The proof of Theorem 1.2 uses the full normalization technique developed by Jensen, Schlutzenberg and Steel (for example see [2], [6], [8] or [9]). While some of the key ideas predate full normalization, it seems the main reason that it was not proven before is exactly the fact that full normalization was missing from the literature. The author was simply unable to carry out many of the calculations used in the proof before the full normalization was discovered.

The last step places the strength of the hypothesis of Theorem 1.2 below that of a Woodin cardinal that is a limit of Woodin cardinals.

[1]Which we will confuse with its natural extension to generic extension of $\mathcal{P}$.

Step 4: Let Hypo be the statement that there is an lbr premouse $\mathcal{P}$ as in Theorem 1.2. Then assuming there is a Woodin cardinal that is a limit of Woodin cardinals it is consistent that there is a transitive model $M \vDash \mathsf{Hypo}$.

A simple application of modus pones gives Theorem 0.1. The above theorem will be proved in [4]. The proof, once again, uses the full normalization technique discovered by Jensen, Schlutzenberg and Steel.

This line of research most likely will also rule out Steel's K^{c}, which we denote by $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$, introduced in [7]. $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$ uses stronger backgrounding condition, thus blocking many extenders from being put on the sequence of $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$. For example, all extenders used in a $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$ construction have inaccessible critical points, which is certainly not the case in $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$ constructions. Moreover, $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$ acquires its desired covering properties as a consequence of the presence of a measurable cardinal (see [7, Theorem 1.4]), and so to establish the consistency of non-convergence of a $\mathsf{K}^{\mathsf{c}}_{\mathsf{s}}$ construction, we will need to build a Chang model with a measurable cardinal and then adopt the usual Easton support iterations and large-cardinal-preservation arguments to such Chang models. All of this is currently a work in progress, and it involves non-trivial generalizations of the arguments presented in this paper.

A personal view: While it is tempting to interpret the above results as devastating for the portion of the inner model program that deals with calculations of lower bounds and covering properties of canonical structures in the short extender region, my interpretation is far from it. For me personally the most important consequence of Theorem 0.1 is the clarification between our two main approaches to the problem of finding canonical structures with covering properties. It has been puzzling that descriptive inner model theoretic approach leads to conjectures involving derived models while the classical approach leads to stacking mice, which seem to be two unrelated structures. It seems the main message of Theorem 0.1 is that one cannot avoid descriptive set theory. Though this is not definite as one can still hope to prove that under PFA some model of $\mathsf{K}^{\mathsf{c}}_{\mathsf{jsss}}$ reaches a superstrong cardinal, though clearly the ideas behind such a proof have to be different than the ideas of [3].

References

[1] Andrés Eduardo Caicedo, Paul Larson, Grigor Sargsyan, Ralf Schindler, John Steel, and Martin Zeman. Square principles in $\mathbb{P}_{\max}$ extensions. *Israel J. Math.*, 217(1):231–261, 2017.

[2] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[3] Ronald Jensen, Ernest Schimmerling, Ralf Schindler, and John Steel. Stacking mice. *The Journal of Symbolic Logic*, 74(01):315–335, 2009.

[4] G. Sargsyan. Generic generators. 2019. available upon request.

[5] Ernest Schimmerling and Martin Zeman. Characterization of $\square_\kappa$ in core models. *Journal of Mathematical Logic*, 4(01):1–72, 2004.

[6] F. Schlutenberg. Iterability for (transfinite) stacks. 2018. Available at https://sites.google.com/site/schlutzenberg/home-1/research/papers-and-preprints?authuser=0.

[7] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[8] John R. Steel. Mouse pairs and suslin cardinals. 2018. Available at math.berkeley.edu/∼steel/papers/Publications.html.

[9] John R. Steel. Normalizing iteration trees and comparing iteration strategies. 2019. Available at math.berkeley.edu/∼steel/papers/Publications.html.

[10] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the non-stationary ideal*, volume 1 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010.