Varsovian models I*†

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Abstract

Let $M_{\rm sw}$ denote the least iterable inner model with a strong cardinal above a Woodin cardinal. By [8], $M_{\rm sw}$ has a fully iterable core model, $K^{M_{\rm sw}}$, and $M_{\rm sw}$ is thus the least iterable extender model which has an iterable core model with a Woodin cardinal. In V, $K^{M_{\rm sw}}$ is an iterate of $M_{\rm sw}$ via its iteration strategy Σ.

We here show that $M_{\rm sw}$ has a bedrock which arises from $K^{M_{\rm sw}}$ by telling $K^{M_{\rm sw}}$ a specific fragment $\bar{\Sigma}$ of its own iteration strategy, which in turn is a tail of Σ. Hence $M_{\rm sw}$ is a generic extension of $L[K^{M_{\rm sw}}, \bar{\Sigma}]$, but the latter model is not a generic extension of any inner model properly contained in it.

These results generalize to models of the form $M_{\rm s}(x)$ for a cone of reals x, where $M_{\rm s}(x)$ denotes the least iterable inner model with a strong cardinal containing x. In particular, the least iterable inner model with a strong cardinal above two (or seven, or boundedly many) Woodin cardinals has a 2–small core model K with a Woodin cardinal and its bedrock is again of the form $L[K, \bar{\Sigma}]$.

1 Introduction.

By a theorem of W. Hugh Woodin, every pure extender model W with a Woodin cardinal has a non–trivial ground,[1] i.e., there is some inner model $\bar{W} \subsetneq W$ such that W is a generic extension of $\bar{W}$. E.g., let $\bar{W} = \mathcal{P}^W(\mathcal{M})$, where $\mathcal{M}$ arises from an

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[1]The terms "ground," "bedrock," and "mantle" are taken from [2]. If $\bar{W} \subset W$ are both inner models, then $\bar{W}$ is a ground of W iff W is a generic extension of $\bar{W}$. W is a bedrock iff W itself is the only ground of W.

$L[E]$–construction inside W up to its first Woodin cardinal and $\mathcal{P}^W(\mathcal{M})$ denotes the $\mathcal{P}$–construction above $\mathcal{M}$ and performed inside W, cf. [9].

The situation is different for hod mice, also called "strategic mice." Woodin showed that there are strategic mice which are bedrocks, i.e., which don't admit any non–trivial grounds, cf. [19]. Strategic mice naturally arise as HODs of models of determinacy, cf. [6].

The current paper produces a minimal example of an extender model with a Woodin cardinal which, when equipped with a fragment of its own iteration strategy, is a bedrock, and it will also be the HOD of a homogeneous generic extension of an extender model.

By a theorem of John Steel, extender models with no strong cardinals cannot have a fully iterable core model with a Woodin cardinal. The paper [3] analyzes the mantle[2] of (tame) extender models with Woodin cardinals but no strong cardinals and shows that it is always a lower part model; in particular, their mantles are not grounds. On the other hand, writing $M_{\rm sw}$ for the least iterable inner model with a strong cardinal above a Woodin cardinal, [8] shows that $M_{\rm sw}$ does have a fully iterable core model $K^{M_{\rm sw}}$ which in turn has a strong cardinal above a Woodin cardinal, so that the mantle of $M_{\rm sw}$ should contain $K^{M_{\rm sw}}$ and *not* be a lower part model.

The current paper analyzes the mantle of $M_{\rm sw}$ and shows that it is a ground, hence the smallest ground, and thus a bedrock. The mantle turns out to be $L[K^{M_{\rm sw}}, \bar{\Sigma}]$, where $\bar{\Sigma}$ is a fragment of the iteration strategy of $K^{M_{\rm sw}}$ which $M_{\rm sw}$ can see and which in turn is a fragment of the tail of $M_{\rm sw}$'s own iteration strategy. $K^{M_{\rm sw}}$ is fully iterable inside $L[K^{M_{\rm sw}}, \bar{\Sigma}]$.

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2 The mantle of $M_{\rm sw}$.

For the record, a *mouse* is a premouse which is countably iterable, i.e., all transitive collapses of sufficiently elementary countable substructures are supposed to be

[2]The mantle of an inner model is defined to be the intersection of all of its grounds.

$(\omega, \omega_1, \omega_1 + 1)$–iterable. Cf. [15, Definition 4.4].

Throughout our paper, we shall assume that V is closed under the operation $a \mapsto a^{\P}$ mapping a to a–pistol, the least active a–mouse with a strong cardinal. For any transitive s.w.o.'d[3] set a, we let $M_{\rm s}(a)$ be the minimal proper class a-mouse with a strong cardinal. $M_{\rm s}(a)$ is obtained from $a^{\P}$ by iterating its top measure out of the universe.

For the purposes of the current paper, a premouse $\mathcal{N}$ is called *suitable* if for some $\delta \in \mathcal{N}$,

1. $\mathcal{N} \models$ "δ is a Woodin cardinal,"
2. $\mathcal{N} = M_{\rm s}(\mathcal{N}|\delta)|\delta^{+M_{\rm s}(\mathcal{N})}$,
3. for every $\eta < \delta$, $M_{\rm s}(\mathcal{N}|\eta) \models$ "η is not Woodin," and
4. $\mathcal{N} \models$ "I'm (ω, δ, δ)–iterable."

We shall now also assume that there is a suitable premouse, and more: Let us call a premouse $\mathcal{M}$ sw–*small* iff for all extenders F from $\mathcal{M}$'s sequence,

$$\mathcal{M}|\text{crit}(F) \models \text{"there is no strong cardinal above a Woodin cardinal."}$$

Let us assume that there is a non–sw–small mouse, and let $\mathcal{M}$ be a non–sw–small mouse such that every proper initial segment of $\mathcal{M}$ is sw–small. As we assume V to be closed under $a \mapsto a^{\P}$, the $(\omega, \omega_1, \omega_1)$–iterability of $\mathcal{M}$ implies that $\mathcal{M}$ be fully iterable with respect to arbitrary stacks of normal trees. Let us denote by

$$M_{\rm sw}$$

the result of iterating $\mathcal{M}$'s top measure out of the universe. Let $\delta = \delta^{M_{\rm sw}}$ be the Woodin cardinal of $M_{\rm sw}$, and let $\kappa = \kappa^{M_{\rm sw}}$ be the strong cardinal of $M_{\rm sw}$. We have that $M_{\rm sw} = M_{\rm s}(M_{\rm sw}|\delta)$, and $M_{\rm sw}|\delta^{+M_{\rm sw}}$ is suitable.

By way of notation, if W is any extender model, then we will denote by δ^W the least Woodin cardinal of W (if it exists), we will denote by $\mathbb{B}^W$ the δ–generator version of the extender algebra of W at δ^W (cf. [9, Lemma 1.3]) given by all total extenders of W's sequence up to δ^W (if it exists), and we will denote by κ^W the least strong cardinal of W (if it exists).

In what follows, the relevant W will always be an iterate of $M_{\rm sw}$, so that δ^W will also be the unique Woodin cardinal of W, and κ^W will be the unique strong cardinal of W.

[3]self–well–ordered

The iteration strategy for $\mathcal{M}$ with respect to finite stacks of normal trees induces an iteration strategy, call it Σ, for M_{sw} with respect to finite stacks of normal trees. We have the following.

(1) Σ satisfies hull condensation, cf. [6, Definition 1.31],

(2) Σ satisfies branch condensation, cf. [6, Definition 2.14], and

(3) Σ is positional, cf. [6, Definition 2.35 (4)].[4]

We shall need the following slight refinement of (2):

Lemma 2.1 *Let M be a proper class sized Σ–iterate of M_{sw}. Let $\mathcal{U}$ be an iteration tree on M living on $M|\delta^M$ with a last model $\mathcal{M}^{\mathcal{U}}_\theta$ such that $[0,\theta]_{\mathcal{U}}$ does not drop and $\mathcal{U}$ is according to Σ_M. Let $\mathcal{T}$ be an iteration tree on M living on $M|\delta^M$ and of limit length which is according to Σ_M. If b and k are in some generic extension of V such that*

(a) b is a cofinal non–dropping branch through $\mathcal{T}$, and

(b) $k\colon \mathcal{M}^{\mathcal{T}}_b|\delta^{\mathcal{M}^{\mathcal{T}}_b} \to \mathcal{M}^{\mathcal{U}}_\theta|\delta^{\mathcal{M}^{\mathcal{U}}_\theta}$ is elementary with

$$\pi^{\mathcal{U}}_{0,\theta} \restriction M|\delta^M = k \circ \pi^{\mathcal{T}}_{0,b} \restriction M|\delta^M, \tag{1}$$

then $b = \Sigma_M(\mathcal{T})$.

Proof. Write $c = \Sigma_M(\mathcal{T})$. If $\delta(\mathcal{U}) \neq \pi^{\mathcal{T}}_{0,b}(\delta^M) = \delta^{\mathcal{M}^{\mathcal{T}}_b}$, then $\mathcal{M}^{\mathcal{T}}_b$ comes with a $\mathcal{Q}$–structure which by the existence of k is iterable, and this gives that $b = c$.

Let us now assume that $\delta(\mathcal{U}) = \pi^{\mathcal{T}}_{0,b}(\delta^M)$. The key fact is that k may be extended to $k^+\colon \mathcal{M}^{\mathcal{T}}_b \to \mathcal{M}^{\mathcal{U}}_\theta$ by setting

$$k^+(\pi^{\mathcal{T}}_{0,b}(f)(a)) = \pi^{\mathcal{U}}_{0,\theta}(f)(k(a)).$$

It is easy to verify that k^+ is well–defined and elementary. Also,

$$\pi^{\mathcal{U}}_{0,\theta} = k^+ \circ \pi^{\mathcal{T}}_{0,b}. \tag{2}$$

Now let λ be a sufficiently large V–cardinal, and let λ^{+n} denote the n^{th} cardinal successor of λ as being computed in V.

[4] The last "positional" in [6, Definition 2.35 (4)] should read "weakly positional," though.

We have that

$$X = \mathrm{Hull}^M(\{\lambda^{+n} \colon 0 < n < \omega\}) \cap \delta^M$$

is cofinal in δ^M. Also,

$$\pi^{\mathcal{T}}_{0,c}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega, \tag{3}$$

and

$$\pi^{\mathcal{U}}_{0,\theta}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega,$$

and by (2) the latter implies that

$$\pi^{\mathcal{T}}_{0,b}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega. \tag{4}$$

But (3) and (4) give that

$$\pi^{\mathcal{T}}_{0,c} \restriction X = \pi^{\mathcal{T}}_{0,b} \restriction X,$$

which implies that $b = c$ by the "zipper argument," cf. e.g. [15, p. 1645f.], as desired.
□ (Lemma 2.1)

Some of the arguments to follow will look pretty familiar to researchers working in the area of descriptive inner model theory, cf. e.g. [17, Section 3].

Let us consider the set $\mathbb{U}$ consisting of all $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$, some $n < \omega$, such that either $n = 0$ and $\mathrm{lh}(\mathcal{U}_0) = 1$ (i.e., $\mathcal{U}$ is trivial), or else there is a sequence $\eta_0 < \ldots < \eta_n < \kappa$ of cutpoints of M_{sw} and:

(a) $\mathcal{U} \in M_{\mathrm{sw}}|\kappa$,

(b) $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ is a finite stack of normal iteration trees $\mathcal{U}_k$,

(c) $\mathcal{U}_0$ is on M_{sw} and lives below δ,

and for every $k \leq n$,

(d) $\mathrm{lh}(\mathcal{U}_k) = (\eta_k)^{+M_{\mathrm{sw}}} = \delta(\mathcal{U}_k)$,

(e) $\mathcal{U}_k$ is defiable over $M_{\mathrm{sw}}|(\eta_k)^{+M_{\mathrm{sw}}}$ and is guided by $\mathcal{Q}$–structures which are obtained via $\mathcal{P}$–constructions, cf. [9, Section 1],

(f) $P(\mathcal{M}(\mathcal{U}_k))$ is a proper class,[5] $\delta(\mathcal{U}_k)$ is a Woodin cardinal of $P(\mathcal{M}(\mathcal{U}))$, and

$$P(\mathcal{M}(\mathcal{U}))[G] = M_{\mathrm{sw}}$$

for some G which is $\mathbb{B}^{P(\mathcal{M}(\mathcal{U}))}$–generic over $P(\mathcal{M}(\mathcal{U}))$, and

[5] Here and in what follows we write $P(M)$ for the $\mathcal{P}$–construction over M as being performed inside M_{sw}. [9, Section 1] would write $\mathcal{P}(M_{\mathrm{sw}}, M, -)$ for this model.

(g) if $k > 0$, then $\mathcal{U}_k$ is on $P(\mathcal{M}(\mathcal{U}_{k-1}))$ and lives below $\delta(\mathcal{U}_{k-1})$.

Let $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ be as above, where $\mathcal{U}_n$ is not trivial. For every $k \leq n$ and inside M_{sw}, $P(\mathcal{M}(\mathcal{U}_k))$ is a universal weasel over $\mathcal{M}(\mathcal{U}_k)$ below $\mathcal{M}(\mathcal{U}_k)^{\P}$. Let us write $K(\mathcal{M}(\mathcal{U}_k))$ for the $\mathcal{M}(\mathcal{U}_k)^{\P}$–small core model over $\mathcal{M}(\mathcal{U}_k)$ as constructed inside M_{sw}. In V, let $b_k = \Sigma(\mathcal{U}_k)$. We then have:

Lemma 2.2 *Let $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{U}$, where $\mathcal{U}_n$ is not trivial. For every $k \leq n$, $P(\mathcal{M}(\mathcal{U}_k)) = K(\mathcal{M}(\mathcal{U}_k)) = M_s(\mathcal{M}(\mathcal{U}_k)) = \mathcal{M}_{b_k}^{\mathcal{U}_k}$.*

Proof. Let us write $M = \mathcal{M}(\mathcal{U}_k)$. As $P(M)[G] = M_{\mathrm{sw}}$ for some generic G, $K(M) = K(M)^{M_{\mathrm{sw}}} = K(M)^{P(M)[G]} = K(M)^{P(M)} \subset P(M)$. On the other hand, $P(M)$ is a universal weasel over M, so that there is an elementary embedding $j \colon K(M) \to P(M)$, which, as $K(M)$ and $P(M)$ are below $M^{\P}$, is given by an iteration of $K(M)$. But then $K(M) \subset P(M)$ gives $K(M) = P(M)$.

$M_{\mathrm{sw}} = \mathrm{Hull}^{M_{\mathrm{sw}}}(I)$, where I is the class of generating indiscernibles given by iterating the top measure of $(M_{\mathrm{sw}}|\delta)^{\P}$ out of the universe. We claim that

$$P(M) = \mathrm{Hull}^{P(M)}(\delta(\mathcal{U}_k) \cup I). \tag{5}$$

To show (5), notice first that the extender sequence of M_{sw} may be defined over $P(M)[G]$ from the parameter $M_{\mathrm{sw}}|\delta(\mathcal{U}_k) \in P(M)[G]$ and the extender sequence of $P(M)$. The forcing language associated with forcing with $\mathbb{B}^{P(M)}$ over $P(M)$ thus has a term for the extender sequence of M_{sw} and therefore also a term for the canonical Σ_1 Skolem function $h_{M_{\mathrm{sw}}}$ of M_{sw}, cf. [10, Theorem 10.16]. Writing h for this term for $h_{M_{\mathrm{sw}}}$, we have that the function $h^* \colon \mathbb{B}^{P(M)} \times \omega \times [M : \mathrm{sw}]^{<\omega} \to P(M)$ with

$$h^*(p, n, \mathsf{a}) = \begin{cases} y & \text{if } p \Vdash^{\mathbb{B}^{P(M)}}_{P(M)} h(\check{n}, \check{\mathsf{a}}) = \check{y}, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

is definable over $P(M)$ using a name for $M_{\mathrm{sw}}|\delta(\mathcal{U}_k)$. But G and $M_{\mathrm{sw}}|\delta(\mathcal{U}_k)$ are computable from each other, so that $\mathrm{Hull}^{P(M)}(X)$ is closed under h^* for *any* X and by $\mathbb{B}^{P(M)} \subset \mathrm{Hull}^{P(M)}(\delta(\mathcal{U}_k) \cup I)$ and $M_{\mathrm{sw}} = \mathrm{Hull}^{M_{\mathrm{sw}}}(I)$, we obtain (5).

It is easy to see that I is also a class of indiscernibles for $P(M)$, so that by (5) is a class of generating indiscernibles. But now $M_s(\mathcal{M}(\mathcal{U}_k))$ is also a least inner model with a strong cardinal end–extending $\mathcal{M}(\mathcal{U}_k)$ and having a proper class of generating indiscernibles relative to $\mathcal{M}(\mathcal{U}_k)$. It follows that $P(M) = M_s(\mathcal{M}(\mathcal{U}_k))$.

Virtually the same argument shows $P(M) = \mathcal{M}_{b_k}^{\mathcal{U}_k}$ by induction on $k \leq n$. □ (Lemma 2.2)

Let $\mathcal{U} = (\mathcal{U}\colon k \leq n) \in \mathbb{U}$. If $\mathcal{U}_n$ is not trivial, then we shall write $\mathcal{M}(\mathcal{U})$ for $\mathcal{M}(\mathcal{U}_n)$. To uniformize the notation, if $n = 0$ and $\mathcal{T}_0$ is trivial, then we shall denote by $P(\mathcal{M}(\mathcal{U}))$ the model M_{sw}. Let us write $\mathcal{F}$ for the family of all proper class mice of the form $P(\mathcal{M}(\mathcal{U}))$, where $\mathcal{U} \in \mathbb{U}$. For the record, $\mathcal{F}$ is definable inside M_{sw} using M_{sw}'s extender sequence as a predicate.

Let $\mathcal{T}, \mathcal{U} \in \mathbb{U}$, and write $N = P(\mathcal{M}(\mathcal{T}))$ and $N' = P(\mathcal{M}(\mathcal{U}))$. By Lemma 2.2, N is a Σ–iterate of M_{sw}. Let Σ_N denote the iteration strategy for N which is induced by Σ. As Σ is positional, Σ_N only depends on N, not on the particular iteration tree which witnesses that N is a Σ–iterate of M_{sw}.

Assume for now that N' is a Σ_N–iterate of N via a finite stack of normal trees, which is tantamount to saying that there is a finite stack $\mathcal{T}_0{}^\frown \dots {}^\frown\mathcal{T}_k$ of normal trees on M_{sw} such that N is the last model of one of the $\mathcal{T}_i$, $i < k$, and N' is the last model of $\mathcal{T}_k$. As Σ satisfies hull condensation, Σ is commuting, cf. [6, Definition 2.35 (9)], so that Σ_N satisfies the Dodd–Jensen property, cf. [6, Proposition 2.36], and hence there is a *unique* iteration map from N to N'. In what follows, we let $\pi_{N,N'}$ denote this unique iteration map from N to N'.

Let's now drop the assumption that N' be a Σ_N–iterate of N. Let $\eta < \kappa$, $\eta > \max(\delta(\mathcal{T}), \delta(\mathcal{U}))$, be a cutpoint of M_{sw}. Let $\mathcal{T}^*, \mathcal{U}^*$ be normal iteration trees on N, N', respectively, such that both start out by iterating the least measurable cardinal and its images $\eta + 1$ times, and from then on $\mathcal{T}^*$ and $\mathcal{U}^*$ result from comparison, simultaneously making an initial segment of the background model generic over the respective iterate; more precisely, if $\mathcal{T}^* \restriction \alpha$ and $\mathcal{U}^* \restriction \alpha$ have already been defined, where $\eta + 2 \leq \alpha \leq \eta^{+M_{\text{sw}}}$, then if α is a successor ordinal, then we let ν be least such that

(a) $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}} \neq E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$ if there is some such ν, and

(b) $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}} = E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$, and writing $F = E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}}$ and $\mu = \text{crit}(F)$ there is some sequence $\vec{\varphi} = (\varphi_i\colon i < \mu) \in \mathcal{M}^{\mathcal{T}^*}_{\alpha-1}|\nu = \mathcal{M}^{\mathcal{U}^*}_{\alpha-1}|\nu$ of formulae associated with the δ–version of the extender algebra of the current models such that the extender sequence of M_{sw} satisfies $\bigvee i_F(\vec{\varphi}) \cap \mathcal{M}^{\mathcal{T}^*}_{\alpha-1}|\nu$ but not $\bigvee \vec{\varphi}$, if there is no ν as in (a) and there is no drop along $[0, \alpha-1]_{\mathcal{T}^*}$ and no drop along $[0, \alpha-1]_{\mathcal{U}^*}$,

and then we let $\mathcal{T}^* \restriction (\alpha+1)$ and $\mathcal{U}^* \restriction (\alpha+1)$ arise by applying $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}}$ and $E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$ (and padding on one side if ν was chosen according to (a) and on this one side the extender is empty), with the understanding that we stop the construction if there is no such ν, and if α is a limit ordinal, then we pick the unique cofinal branches through $\mathcal{T}^* \restriction \alpha$ and $\mathcal{U}^* \restriction \alpha$ whose limit models have $\mathcal{Q}$–structures as initial segments which are given by $P(\mathcal{M}(\mathcal{T}^* \restriction \alpha)) = P(\mathcal{M}(\mathcal{T}^* \restriction \alpha))$, and we let $\mathcal{T}^* \restriction (\alpha+1)$ and

$\mathcal{U}^* \restriction (\alpha+1)$ arise by adding those branches, again with the understanding that we stop the construction if such branches don't exist. Notice that $\mathcal{T}^*$ and U^* are defined inside $M_{\rm sw}$. By [9, Lemmata 1.3 and 1.5], the construction of $\mathcal{T}^*$ and $\mathcal{U}^*$ will stop exactly at stage $\eta^{+M_{\rm sw}}$, which means that we produced $P(\mathcal{M}(\mathcal{T}^*)) = P(\mathcal{M}(\mathcal{U}^*)) \in \mathcal{F}$ such that by Lemma 2.2, writing $R = P(\mathcal{M}(\mathcal{T}^*)) = P(\mathcal{M}(\mathcal{U}^*))$, R is a Σ_N–iterate of N as well as a $\Sigma_{N'}$–iterate of N'.

We may now let

$$(\mathcal{M}_\infty, (\pi_{N,\infty}\colon N \in \mathcal{F})) = \mathrm{dirlim}(N, (\pi_{N,N'}\colon N, N' \in \mathcal{F})).$$

Notice that even though $\mathcal{F} \in M_{\rm sw}$, this system is not in $M_{\rm sw}$, as the maps $\pi_{N,N'}$ are not in $M_{\rm sw}$.

We are now going to show that we may "catch" $\mathcal{F}$ by a system which does exist in $M_{\rm sw}$.

In what follows, we shall write $\delta_\infty = \delta^{\mathcal{M}_\infty}$ and $\kappa_\infty = \kappa^{\mathcal{M}_\infty}$.

Let s be a non–empty finite set of ordinals. Write $s^- = s \setminus \max(s)$. For $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$ write

$$T_s^N = \{\varphi(\vec{x}, v)\colon \vec{x} \in \delta(\mathcal{U}) \wedge N|\max(s) \models \varphi(\vec{x}, s^-)\}.$$

We call $N \in \mathcal{F}$ *s–iterable* iff for all $\mathcal{T} \in M_{\rm sw}$ on $\mathcal{M}(\mathcal{U})$ of limit length $\lambda < \kappa$ such that $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$, say $\mathcal{T} = (\mathcal{T}_k\colon k < n)$, $n < \omega$, there are for every $i < n$ cofinal branches

$$b_i \in (M_{\rm sw})^{\mathrm{Col}(\omega, <\kappa)}$$

through $\mathcal{T}_i$ such that, writing N_0 for the starting model of $\mathcal{T}_0$ and $N_{i+1} = P(\mathcal{M}(\mathcal{T}_i))$,

(1) $\pi^{\mathcal{T}_i}_{0,b_i}(N_i|\max(s)) = N_{i+1}|\max(s)$, and

(2) $\pi^{\mathcal{T}_i}_{0,b_i}(T_s^{N_i}) = T_s^{N_{i+1}}$.

Writing b for the composition of the branches b_i, $i < n$, and then writing

$$\gamma_s^N = \sup(\delta^N \cap \mathrm{Hull}^{N|\max(s)}(s^-)),$$

the "zipper argument," cf. e.g. the proof of [15, Theorem 6.10], shows that the map

$$\pi^{\mathcal{T}}_{0,b} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) \tag{6}$$

is independent from the particular choice of b and hence is in $M_{\rm sw}$, and moreover

$$\pi^{\mathcal{T}}_{0,b} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) = \pi_{N,N'} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-).$$

We shall denote the map from (6) by $\pi^s_{N,N'}$. Let us write

$$(N,s) \preceq_{\mathcal{F}} (N',t)$$

to express the fact that $N \in \mathcal{F}$ is s–iterable, $N' \in \mathcal{F}$ is t–iterable, $t \supset s$, and there is a tree $\mathcal{T}$ on N as above such that $N' = P(\mathcal{M}(\mathcal{T}))$.

Notice that for N and s as above, the s–iterability of N is uniformly defined in a way which is first order over $M_{\rm sw}$.

Let s be a non–empty finite set of ordinals, $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, and $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$. Write $c = \Sigma_N(\mathcal{T})$. If $\pi^{\mathcal{T}}_{0,c}(s) = s$, then an easy absoluteness argument shows that there is also some $b \in (M_{\rm sw})^{{\rm Col}(\omega,<\kappa)}$ with (1) and (2) above.

This yields that for every non–empty finite set s of ordinals there is some s–iterable $N \in \mathcal{F}$: otherwise there would some non–empty finite set s of ordinals and some infinite sequence $(N_n \colon n < \omega)$ such that $N_0 = M_{\rm sw}$, and N_{n+1} is a Σ_{N_n}–iterate of N_n via some tree $\mathcal{T}_n$ such that $\mathcal{T}_0{}^\frown \ldots {}^\frown\mathcal{T}_n \in \mathbb{U}$ and $\pi_{N_n,N_{n+1}}(s) > s$ for all $n < \omega$. This contradicts the $(\omega,\omega,{\rm OR})$–iterability of $M_{\rm sw}$ in V.

Also, the collection of all s–iterable $N \in \mathcal{F}$ is finitely directed in that if $N \in F$ is s–iterable and $N' \in \mathcal{F}$ is t–iterable, then there is $N^* \in \mathcal{F}$ which is $(s \cup t)$–iterable and

$$(N,s),(N',t) \preceq_{\mathcal{F}} (N^*, s \cup t).$$

This is true because given (N,s) and (N',t), we may pick some $R \in \mathcal{F}$ which is $s \cup t$–iterable. A joint comparison process as defined above will then produce some $s \cup t$–iterable $N^* \in \mathcal{F}$ which in V is Σ_N–iterate of N, a $\Sigma_{N'}$–iterate of N', as well as a Σ_R–iterate of R.

We may then let

$$(\mathcal{M}'_\infty, (\pi^s_{N,\infty} \colon N \in \mathcal{F}, N \text{ is } s\text{–iterable})) = {\rm dirlim}(N, (\pi^s_{N,N'} \colon (N,s) \preceq_{\mathcal{F}} (N',s))). \quad (7)$$

We have that $\mathcal{M}_\infty = \mathcal{M}'_\infty$. To see this, let ρ' be any ordinal, and let $\rho' = \pi_{N,\infty}(\rho)$, where $N \in \mathcal{F}$. Let $\chi < \delta^N$ and let $\bar{s}$ be a finite set of indiscernibles for $M_{\rm sw}$ such that

$$\rho \in {\rm Hull}^N(\chi \cup \{\bar{s}\}).$$

Such χ and $\bar{s}$ exist, as $M_{\rm sw}$ is the hull generated from the class of all indiscernibles for $M_{\rm sw}$, those indiscernibles are not moved by $\pi_{M_{\rm sw},N}$, and $N = {\rm Hull}^N(\delta^N \cup {\rm ran}(\pi_{M_{\rm sw},N}))$. As ${\rm ran}(\pi_{M_{\rm sw},N}) \cap \delta^N$ is cofinal in δ^N, we may in addition assume (by enlarging χ and $\bar{s}$ if necessary) that

$$[\chi, \delta^N) \cap {\rm Hull}^N(\{\bar{s}\}) \neq \emptyset.$$

Let $s = \bar{s} \cup \{\tau\}$, where τ is any V–cardinal strictly above $\max(\bar{s})$. Then N is s–iterable, and $\gamma^N_s > \chi$, so that $\rho \in {\rm dom}(\pi^s_{N,\infty})$.

The following is straightforward to verify.

Lemma 2.3 *In V, $\mathcal{M}_\infty$ is a Σ–iterate of $M_{\rm sw}$ via an ω–stack of normal trees each of which are individually in $M_{\rm sw}$.*

Moreover, let E be a total extender from the $M_{\rm sw}$–sequence with $\operatorname{crit}(F) = \kappa$, *and write $j\colon M_{\rm sw} \to_F \operatorname{ult}(M_{\rm sw}; F)$. Then $j(\mathcal{M}_\infty)$ is a $\Sigma_{\mathcal{M}_\infty}$–iterate of $\mathcal{M}_\infty$ via an ω–stack of normal trees.*

Proof. Let $(\mathcal{U}_k\colon k < \omega)$ be such that $\mathcal{U}_k \in \mathbb{U}$ for all $k < \omega$ and setting $N_k = P(\mathcal{M}(\mathcal{U}_k))$ for $k < \omega$, $(N_k\colon k < \omega)$ is cofinal in $\mathcal{F}$, i.e., if $P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, then there is some $k < \omega$ such that N_k is a $\Sigma_{P(\mathcal{M}(\mathcal{U}))}$–iterate of N_k. The direct limit of the N_k, along with the maps π_{N_k,N_ℓ}, $k \leq \ell < \omega$, must yield $\mathcal{M}_\infty$.

Next, we have for every $N \in \mathcal{F}$, $j(N) \in j(\mathcal{F})$ and $j(N) = \operatorname{ult}(N; E \restriction N)$, where $E \restriction N$ is on the sequence of N. The direct limit of the $\operatorname{ult}(N; E \restriction N)$, along with $j(\pi_{N,N'})$, with N, $N' \in \mathcal{F}$, N' being a Σ_N–iterate of N, is then equal to $\operatorname{ult}(\mathcal{M}_\infty; \pi_{M_{\rm sw},\infty}(F))$ and canonically embed into $j(\mathcal{M}_\infty)$. If $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, then $\operatorname{ult}(N; E \restriction N)$ is an iterate of $M_{\rm sw}$ via $\mathcal{U}^\frown E \restriction N$, and if N, $N' \in \mathcal{F}$, where N' is a Σ_N–iterate of N via $\mathcal{T}$, and if $\mathcal{T} = \mathcal{U}_0{}^\frown \ldots {}^\frown\mathcal{U}_{k-1}$, where all $\mathcal{U}_i$, $i < k$, are normal, then $j(\mathcal{U}_i)$ has the very same tree structure as $\mathcal{U}_i$, and, as $\mathcal{U}_i$ is a hull of $j(\mathcal{U}_i)$, the fact that Σ satisfies branch condensation implies that $j(\mathcal{U}_i)$ is according to Σ and $\Sigma(\mathcal{U}_i) = \Sigma(j(\mathcal{U}_i))$ for $i < k$.

We may conclude that the collection of all $j(N)$, for $N \in \mathcal{F}$, is definable in $\operatorname{ult}(M_{\rm sw}; F)$, and for $\eta = \kappa$ which is a cutpoint of $\operatorname{ult}(M_{\rm sw}; F)$ below $j(\kappa)$ we may work in $\operatorname{ult}(M_{\rm sw}; F)$ to simultaneously compare all $j(N)$, $N \in \mathcal{F}$, in a fashion as on p. 7f. to produce some $M = P^{\operatorname{ult}(M_{\rm sw};F)}(\mathcal{M}(\mathcal{U}')) \in j(\mathcal{F})$ with $\delta(\mathcal{U}') = \kappa^{+\operatorname{ult}(M_{\rm sw};F)} = \kappa^{+M_{\rm sw}}$ and such that M is a $\Sigma_{j(N)}$–iterate of $j(N)$ for all $N \in \mathcal{F}$.

$\operatorname{ult}(\mathcal{M}_\infty; \pi_{M_{\rm sw},\infty}(F))$ is a definable inner model of $\operatorname{ult}(M_{\rm sw}; F)$ and the former must now canonically embed into M. We may then choose some $\eta > \kappa$ which is a cutpoint of $\operatorname{ult}(M_{\rm sw}; F)$ and work in $\operatorname{ult}(M_{\rm sw}; F)$ to compare M with $\operatorname{ult}(\mathcal{M}_\infty; \pi_{M_{\rm sw},\infty}(F))$ in a fashion as on p. 7f. to produce some $M^* = P^{\operatorname{ult}(M_{\rm sw};F)}(\mathcal{M}(\mathcal{U}^*)) \in j(\mathcal{F})$ with $\delta(\mathcal{U}^*) = \eta^{+\operatorname{ult}(M_{\rm sw};F)}$ and such that M^* is a Σ_M–iterate of M and also an iterate of $\operatorname{ult}(\mathcal{M}_\infty; \pi_{M_{\rm sw},\infty}(F))$ via $\Sigma_{\operatorname{ult}(\mathcal{M}_\infty;\pi_{M_{\rm sw},\infty}(F))}$. We may actually produce an ω–sequence of such M^* which is cofinal in $\mathcal{F}^{\operatorname{ult}(M_{\rm sw};F)}$.

$j(\mathcal{M}_\infty)$ may thus be represented as an iterate of $\mathcal{M}_\infty$ via using $\pi_{M_{\rm sw},\infty}(F)$, followed by an ω–stack of normal iteration trees which are according to $\Sigma_{\operatorname{ult}(\mathcal{M}_\infty;\pi_{M_{\rm sw},\infty}(F))}$.
□ (Lemma 2.3)

Inside $\mathcal{M}_\infty$, we may look at the image of the system (7) under the map $\pi_{0,\infty}$. Let us write $\mathcal{M}_\infty^\infty$ for the direct limit model, i.e.,

$$\mathcal{M}_\infty^\infty = \pi_{M_{\rm sw},\infty}(\mathcal{M}_\infty),$$

which is a definable subclass of $\mathcal{M}_\infty$, defined in the same way over $\mathcal{M}_\infty$ as $\mathcal{M}_\infty$ was defined over $M_{\rm sw}$ by (7). In analogy to Lemma 2.3, we have:

Lemma 2.4 *If $N \in \mathcal{F}^{M_\infty}$,then N is a $\Sigma_{\mathcal{M}_\infty}$–iterate of $\mathcal{M}_\infty$, and $\mathcal{M}_\infty^\infty$ is a $\Sigma_{\mathcal{M}_\infty}$–iterate of $\mathcal{M}_\infty$ via an ω–stack of normal trees on $\mathcal{M}_\infty^\infty$.*

In particular, we get a unique iteration map, call it $\pi_{0,\infty}^\infty$, from $\mathcal{M}_\infty$ into $\mathcal{M}_\infty^\infty$, which is given by Σ_{M_∞}. A priori, there doesn't seem to be a reason why $\pi_{0,\infty}^\infty$ should be definable in $M_{\rm sw}$.

However, for each ordinal ρ let us denote by ρ^* the minimum of the set of all $\pi_{N,\infty}(\rho)$ for $N \in \mathcal{F}$. The argument for $\mathcal{M}_\infty = \mathcal{M}'_\infty$ we gave above shows that for every ρ and every $N \in \mathcal{F}$ there is some finite set s of ordinals such that N is s–iterable and $\rho \in {\rm dom}(\pi_{N,\infty}^s)$. In other words, we may define $\rho \mapsto \rho^*$ *inside* $M_{\rm sw}$ by

$$\rho^* = \min(\{\pi_{N,\infty}^s(\rho) \colon N \text{ is } s\text{–iterable and } \rho \in {\rm dom}(\pi_{N,\infty}^s)\}). \tag{8}$$

We have that if $\rho = \pi_{N,\infty}(\bar\rho)$, where N which is s–iterable for some s such that $\rho \in {\rm ran}(\pi_{N,\infty}^s)$, then

$$\begin{aligned} \pi_{N,\infty}(\rho) &= \pi_{N,\infty}(\pi_{N,\infty}(\bar\rho)) \\ &= \pi_{N,\infty}(\pi_{N,\infty}^s(\bar\rho)) \\ &= \pi_{N,\infty}(\pi_{N,\infty}^s)(\pi_{N,\infty}(\bar\rho)) \\ &= \pi_{0,\infty}^\infty(\rho), \end{aligned}$$

which means that

$$\rho^* = \pi_{0,\infty}^\infty(\rho).$$

Notice that $\pi_{0,\infty}^\infty$ is also equal to the ultrapower map produced by applying the long extender derived from $\pi_{0,\infty}^\infty \restriction \mathcal{M}_\infty|\delta_\infty$ to the model $\mathcal{M}_\infty$. In other words,

$$\rho \mapsto \rho^* \text{ may be defined inside the model } L[\mathcal{M}_\infty, (\rho \mapsto \rho^*) \restriction \delta_\infty], \tag{9}$$

and in particular

$$L[\mathcal{M}_\infty, (\rho \mapsto \rho^*)] = L[\mathcal{M}_\infty, (\rho \mapsto \rho^*) \restriction \delta_\infty].$$

Lemma 2.5 *(a) κ is the least measurable cardinal of $\mathcal{M}_\infty$.*

(b) $\delta_\infty = \kappa^{+M_{\rm sw}}$.

(c) $\kappa^{+M_{\rm sw}} < \kappa_\infty < (\kappa_\infty)^{+\mathcal{M}_\infty} < (\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\rm sw}}$.

Proof. (a): This is easy.

(b): Cf. [17, Lemma 3.38 (2)]. To show that $\delta_\infty \leq \kappa^+$ in $M_{\rm sw}$, let $\eta < \delta_\infty$, say $\eta = \pi^s_{N,\infty}(\bar\eta)$, where $N \in \mathcal{F}$ is s–iterable and $\bar\eta < \gamma^N_s$. Then each ordinal below η is of the form $\pi^s_{N',\infty}(\zeta)$ for some $N' \in \mathcal{F}$ with $(N,s) \preceq_\mathcal{F} (N',s)$ and $\zeta < \pi^s_{N,N'}(\bar\eta)$. As $\mathcal{F}$ has cardinality κ, this shows that $\eta < \kappa^+$ in $M_{\rm sw}$.

Let us now show that $\kappa^{+M_{\rm sw}} \leq \delta_\infty$. Let $\alpha < \kappa^{+M_{\rm sw}}$, and let $f \colon \kappa \to \alpha$, $f \in M_{\rm sw}$, be bijective, say $f = \tau^{M_{\rm sw}|\max(s)}(s^-)$, where τ is a Σ_1–Skolem term and s is a finite set of $M_{\rm sw}$–indiscernibles.

Let $\beta < \alpha$, and let $\lambda < \kappa$ be such that $\beta = f(\lambda)$. Let $N \in \mathcal{F}$ be such that

$$\lambda < \min(\gamma^N_s, \text{ the least measurable cardinal of } N)$$

and $\pi^s_{N,N'}(\beta) = \beta$ for all $N' \in \mathcal{F}$ where $\pi^s_{N,N'}$ is defined. Let

$$S^N = \{\epsilon \colon \exists\mu < \text{the least measurable of } N \,\exists p \in \mathbb{B}^N\, p \Vdash^{\mathbb{B}^N}_N \tau^{N[\dot G]|\max(s)}(\check s^-)(\check\mu) = \check\epsilon\}.$$

We have that $\beta \in S^N$ and $\operatorname{otp}(S^N) < \delta^N$. Let γ^N_β be the unique γ such that β is the $\gamma^{\rm th}$ element of S^N. In particular, $\gamma^N_\beta < \delta^N$.

We claim that $\beta \mapsto \pi^s_{N,\infty}(\gamma^N_\beta)$ is well–defined, i.e., that it is independent from the particular choice of an N as above, and that it is also order–preserving. Well, this is because if $\beta \leq \beta' < \alpha$ and γ^N_β and $\gamma^{N'}_{\beta'}$ are defined, then there is some $Q \in \mathcal{F}$ such that $\pi^s_{N,Q}$ and $\pi^s_{N',Q}$ are both defined and $\pi^s_{N,Q}(S^N) = Q^N = \pi^s_{N',Q}(S^{N'})$, and hence $\gamma^Q_\beta \leq \gamma^Q_{\beta'}$.

But now $\beta \mapsto \pi^s_{N,\infty}(\gamma^N_\beta)$ is an injection from α into δ_∞ which exists in $M_{\rm sw}$.

(c): $\kappa^{+M_{\rm sw}} < \kappa_\infty$ is obviously given by (b).

To show that $(\kappa_\infty)^{+\mathcal{M}_\infty} < \kappa^{++M_{\rm sw}}$, we use the argument from the proof of Lemma 2.3 and let $F = E^{M_{\rm sw}}_\nu$ be the least total extender of the $M_{\rm sw}$–sequence which has critical point κ. Write $i_F \colon M_{\rm sw} \to_F W = \operatorname{ult}(M_{\rm sw}; F)$, so that $i_F(\kappa)^{+W} < \kappa^{++M_{\rm sw}} = \kappa^{++W}$. For each $N \in \mathcal{F}$, $F \cap N$ is the least total extender of the N–sequence which has critical point $\kappa = \kappa^N$, and $\operatorname{ult}(N; F \cap N) \in \mathcal{F}^W$. A joint comparison process as defined above on p. 7f. allows us to produce some $N^* \in \mathcal{F}^W$ such that

1. in V, N^* is a $\Sigma_{\operatorname{ult}(N;F\cap N)}$–iterate of $\operatorname{ult}(N; F \cap N)$ for all $N \in \mathcal{F} = \mathcal{F}^{M_{\rm sw}}$, and
2. $\delta^{N^*} = \kappa^{+W} = \kappa^{+M_{\rm sw}}$.

As Σ is commuting, for each $N \in \mathcal{F}$ there is a unique iteration map, call it π_{N,N^*}, from N to N^*, namely the ultrapower map $N \to \text{ult}(N; F \cap N)$ followed by the iteration map from $\text{ult}(N; F \cap N)$ to N^*, and if N, $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists, then

$$\pi_{N',N^*} \circ \pi_{N,N'} = \pi_{N,N^*}.$$

Therefore, there is a canonical elementary embedding

$$k \colon \mathcal{M}_\infty \to N^*.$$

But $N^* = P(N^* | \kappa^{+M_{\text{sw}}})$, as being constructed inside W. Therefore,

$$k(\kappa_\infty) = \kappa^{N^*} = \kappa^W = i_F(\kappa),$$

and

$$(\kappa_\infty)^{+\mathcal{M}_\infty} \leq i_F(\kappa)^{+W} < \kappa^{++M_{\text{sw}}}.$$

Finally, $(\kappa_\infty)^{++\mathcal{M}_\infty} = \pi_{M_{\text{sw}},\infty}(\kappa^{++M_{\text{sw}}}) \geq \kappa^{++M_{\text{sw}}}$. As $\kappa^{++M_{\text{sw}}}$ is a cardinal in $\mathcal{M}_\infty$, this gives $(\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\text{sw}}}$. □ (Lemma 2.5)

The following key lemma makes up the first key step in analyzing the mantle of M_{sw}.

Lemma 2.6 *Let us write $\kappa^+ = \kappa^{+M_{\text{sw}}}$ and $\kappa^{++} = \kappa^{++M_{\text{sw}}}$.*[6] *$M_{\text{sw}}$ is a forcing extension of $L[M_\infty, \rho \mapsto \rho^*]$ via some $\mathbb{P}$ which satisfies the κ^+–c.c.*

In fact,

$$M_{\text{sw}} = L[M_\infty, \rho \mapsto \rho^*][M_{\text{sw}} | \kappa^{++}],$$

where $M_{\text{sw}} | \kappa^{++}$ is $\mathbb{P}$–generic over $L[M_\infty, \rho \mapsto \rho^*]$ for some $\mathbb{P} \in L[M_\infty, \rho \mapsto \rho^*]$ such that $L[M_\infty, \rho \mapsto \rho^*] \models$ "$\mathbb{P}$ has the κ^+-c.c. and is of size κ^{++}."

Proof. We shall make use of Bukovský's theorem from [1]. For the reader's convenience, we give a proof sketch in the appendix to the current paper, cf. Theorem 3.5, cf. also [11].

We claim that $L[M_\infty, \rho \mapsto \rho^*]$ uniformly κ^+–covers M_{sw}, cf. Definition 3.1, i.e., for all functions $f \in M_{\text{sw}}$ with $\text{dom}(f) \in L[M_\infty, \rho \mapsto \rho^*]$ and $\text{ran}(f) \subset L[M_\infty, \rho \mapsto \rho^*]$ there is some function $g \in L[M_\infty, \rho \mapsto \rho^*]$ with $\text{dom}(g) = \text{dom}(f)$ such that for all $x \in \text{dom}(g)$,

(a) $f(x) \in g(x)$ and

[6] Making use of this notation, we will later show that $\kappa^{++} = (\kappa_\infty)^{++\mathcal{M}_\infty}$, cf. Lemma 2.7.

(b) $\mathrm{Card}(g(x)) < \kappa^+$ for all $x \in \mathrm{dom}(g)$.

It obviously suffices to prove this for all f whose domain is an ordinal and whose range is contained in the class of all ordinals.

Suppose what we claim would not be true. As $L[M_\infty, \rho \mapsto \rho^*]$ is definable inside M_{sw} (from M_{sw}'s extender sequence[7]), there is then some counterexample $f\colon \theta \to \mathrm{OR}$ which is parameter–free definable inside M_{sw} (again, from M_{sw}'s extender sequence). Let us fix such an f, $f\colon \theta \to \mathrm{OR}$, and let φ be a formula in the language of M_{sw} such that for all ξ, η, $f(\xi) = \eta$ iff $M_{\mathrm{sw}} \vDash \varphi(\xi, \eta)$.

If $N \in \mathcal{F}$, then $M_{\mathrm{sw}} = N[h]$ for some h which is $\mathbb{B}^N$–generic over N; in fact, $h = M_{\mathrm{sw}}|\delta^N$. The extender sequence of M_{sw} is then uniformly definable inside $N[h]$ from the extender sequence of N and the parameter $M_{\mathrm{sw}}|\delta^N$. There is then a formula ψ such that for all $N \in \mathcal{F}$, ψ is a formula of the forcing language of N associated to forcing with $\mathbb{B}^N$ over N such that if $M_{\mathrm{sw}} = N[h]$, where h which is $\mathbb{B}^N$–generic over N, then for all ξ, η, $M_{\mathrm{sw}} \vDash \varphi(\xi, \eta)$ iff there is some $p \in h$ such that $p \Vdash_N^{\mathbb{B}^h} \psi(\check{\xi}, \check{\eta})$. Of course, the formula ψ is also a formula of the forcing language of $\mathcal{M}_\infty$ associated to forcing with $\mathbb{B}^{\mathcal{M}_\infty}$ over $\mathcal{M}_\infty$.

Let $N \in \mathcal{F}$ or $N = \mathcal{M}_\infty$. If $p \in \mathbb{B}^N$, then we write

$$p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function”}$$

to mean that

$$p \Vdash_N^{\mathbb{B}^N} \forall v \forall w \forall w' \, \psi(v, w) \wedge \psi(v, w') \to w = w'.$$

Let $g_N \in N$ be the function with domain $\pi_{M_{\mathrm{sw}},N}(\theta)$ (in case $N = \mathcal{M}_\infty$ by this we mean $\pi_{M_{\mathrm{sw}},\infty}(\theta)$) such that for all $\xi < \pi_{M_{\mathrm{sw}},N}(\theta)$,

$$g_N(\xi) = \{\eta \colon \exists p \in \mathbb{B}^N \, p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function and } \psi(\check{\xi}, \check{\eta})\text{”}\} \tag{10}$$

As $\mathbb{B}^N$ has the δ^N–c.c. inside N, $\mathrm{Card}(\tilde{g}(\xi)) < \delta^N$ in N for all $\xi < \pi_{M_{\mathrm{sw}},N}(\theta)$.

Of course, if $N \in \mathcal{F}$, then $\pi_{N,\infty}(g_N) = g_{\mathcal{M}_\infty}$.

Let $g \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ be the function with domain θ such that for all $\xi < \theta$,

$$g(\xi) = \{\eta \colon \eta^* \in g_{\mathcal{M}_\infty}(\xi^*)\}. \tag{11}$$

Obviously, $\mathrm{Card}(g(\xi)) \leq \mathrm{Card}(g_{\mathcal{M}_\infty}(\xi^*)) < \delta_\infty$ in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

Let $\xi < \theta$ and $\eta = f(\xi)$, i.e., $M_{\mathrm{sw}} \vDash \varphi(\xi, \eta)$. Pick $N \in \mathcal{F}$ such that $\xi^* = \pi_{N,\infty}(\xi)$ and $\eta^* = \pi_{N,\infty}(\eta)$. As $M_{\mathrm{sw}} = N[h]$, for some h which is $\mathbb{B}^N$–generic over N, there is some $p \in h \subset \mathbb{B}^N$ with

$$p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function and } \psi(\check{\xi}, \check{\eta})\text{,”} \tag{12}$$

[7]Claim 2.10 (a) will in fact prove a stronger definability fact, but this is not needed here.

so that $\eta \in g_N(\xi)$. But then

$$\eta^* = \pi_{N,\infty}(\eta) \in \pi_{N,\infty}(g_N)(\pi_{N,\infty}(\xi)) = g_{\mathcal{M}_\infty}(\xi^*),$$

and hence $\eta \in g(\xi)$. Because $\delta_\infty = \kappa^+$ by Lemma 2.5, we have shown that $L[M_\infty, \rho \mapsto \rho^*]$ κ^+–globally covers M_{sw}. □ (Lemma 2.6)

Lemma 2.7 *(a) M_∞ is fully iterable inside M_{sw}.*

(b) If $\mathbb{P}$ is a poset in M_{sw} and if g is $\mathbb{P}$–generic over M_{sw}, then $\mathcal{M}_\infty$ is fully iterable inside M_{sw}.

(c) M_∞ is fully iterable inside $L[M_\infty, \rho \mapsto \rho^*]$.

(d) $\kappa^{+M_{\mathrm{sw}}} = \delta_\infty < (\delta_\infty)^{+L[\mathcal{M}_\infty,\rho\mapsto\rho^*]} = \kappa^{++M_{\mathrm{sw}}}$.

(e) If λ is a cardinal of $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ with $\lambda \geq \delta_\infty$, then λ is also a cardinal of M_{sw}.

Proof. (a): Cf. [8]. We aim to show that $\Sigma_{\mathcal{M}_\infty} \restriction M_{\mathrm{sw}}$ is definable in M_{sw}. To this end, let $\mathcal{T} \in M_{\mathrm{sw}}$ be a tree of limit length on $\mathcal{M}_\infty$ which is according to $\Sigma_{\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$.

If there is a drop along c, or if there is no drop along c and $\delta(\mathcal{T}) \neq \delta^{\mathcal{M}_c^\mathcal{T}}$, then there is a $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_c^\mathcal{T}$ which is ¶–small above $\delta(\mathcal{T})$. But then $\mathcal{Q} \in M_{\mathrm{sw}}$, as M_{sw} is closed under the operator $X \mapsto M_{\mathrm{s}}(X)$, and therefore $b \in M_{\mathrm{sw}}$.

Let us now assume that there is no drop along c and $\delta(\mathcal{T}) = \delta^{\mathcal{M}_c^\mathcal{T}}$. We have that $\mathcal{M}_c^\mathcal{T}$ is an iterate of $K(\mathcal{M}(\mathcal{T}))^{M_{\mathrm{sw}}}$. Let us assume that $\mathcal{M}_c^\mathcal{T} = K(\mathcal{M}(\mathcal{T}))^{M_{\mathrm{sw}}}$ and leave the other case to the reader's discretion.

We then have that $\mathcal{M}_c^\mathcal{T}$ is definable in $\mathcal{M}_{\mathrm{sw}}$. Let E be a total extender on the M_{sw}–sequence such that $\mathrm{crit}(E) = \kappa$ and $\mathcal{T} \in \mathrm{ult}(M_{\mathrm{sw}}; E)$. Let us write

$$j\colon M_{\mathrm{sw}} \to_E W = \mathrm{ult}(M_{\mathrm{sw}}; E).$$

We may produce some $N \in \mathcal{F}^W$ such that in V, $N|\delta^N$ is a normal iterate of $\mathcal{M}_c^\mathcal{T}|\delta(\mathcal{T})$. There is hence some elementary

$$k'\colon \mathcal{M}_c^\mathcal{T}|\delta(\mathcal{T}) \to j(\mathcal{M}_\infty|\delta_\infty) = (\mathcal{M}_\infty)^W|\delta^{\mathcal{M}_\infty^W}. \tag{13}$$

Let g be $\mathrm{Col}(\omega, \delta(\mathcal{T}))$–generic over V. Inside $M_{\mathrm{sw}}[g]$ let us consider a tree T searching for a cofinal branch b through $\mathcal{T}$ such that b does not drop and there is an elementary embedding

$$k\colon \mathcal{M}_b^\mathcal{T}|\delta(\mathcal{T}) \to j(\mathcal{M}_\infty|\delta_\infty)$$

such that

$$k \circ \pi_{0,b}^\mathcal{T} \restriction \mathcal{M}_\infty|\delta_\infty = j \restriction \mathcal{M}_\infty|\delta_\infty \tag{14}$$

We claim that $c = \Sigma_{\mathcal{M}_\infty}$ is given by a branch through T. To see this, let $x \in \mathcal{M}_\infty|\delta_\infty$. Let $x \in \mathrm{ran}(\pi_{N,\infty})$, where $N \in \mathcal{F}$, and write $\bar{x} = {\pi_{N,\infty}}^{-1}(x)$. Pick s, a finite set of M_{sw}–indiscernibles which is moved neither by $\pi_{M_{\mathrm{sw}},\infty}$ nor by j and such that $\bar{x} \in \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) = \mathrm{dom}(\pi^s_{N,\infty})$. Notice that $j(\bar{x}) = \bar{x}$, and $j(N) = \mathrm{ult}(N; E \cap N) \in \mathcal{F}^W$. We may copy $\mathcal{T}$ onto $\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(E))$ via the map $i = i_{\pi_{M_{\mathrm{sw}},\infty}(E)}$, write $i\mathcal{T}$ for the resulting tree. Let

$$i^*\colon \mathcal{M}^{\mathcal{T}}_c \to \mathrm{ult}(\mathcal{M}^{\mathcal{T}}_c; i^{\mathcal{T}}_c \circ i(E)) = \mathcal{M}^{i\mathcal{T}}_c.$$

We may produce some $N^* \in \mathcal{F}^W$ such that in V, N^* is a $\Sigma_{j(N)}$–iterate of $j(N)$ as well as a $\Sigma_{\mathcal{M}^{i\mathcal{T}}_c}$–iterate of $\mathcal{M}^{i\mathcal{T}}_c$. We write $\pi_{j(N),N^*}$ and $\pi_{\mathcal{M}^{i\mathcal{T}}_c,N^*}$ for the iteration maps, and we also write $\pi_{N^*,j(\mathcal{M}_\infty)}$ for the iteration map from N^* to $j(\mathcal{M}_\infty)$.

We now get that

$$\begin{aligned}
j(x) &= j(\pi_{N,\infty}(\bar{x})) \\
&= j(\pi^s_{N,\infty}(\bar{x})) \\
&= j(\pi^s_{N,\infty})(j(\bar{x})) \\
&= \pi^s_{j(N),j(\mathcal{M}_\infty)}(\bar{x}) \\
&= \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}^{i\mathcal{T}}_c,N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ \pi_{j(N),\mathrm{ult}(\mathcal{M}_\infty;\pi_{M_{\mathrm{sw}},\infty}(E))}(\bar{x}) \\
&= \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}^{i\mathcal{T}}_c,N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ i^* \circ \pi^{\mathcal{T}}_{0,c}(x),
\end{aligned}$$

so that $k = \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}^{i\mathcal{T}}_c,N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ i^*$ witnesses that c is indeed given by a branch through T.

Notice that (14) implies that

$$k \circ \pi^{\mathcal{T}}_{0,b} \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta = j \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta. \tag{15}$$

Let $x \in M_{\mathrm{sw}}|\delta$, and let s be a finite set of M_{sw}–indiscernibles which are moved neither by $\pi_{M_{\mathrm{sw}},\infty}$ nor by j and such that $x \in \mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}} \cup s^-) = \mathrm{dom}(\pi^s_{M_{\mathrm{sw}},\infty})$. Then $\pi^s_{M_{\mathrm{sw}},\infty} \in M_{\mathrm{sw}}$ and $j \circ \pi_{M_{\mathrm{sw}},\infty}(x) = j \circ \pi^s_{M_{\mathrm{sw}},\infty}(x) = j(\pi^s_{M_{\mathrm{sw}},\infty})(j(x)) = \pi^s_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}(x) = \pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}(x)$, where $\pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}$ is the iteration map from M_{sw} to $j(\mathcal{M}_\infty)$. Hence the right hand side of (15) is equal to $\pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}$. The left hand side of (15) is equal to the iteration map $\pi^{\mathcal{T}}_{0,b} \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta$ followed by k.

By Lemmas 2.3 and 2.1, b must therefore be equal to c, so that in fact $c \in M_{\mathrm{sw}}$.

We have shown that $\Sigma_{\mathcal{M}_\infty}(\mathcal{T}) \in M_{\mathrm{sw}}$ for every $\mathcal{T} \in M_{\mathrm{sw}}$. But recall that $\delta_\infty = \kappa^{+M_{\mathrm{sw}}}$, cf. Lemma 2.5 (b), and δ_∞ is hence regular in M_{sw}. Hence if $\mathcal{T}$ is a tree on $\mathcal{M}_\infty$ with $\delta(\mathcal{T}) = \pi_{0,\Sigma(\mathcal{T})}(\delta_\infty)$, then M_{sw} will have exactly one cofinal branch through $\mathcal{T}$, namely $\Sigma(\mathcal{T})$. $\Sigma_{\mathcal{M}_\infty} \restriction M_{\mathrm{sw}}$ is therefore definable in M_{sw}.

(b): Let $\mathcal{T} \in M_{\text{sw}}[g]$ be a tree of limit length on $\mathcal{M}_\infty$ which is according to $\Sigma_{\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$. Assume that there is no drop along c and $\delta(\mathcal{T}) = \delta^{\mathcal{M}_c^{\mathcal{T}}}$.

Let θ be an appropriate ordinal, and let h be $\text{Col}(\omega, \theta)$–generic over V such that $M_{\text{sw}}[g] \subset M_{\text{sw}}[h]$. Say $p \Vdash^{\text{Col}(\omega,\theta)}_{M_{\text{sw}}}$ "$\dot{\mathcal{T}}$ is a tree of limit length on $\mathcal{M}_\infty$ which is guided by ¶–small iterable $\mathcal{Q}$–structures, and $\delta(\dot{\mathcal{T}})$ is Woodin in $K(\mathcal{M}(\dot{\mathcal{T}}))$."

For any $q \leq_{\text{Col}(\omega,\theta)} p$ let h_q denote the unique $\text{Col}(\omega, \theta)$–generic filter over N such that for $n < \omega$,

$$(\bigcup h_q)(n) = \begin{cases} q(n) & \text{if } n \in \text{dom}(q), \text{ and} \\ (\bigcup h)(n) & \text{otherwise.} \end{cases}$$

Inside $M_{\text{sw}}[h]$, we may pseudo–compare all $K(\mathcal{M}(\dot{\mathcal{T}}^{h_q}))$, $q \leq_{\text{Col}(\omega,\theta)} p$, so as to produce $K(\mathcal{M})$ for some $\mathcal{M}$. As $\mathcal{M}$ is definable inside $M_{\text{sw}}[h]$ from $\{h_q : q \leq_{\text{Col}(\omega,\theta)} p\}$ and some parameters from M_{sw}, $\mathcal{M}$ will actually be an element of M_{sw}, and in $V[h]$, $K(\mathcal{M})$ is a $\Sigma_{\mathcal{M}_c^{\mathcal{T}}}$–iterate of $\mathcal{M}_c^{\mathcal{T}}$, a fact which will give rise to the existence of the natural iteration map from $\mathcal{M}_c^{\mathcal{T}} = K(\mathcal{M}(\mathcal{T}))$ into $K(\mathcal{M})$.

Inside M_{sw}, we may now pseudo–compare $\mathcal{M}_\infty$ with $K(\mathcal{M})$, producing a $\Sigma_{\mathcal{M}_\infty}$–iterate $\mathcal{M}^*$ of $\mathcal{M}_\infty$ such that in V, $K(\mathcal{M})$ is also a $\Sigma_{K(\mathcal{M})}$–iterate of $K(\mathcal{M})$, a fact which will give rise to the existence of the natural iteration map from $K(\mathcal{M})$ into $\mathcal{M}^*$. As $\mathcal{M}_\infty$ is iterable in M_{sw} by (a), the iteration map

$$i : \mathcal{M}_\infty \to M^*$$

is definable inside M_{sw}. Inside $M_{\text{sw}}[h]$, we may now construct a tree T searching for a cofinal branch b through $\mathcal{T}$ together with an elementary embedding $k : \mathcal{M}_b^{\mathcal{T}} | \delta(\mathcal{T}) \to \mathcal{M}^* | \delta^{\mathcal{M}^*}$ such that

$$k \circ \pi_{0,c}^{\mathcal{T}} \upharpoonright \mathcal{M}_\infty | \delta_\infty = i \upharpoonright \mathcal{M}_\infty | \delta_\infty.$$

T is ill–founded in $V[h]$, hence in $M_{\text{sw}}[h]$, and by Lemma 2.1 there is a unique b given by a branch through T, so that $b \in M_{\text{sw}}[g]$.

This argument shows that $\Sigma_{\mathcal{M}_\infty} \upharpoonright M_{\text{sw}}[g]$ is definable in $M_{\text{sw}}[g]$.

(c): We aim to show that $\Sigma_{\mathcal{M}_\infty} \upharpoonright L[M_{\text{sw}}, \rho \mapsto \rho^*]$ is definable in $L[M_{\text{sw}}, \rho \mapsto \rho^*]$ and total there. Let $\mathcal{T} \in L[M_{\text{sw}}, \rho \mapsto \rho^*]$ be a tree of limit length on $\mathcal{M}_\infty$ which is according to $\Sigma_{\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$, and let us now assume that there is no drop along c and $\delta(\mathcal{T}) = \delta^{\mathcal{M}_c^{\mathcal{T}}}$.

Let $\mathbb{P} \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ be as in Lemma 2.6, and let g be $\mathbb{P}$–generic over $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ such that $M_{\text{sw}} = L[\mathcal{M}_\infty, \rho \mapsto \rho^*][g]$. Let θ be an appropriate ordinal, and let h and h^* be $\text{Col}(\omega, \theta)$–generic over V such that $M_{\text{sw}} = L[\mathcal{M}_\infty, \rho \mapsto \rho^*][g] \subset$

$L[\mathcal{M}_\infty, \rho \mapsto \rho^*][h^*] = M_{\rm sw}[h]$. By (b), $\mathcal{M}_\infty$ is fully iterable inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*][h^*]$, as being witnessed by $\Sigma_{\mathcal{M}_\infty} \restriction L[\mathcal{M}_\infty, \rho \mapsto \rho^*][h^*]$. In particular,

$$\pi^{\mathcal{T}}_{0,c} \colon \mathcal{M}_\infty \to \mathcal{M}^{\mathcal{T}}_c$$

is in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*][h^*]$, and in fact c is the unique cofinal branch b given by a branch through a tree $T \in M_{\rm sw}[h]$ as in the proof of (b). Hence $c \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

Finally, δ_∞ is regular in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$, so that as in the last paragraph of the proof of (a), $\Sigma_{\mathcal{M}_\infty} \restriction L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ is definable in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ and total inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

(d): Let E be the least extender on the $\mathcal{M}_\infty$–sequence such that E is total and ${\rm crit}(E) = \kappa_\infty$. Inside ${\rm ult}(\mathcal{M}_\infty; E)$, we may pick some $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}^{{\rm ult}(\mathcal{M}_\infty;E)}$ such that $\delta(\mathcal{U}) = (\kappa_\infty)^{+{\rm ult}(\mathcal{M}_\infty;E)} = (\kappa_\infty)^{+\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{U})$.

By the proof of Lemma 2.2, $N = \mathcal{M}^{\mathcal{U}}_c$. But $c \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ by (b), and hence $\pi_{0,c}"\delta_\infty \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ witnesses that $(\kappa_\infty)^{+\mathcal{M}_\infty}$ has cofinality δ_∞ inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

Because N is also the ¶–small core model over $\mathcal{M}(\mathcal{U})$ inside ${\rm ult}(\mathcal{M}_\infty; E)$, again by the proof of Lemma 2.2, the Weak Covering Lemma (cf. e.g. [4]) therefore gives that ${\rm Card}((\kappa_\infty)^{+\mathcal{M}_\infty}) = \delta_\infty$ inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. By Lemma 2.5 (c), $(\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\rm sw}}$, so that now $(\delta_\infty)^{+L[\mathcal{M}_\infty, \rho \mapsto \rho^*]} = \kappa^{++M_{\rm sw}}$.

(e): This now immediately follows from(d) and Lemma 2.6. □ (Lemma 2.7)

In what follows we let K denote the unique weasel W, if it exists, such that for every α, $W|\alpha$ is isomorphic to an initial segment of

$$\bigcap\{{\rm Hull}^{K^c}(\Gamma) \colon \Gamma \text{ is thick in } K^c\},$$

for any K^c with sufficiently big additivity, cf. [14]. K is called the *Steel core model*. The paper [8] shows that $M_{\rm sw}$ has a K. We here prove that this K is actually equal to $\mathcal{M}_\infty$.

Lemma 2.8 *$\mathcal{M}_\infty = K$.*

Proof. Let us fix g which is ${\rm Col}(\omega, < \kappa)$–generic over $M_{\rm sw}$. Let us write

$$H = \mathsf{HOD}^{M_{\rm sw}[g]}.$$

Claim 2.9 *$L[M_\infty, \rho \mapsto \rho^*] \subset H$.*

Proof. Let us write $\mathcal{C}$ for the collection, as being defined inside $M_{\rm sw}[g]$, of all extender models N with a Woodin cardinal, δ^N, and a strong cardinal, κ^N, such that the following conditions (1) through (6) are met.

(1) $N|(\delta^N)^{+N}$ is suitable,

(2) $\kappa^N = \kappa$,

(3) $N[h] = M_{\rm sw}[g]$ for some h which is $\mathrm{Col}(\omega, <\kappa)$–generic over N,

(4) $N = K(N|\delta^N)$ is the ¶–small core model over $N|\delta^N$,

(5) N is *pseudo–iterable* in the following sense. Let $\mathbb{T}(N)$ be the collection of all $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in N$, some $n < \omega$, such that either $n = 0$ and $\mathrm{lh}(\mathcal{U}_0) = 1$ (i.e., $\mathcal{U}$ is trivial), or else there is a sequence $\eta_0 < \ldots < \eta_n < \kappa$ of cutpoints of N and:

   (a) $\mathcal{U} \in N|\kappa$,

   (b) $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ is a finite stack of normal iteration trees $\mathcal{U}_k$,

   (c) $\mathcal{U}_0$ is on N and lives below δ^N,

   and for every $k < n$,

   (d) if $k < n$, then $\mathrm{lh}(\mathcal{U}_k) = (\eta_k)^{+N} = \delta(\mathcal{U}_k)$, and $\mathrm{lh}(\mathcal{U}_n) = (\eta_n)^{+N} = \delta(\mathcal{U}_n)$,

   (e) $\mathcal{U}_k$ is definable over $N|(\eta_k)^{+N}$ and is guided by $\mathcal{Q}$–structures which are obtained via $\mathcal{P}$–constructions inside N, cf. [9, Section 1],

   (f) if $k < n$, then $P^N(\mathcal{M}(\mathcal{U}_k))$ is a proper class, $\delta(\mathcal{U}_k)$ is a Woodin cardinal of $P^N(\mathcal{M}(\mathcal{U}))$, and
   $$P^N(\mathcal{M}(\mathcal{U}))[G] = N$$
   for some G which is $\mathbb{B}^{P(\mathcal{M}(\mathcal{U}))}$–generic over $P(\mathcal{M}(\mathcal{U}))$, and

   (g) if $k > 0$, then $\mathcal{U}_k$ is on $P^N(\mathcal{M}(\mathcal{U}_{k-1}))$ and lives below $\delta(\mathcal{U}_{k-1})$. (We allow $\mathcal{U}_n$ to consist of only one model, namely $P^N(\mathcal{M}(\mathcal{U}_{n-1}))$.)

   For N to be pseudo–iterable we demand that if $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$, then

   (a) if $\mathcal{U}_n$ has a last model, say $\mathcal{M}_\theta^{\mathcal{U}_n}$ and if F is an extender from the sequence of $\mathcal{M}_\theta^{\mathcal{U}_n}$ such that if $[0,\theta]_{\mathcal{U}_n}$ does not drop, then the index of F is below $\delta^{\mathcal{M}_\theta^{\mathcal{U}_n}}$, then $(\mathcal{U}_k \colon k < n)^\frown(\mathcal{U}_n{}^\frown F) \in \mathbb{T}(N)$, where $(\mathcal{U}_n{}^\frown F)$ is the normal extension of $\mathcal{U}_n$, and

   (b) if $\mathcal{U}_n$ is of limit length, then there is either a cofinal branch b through $\mathcal{U}_n$ such that $(\mathcal{U}_k \colon k < n)^\frown(\mathcal{U}_n{}^\frown b) \in \mathbb{T}(N)$, or else letting $\mathcal{U}^*$ be the trivial tree consisting only of the model $P^N(\mathcal{U}_n)$, $(\mathcal{U}_k \colon k \leq n)^\frown\mathcal{U}^* \in \mathbb{T}(N)$.

Before stating condition (6) let us say that we call M a *pseudo–iterate* of N iff there is some $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$ such that $\mathcal{U}_n$ consists of only one model, namely N. We will write $\mathcal{F}^N$ for the collection of all pseudo–iterates of N.[8] Let s be a non–empty finite set of ordinals. For $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}^N$ write

$$T_s^N = \{\varphi(\vec{x}, v) \colon \vec{x} \in \delta(\mathcal{U}) \wedge N|\max(s) \vDash \varphi(\vec{x}, s^-)\}.$$

We call $M \in \mathcal{F}^N$ *s–iterable inside* N iff for all $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$, writing M_k for the starting model of $\mathcal{U}_k$, $k \leq n$, if $M = M_{k_0}$ for some $k_0 < n$, there are for every $i \geq k_0$, $i + 1 < n$, cofinal branches

$$b_i \in (M_{\rm sw})^{{\rm Col}(\omega, <\kappa)}$$

through $\mathcal{U}_i$ such that

(1) $\pi^{\mathcal{U}_i}_{0,b_i}(M_i|\max(s)) = M_{i+1}|\max(s)$, and

(2) $\pi^{\mathcal{U}_i}_{0,b_i}(T_s^{M_i}) = T_s^{M_{i+1}}$.[9]

Our last condition on N now runs:

(6) For every finite set s of ordinals there is some $M \in \mathcal{F}^N$ such that M is s–iterable in N.

Given $N \in \mathcal{C}$, we may define a direct limit system inside N in much the same way as the system was defined in $M_{\rm sw}$ to give rise to $\mathcal{M}_\infty$. We write $(\mathcal{M}_\infty)^N$ for the direct limit of that system as being defined in N.

We claim that if $N \in \mathcal{C}$, then

$$(\mathcal{M}_\infty)^N = \mathcal{M}_\infty$$

and that in fact the systems giving rise to $\mathcal{M}_\infty$ and $(\mathcal{M}_\infty)^N$, respectively, have cofinally many common points. As $\mathcal{C}$ is ordinal definable inside $M_{\rm sw}[g]$, this immediately establishes Claim 2.9.

Let us thus fix some $N \in \mathcal{C}$. Let $\xi < \kappa$ be least such that $N|\delta^N \in M_{\rm sw}[g \restriction \xi]$. We have, by the forcing absoluteness of the ¶–small K over $N|\delta^N$,

$$N = (K(N|\delta^N))^N = (K(N|\delta^N))^{N[h]} = (K(N|\delta^N))^{M_{\rm sw}[g]} = (K(N|\delta^N))^{M_{\rm sw}[g \restriction \xi]}, \quad (16)$$

[8]We have that $\mathcal{F}^{M_{\rm sw}}$, defined this way, is equal to $\mathcal{F}$ as being defined earlier.

[9]The two notions of being s–iterable in $M_{\rm sw}$ we have now defined coincide.

so that in particular N exists in $M_{\rm sw}[g \restriction \xi]$ as a subclass which is definable there from the parameter $N|\delta^N$. Symmetrically, if $\xi' < \kappa$ is least such that $M_{\rm sw}|\delta \in N[h \restriction \xi']$, then

$$M_{\rm sw} = (K(M_{\rm sw}|\delta))^{N[h \restriction \xi']} \tag{17}$$

and $M_{\rm sw}$ exists in $N[h \restriction \xi']$ as a subclass which is definable there from the parameter $M_{\rm sw}|\delta$.

Let us denote by F_1 the $M_{\rm sw}$–extender of Mitchell order 0 and with critical point κ, and let us denote by F_2 the N–extender of Mitchell order 0 with critical point κ. Let $\pi_1 \colon M_{\rm sw} \to \mathrm{ult}(M_{\rm sw}; E_1)$ and $\pi_2 \colon N \to \mathrm{ult}(N; E_2)$ denote the ultrapower maps. Let us write

$$\bar{H} = (H_{\kappa^+})^{\mathrm{ult}(M_{\rm sw};E_1)[g]} = (H_{\kappa^+})^{M_{\rm sw}[g]} = (H_{\kappa^+})^{N[h]} = (H_{\kappa^+})^{\mathrm{ult}(N;E_2)[h]}.$$

We have that

$$\mathrm{ult}(M_{\rm sw}; E_1)[g] = K(\bar{H})^{M_{\rm sw}[g]} = K(\bar{H})^{\mathrm{ult}(M_{\rm sw};E_1)[g]},$$

and

$$\mathrm{ult}(N; E_2)[h] = K(\bar{H})^{N[h]} = K(\bar{H})^{\mathrm{ult}(N;E_2)[h]}$$

Let us write $K(\bar{H})$ for this common value of the ¶–small K over $\bar{H}$. Then

$$\mathrm{ult}(M_{\rm sw}; E_1)[g] = K(\bar{H}) = \mathrm{ult}(N; E_2)[h]. \tag{18}$$

This immediately gives

$$\pi_1(\kappa) = \pi_2(\kappa). \tag{19}$$

But also, $M_{\rm sw}|\kappa^{+M_{\rm sw}}$ may be defined over $\bar{H}$ from the parameter $M_{\rm sw}|\kappa$ as the stack of all ¶–small sound mice end–extending $M_{\rm sw}|\kappa$ and projecting to κ, and

$$\mathrm{ult}(M_{\rm sw}; E_1) = \mathcal{P}^{\mathrm{ult}(M_{\rm sw};E_1)[g]}(M_{\rm sw}|\kappa^{+M_{\rm sw}}) = \mathcal{P}^{K(\bar{H})}(M_{\rm sw}|\kappa^{+M_{\rm sw}}). \tag{20}$$

In the same way, $N|\kappa^{+N}$ may be defined over $\bar{H}$ from the parameter $N|\kappa$ as the stack of all ¶–small sound mice end–extending $N|\kappa$ and projecting to κ, and

$$\mathrm{ult}(N; E_2) = \mathcal{P}^{\mathrm{ult}(N;E_2)[h]}(N|\kappa^{+N}) = \mathcal{P}^{K(\bar{H})}(N|\kappa^{+N}). \tag{21}$$

Let k be $\mathrm{Col}(\omega,[\kappa,\pi_1(\kappa)))$–generic over the common model from (18), cf. (19). Then π_1 and π_2 lift to

$$\tilde{\pi}_1\colon M_{\mathrm{sw}}[g] \to \mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k] = K(\bar{H})[k]$$

and

$$\tilde{\pi}_2\colon N[h] \to \mathrm{ult}(N;E_2)[h^\frown k] = K(\bar{H})[k],$$

respectively. The maps $\tilde{\pi}_1$ and $\tilde{\pi}_2$ might be different, but the universes of their domains and target models are the same, and by (19), any objects defined in $M_{\mathrm{sw}}[g] = N[h]$ from parameters in $(H_\kappa)^{M_{\mathrm{sw}}[g]} \cup \{\kappa\} = (H_\kappa)^{N[h]} \cup \{\kappa\}$ will be moved the same way.

In particular, $\tilde{\pi}_1$ maps $N = (K(N|\delta^N))^{M_{\mathrm{sw}}[g]}$ to

$$\begin{aligned}(K(N|\delta^N))^{\mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k]} = (K(N|\delta^N))^{\mathrm{ult}(N;E_2)[h^\frown k]} &= \tilde{\pi}_2(K(N|\delta^N)^{N[h]}) \\ &= \tilde{\pi}_2(N) = \mathrm{ult}(N;E_2),\end{aligned}$$

i.e.,

$$\tilde{\pi}_1(N) = \mathrm{ult}(N;E_2). \tag{22}$$

Let $\rho < \kappa$ be arbitrary. We have that $\mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k]$ thinks that there is some strong cutpoint $\eta < \tilde{\pi}_1(\kappa)$ of both $\mathrm{ult}(M_{\mathrm{sw}};E_1) = \tilde{\pi}_1(M_{\mathrm{sw}}) = K(M_{\mathrm{sw}}|\delta)$ and $\mathrm{ult}(N;E_2) = \tilde{\pi}_1(N) = K(N|\delta^N)$ with $\eta > \rho$ (namely, $\eta = \kappa$) such that setting

$$H' = (H_{\eta^+})^{\tilde{\pi}_1(M_{\mathrm{sw}})[g^\frown k\restriction\eta]}$$

(so $H' = \bar{H}$ for $\eta = \kappa$), $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})}$ may be defined over H' from the parameter $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta$ as the stack of ¶–small sound mice end–extending $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta$ and projecting to η,

$$\tilde{\pi}_1(M_{\mathrm{sw}}) = \mathcal{P}^{\tilde{\pi}_1(M_{\mathrm{sw}})[g^\frown k\restriction\eta]}(\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})}) = \mathcal{P}^{K(H')}(\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})}),$$

$\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}$ may be defined over H' from the parameter $\tilde{\pi}_1(N)|\eta$ as the stack of all ¶–small sound mice end–extending $\tilde{\pi}_1(N)|\eta$ and projecting to η, and finally there is some h^* which is $\mathrm{Col}(\omega,<\eta)$–generic over $\tilde{\pi}_2(N)$ (namely, $h^* = h$) such that $\tilde{\pi}_1(M_{\mathrm{sw}})[g^\frown k \restriction \eta] = \tilde{\pi}_1(N)[h^*]$ and

$$\tilde{\pi}_1(N) = \mathcal{P}^{\tilde{\pi}_1(N)[h^*]}(\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}) = \mathcal{P}^{K(H')}(\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}).$$

By the elementarity of $\tilde{\pi}_1$ and because $\rho < \kappa$ was arbitrary, we then get arbitrarily large $\eta < \kappa$ which are strong cutpoints of both M_{sw} and N such that setting

$$H'' = (H_{\eta^+})^{M_{\mathrm{sw}}[g\restriction\eta]}, \tag{23}$$

$M_{\rm sw}|\eta^{+M_{\rm sw}}$ may be defined over H'' from the parameter $M_{\rm sw}|\eta$ as the stack of all ¶–small sound mice end–extending $M_{\rm sw}|\eta$ and projecting to η,

$$M_{\rm sw} = \mathcal{P}^{M_{\rm sw}[g\restriction\eta]}(M_{\rm sw}|\eta^{+M_{\rm sw}}) = \mathcal{P}^{K(H'')}(M_{\rm sw}|\eta^{+M_{\rm sw}}),$$

$N|\eta^{+N}$ may be defined over H'' from the parameter $N|\eta$ as the stack of all ¶–small sound mice end–extending $N|\eta$ and projecting to η, and there is some h^* which is ${\rm Col}(\omega, <\eta)$–generic over N such that

$$N = \mathcal{P}^{N[h^*]}(N|\eta^{+N}) = \mathcal{P}^{K(H'')}(N|\eta^{+N}), \tag{24}$$

where $K(H'')$ is the ¶–small core model over H'' inside the model

$$M_{\rm sw}[g\restriction\eta] = N[h^*].$$

Let us write $S \subset \kappa$ for the set all of $\eta < \kappa$ with the properties as above, so that S is unbounded in κ.

Let us now suppose that $\mathcal{M}$ is a premouse with a largest limit ordinal $\delta^{\mathcal{M}}$ such that

1. $\eta^{+M_{\rm sw}} < \delta^{\mathcal{M}} \leq \eta^{++M_{\rm sw}}$ for some $\eta \in S$,
2. $\mathcal{M} \in M_{\rm sw} \cap N$,
3. $\mathcal{M} \models$ "$\delta^{\mathcal{M}}$ is a Woodin cardinal," and
4. both $M_{\rm sw}|\delta^{\mathcal{M}}$ and $N|\delta^M$ are $\mathbb{B}^{\mathcal{M}}$–generic over $\mathcal{M}$.

We then have, for H'' as in (23) and h^* being ${\rm Col}(\omega, <\eta)$–generic over N with (24),

$$\begin{aligned} \mathcal{P}^{M_{\rm sw}}(\mathcal{M}) &= \mathcal{P}^{M_{\rm sw}[g\restriction\eta]}(\mathcal{M}) \\ &= \mathcal{P}^{K(H'')}(\mathcal{M}) \\ &= \mathcal{P}^{N[h^*]}(\mathcal{M}) \\ &= \mathcal{P}^{N}(\mathcal{M}), \end{aligned} \tag{25}$$

where $K(H'')$ is the ¶–small K over H'' in $M_{\rm sw}[g\restriction\eta] = N[h^*]$.

Now let $s \in {\rm OR}^{<\omega}$, and let $M \in \mathcal{F} = \mathcal{F}^{M_{\rm sw}}$ be s–iterable in $M_{\rm sw}$, and let $M' \in \mathcal{F}^N$ be s–iterable in N. We aim to find $M^* \in \mathcal{F} \cap \mathcal{F}^N$ such that

$$(M, s) \preceq_{\mathcal{F}} (M^*, s) \text{ and } (M', s) \preceq_{\mathcal{F}^N} (M^*, s).$$

Let $\xi' \leq \xi'' < \kappa$ be such that $g \restriction \xi \in N[h \restriction \xi'']$, so that by (16) and (17)

$$N \subset M_{\mathrm{sw}}[g \restriction \xi] \subset N[h \restriction \xi''],$$

which implies that N is a ground of $M_{\mathrm{sw}}[g \restriction \xi]$, and in fact both M_{sw} and N grounds of $M_{\mathrm{sw}}[g \restriction \xi]$ via posets of size less than κ. Therefore, by [18, Proposition 5.1], there is an inner model $P \subset M_{\mathrm{sw}} \cap N$ such that P is a ground of $M_{\mathrm{sw}}[g \restriction \xi]$ via a poset of size less than κ. We may then pick some $\theta < \kappa$ such that for some $\ell \in M_{\mathrm{sw}}[g]$ which is $\mathrm{Col}(\omega, \theta)$–generic over P,

$$\{M_{\mathrm{sw}}|\delta, N|\delta^N, M|\delta^M, M'|\delta^{M'}\} \subset P[\ell], \tag{26}$$

and in fact all of M_{sw}, N, M, M' exist in $P[\ell]$ as subclasses which are definable there as $K(M_{\mathrm{sw}}|\delta)$, $K(N|\delta^N)$, $K(M|\delta^M)$, and $K(M'|\delta^{M'})$, respectively.

Let τ_0, τ_1, σ_0, $\sigma_1 \in P^{\mathrm{Col}(\omega,\theta)}$ be such that

$$\tau_0^\ell = M_{\mathrm{sw}}|\delta^{+M_{\mathrm{sw}}},\ \tau_1^\ell = N|(\delta^N)^{+N},\ \sigma_0^\ell = M|(\delta^M)^{+M}, \text{ and } \sigma_1^\ell = M'|(\delta^{M'})^{+M'}. \tag{27}$$

Let $p \in \mathrm{Col}(\omega, \theta)$ force over P all the relevant properties about τ_0,τ_1, σ_0,σ_1 for the following to go through. For any $q \leq_{\mathrm{Col}(\omega,\theta)} p$ let ℓ_q denote the unique $\mathrm{Col}(\omega, \theta)$–generic filter over N such that for $n < \omega$,

$$(\bigcup \ell_q)(n) = \begin{cases} q(n) & \text{if } n \in \mathrm{dom}(q), \text{ and} \\ (\bigcup \ell)(n) & \text{otherwise.} \end{cases}$$

Let $\eta \in S$, $\eta > \max\{\xi, \xi'\}$. Notice that $\eta^{++N} \leq \eta^{++M_{\mathrm{sw}}[g \restriction \xi]} = \eta^{++M_{\mathrm{sw}}} \leq \eta^{++N[h \restriction \xi} = \eta^{++N}$ by (16) and (17), so that

$$\eta^{++M_{\mathrm{sw}}} = \eta^{++N}.$$

This is then also the common η^{++} of all $K(\tau_0^{\ell_q})$, $K(\tau_1^{\ell_q})$. Working in $P[\ell]$, let for $q \leq_{\mathrm{Col}(\omega,\theta)} p$,

$$\mathcal{U}_q \text{ and } U'_q \text{ be normal iteration trees on } \sigma_0^{\ell_q} \text{ and } \sigma_1^{\ell_q}, \text{ respectively,}$$

such that

1. $\mathrm{lh}(\mathcal{U}_q) = \mathrm{lh}(\mathcal{U}'_q) = \eta^{++M_{\mathrm{sw}}} = \delta(\mathcal{U}_q) = \delta(U'_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$,

2. $\mathcal{M}(\mathcal{U}_q) = \mathcal{M}(\mathcal{U}'_{q'})$ for all q, $q' \leq_{\mathrm{Col}(\omega,\theta)} p$,

3. every $\mathcal{U}_q$ as well as every $\mathcal{U}'_q$ is guided by ¶–small $\mathcal{Q}$–structures,

4. $K(\tau_0^{\ell_q})|\delta(\mathcal{U}_q)$ is generic over $\mathcal{M}(\mathcal{U}_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$, and

5. $K(\tau_1^{\ell_q})|\delta(\mathcal{U}'_q)$ is generic over $\mathcal{M}(\mathcal{U}'_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$.

Let us write $\mathcal{M}$ for the common value of all $\mathcal{M}(\mathcal{U}_q)$ and $\mathcal{M}(\mathcal{U}'_q)$. Notice that $\mathcal{M} \in P \subset M_{\mathrm{sw}} \cap N$. Set

$$M^* = (K(\mathcal{M}))^P.$$

By (25), we have that

$$\mathcal{M}^* = (\mathcal{P}(\mathcal{M}))^{M_{\mathrm{sw}}} = (\mathcal{P}(\mathcal{M}))^N. \tag{28}$$

Also, $\mathcal{U}_p$ is normal and is a tree on M which produces $\mathcal{M}^*$, so that (modulo potential padding) $\mathcal{U}_p$ can be computed in M_{sw} via the comparison process which tries to coiterate M and $\mathcal{M}^*$. Similarly, $\mathcal{U}'_p$ is normal and is a tree on M' which produces $\mathcal{M}^*$, so that (again modulo potential padding) $\mathcal{U}'_p \in N$. As M is s–iterable in M_{sw} and M' is s–iterable in N, we therefore get that

$$M^* \in \mathcal{F} \cap \mathcal{F}^N,\ (M, s) \preceq_{\mathcal{F}} (M^*, s), \text{ and } (M', s) \preceq_{\mathcal{F}^N} (M^*, s),$$

as desired. □ (Claim 2.9)

Claim 2.10 *(a) $H \subset L[M_\infty, \rho \mapsto \rho^*]$. Hence, $H = L[M_\infty, \rho \mapsto \rho^*]$.*

(b) If $\gamma < \delta_\infty$ and $X \in H \cap \mathcal{P}(\gamma)$, then $X \in \mathcal{M}_\infty$. In particular, δ_∞ is a regular cardinal in H and $(H_{\delta_\infty})^H = \mathcal{M}_\infty|\delta_\infty$.

Proof. (a): Let us fix X, a set of ordinals, such that $X \in H$, say $\xi \subset \gamma$ and $\xi \in X$ iff

$$\Vdash_{M_{\mathrm{sw}}}^{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{\xi}, \check{\alpha}_1, \ldots, \check{\alpha}_k). \tag{29}$$

If $N \in \mathcal{F}$, then there is some h which is $\mathrm{Col}(\omega, <\kappa)$–generic over N such that $N[h] = M_{\mathrm{sw}}[g]$, so that (29) is equivalent with

$$\Vdash_{N}^{\mathrm{Col}(\omega,<\kappa)} \varphi(\check{\xi}, \check{\alpha}_1, \ldots, \check{\alpha}_k). \tag{30}$$

In particular, $X \in \bigcap \mathcal{F}$ and $\pi_{N,N'}(X) = X$ for all N, $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists and

$$\pi_{N,N'}(\alpha_1, \ldots, \alpha_k) = \alpha_1, \ldots, \alpha_k. \tag{31}$$

Let $N \in \mathcal{F}$ be such that (31) holds true for all $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists, and set $\tilde{X} = \pi_{N,\infty}(X) \in \mathcal{M}_\infty$. Then for any $\xi < \gamma$, if $N' \in \mathcal{F}$ is such that $\pi_{N,N'}$ exists and $\pi_{N',N''}(\xi) = \xi$ for all $N'' \in \mathcal{F}$ for which $\pi_{N',N''}$ exists, we have that $\xi \in X$ iff

$$\xi^* = \pi_{N',\infty}(\xi) \in \pi_{N',\infty}(X) = \pi_{N,\infty}(X) = \tilde{X},$$

so that $X \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

We have shown (a). (b): Let $\gamma < \delta_\infty$, say $\gamma \leq \pi_{M_{\text{sw}},\infty}(\bar{\gamma})$. Pick a finite set s of ordinals such that M_{sw} is s–iterable and $\bar{\gamma} < \gamma_s^{M_{\text{sw}}}$, cf. the argument on p. 9. We have that $\pi^s_{M_{\text{sw}},\infty} \restriction \gamma_s^{M_{\text{sw}}} \in M_{\text{sw}}$, so that

$$(\rho \mapsto \rho^*) \restriction \gamma = \pi^\infty_{0,\infty} \restriction \gamma = \pi_{M_{\text{sw}},\infty}(\pi^s_{M_{\text{sw}},\infty} \restriction \gamma_s^{M_{\text{sw}}}) \restriction \gamma$$

is an element of $\mathcal{M}_\infty$. The above argument then shows (b). □ (Claim 2.10)

We are now ready to finish the proof of Lemma 2.8.

As $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ is a ground by Lemma 2.6, $K = K^{M_{\text{sw}}} = K^{L[\mathcal{M}_\infty, \rho \mapsto \rho^*]}$. Inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$, there is a canonical elementary embedding $j \colon K \to \mathcal{M}_\infty$. We aim to show that $j = \text{id}$.

Let us assume that $j \neq \text{id}$, and set $\lambda = \text{crit}(j)$. If $j(\lambda) < \delta_\infty$, then $j \restriction \lambda^{+K}$ is cofinal in $j(\lambda)^{+\mathcal{M}_\infty}$ and witnesses that $j(\lambda)^{+\mathcal{M}_\infty}$ is singular. However, this contradicts Claim 2.10 (b). If $j(\lambda) = \delta_\infty$, then λ is the Woodin cardinal of K, but there is some initial segement $\mathcal{N}$ of $\mathcal{M}_\infty$ projecting to λ which defines a counterexample to the Woodinness of λ. However, by universality, $\mathcal{N}$ would have to be an initial segment of K. Finally, if $j(\lambda) > \delta_\infty$, then j comes from an iteration of K strictly above δ_∞, the common Woodin cardinal of K and $\mathcal{M}_\infty$. But $\mathcal{M}_\infty$ is generated from δ_∞ together with a club class of indiscernibles above κ_∞, which immediately gives $j \restriction \kappa_\infty = \text{id}$ and then $j = \text{id}$. □ (Lemma 2.8)

Lemma 2.11 *$L[M_\infty, \rho \mapsto \rho^*]$ is the mantle of M_{sw}.*

Proof. As $L[M_\infty, \rho \mapsto \rho^*]$ is a ground of M_{sw} by Lemma 2.6, if suffices to prove that $L[M_\infty, \rho \mapsto \rho^*] \subset N$ for every ground N of M_{sw}. But if N is a ground of M_{sw}, then $L[M_\infty, \rho \mapsto \rho^*] \subset L[K, \Sigma] \subset N$. □ (Lemma 2.11)

We call $L[M_\infty, \rho \mapsto \rho^*]$ the *Varsovian model derived from* M_{sw}. If M is a model which is elementarily equivalent to M_{sw}, then the *Varsovian model derived from* M is that inner model of M which is defined over M as $L[M_\infty, \rho \mapsto \rho^*]$ is defined over M_{sw}.

Lemma 2.12 **(**F. Schlutzenberg)

(a) $\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$ is closed under both $\pi^{\infty}_{0,\infty}$ and $(\pi^{\infty}_{0,\infty})^{-1}$.

(b) $\mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}))\cap\mathrm{OR}=\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})\cap\mathrm{OR}$.

Proof. (a) Let ρ be such that $\{\rho,\rho^*\}\cap\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})\neq\emptyset$. Let s be a finite set of M_{sw}–indiscernibles such that

$$\rho\in\mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}}\cup s^-).$$

We have that $\pi^{\infty}_{0,\infty}\restriction\mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty}\cup s^-)\in\mathcal{M}_\infty$ and in fact

$$\pi^{\infty}_{0,\infty}\restriction\mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty}\cup s^-)=\pi_{M_{\mathrm{sw}},\infty}(\pi_{M_{\mathrm{sw}},\infty}\restriction\mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}}\cup s^-),$$

where $\pi_{M_{\mathrm{sw}},\infty}\restriction\mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty}\cup s^-)\in M_{\mathrm{sw}}$. Then if $\rho\in\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, then $\rho^*=(\pi^{\infty}_{0,\infty}\restriction\mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty}\cup s^-))(\rho)\in\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, and if $\rho^*\in\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, then $\rho=(\pi^{\infty}_{0,\infty}\restriction\mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty}\cup s^-))^{-1}(\rho^*)\in\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$.

(b) Let $\rho\in\mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}))\cap\mathrm{OR}$. By (a), it suffices to prove that $\rho^*\in\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$.

We may pick a finite set s of M_{sw}–indiscernibles such that

$$\rho\in\mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(s). \tag{32}$$

Let $N\in\mathcal{F}$ be s–iterable such that $\pi_{N,N'}(\rho)=\rho$ for all $N'\in\mathcal{F}$ with $\pi_{N,N'}\downarrow$. As $L[\mathcal{M}_\infty,\rho\mapsto\rho^*]=\mathrm{HOD}^{N[h]}$ for some/all h which are $\mathrm{Col}(\omega,<\kappa)$–generic over N, cf. Claim 2.10 (a), (32) implies that

$$\rho\in\mathrm{Hull}^N(s).$$

But then

$$\rho^*\in\mathrm{Hull}^{\mathcal{M}_\infty}(s)\subset\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}).$$

□ (Lemma 2.12)

Corollary 2.13 *Let* $\sigma\colon\mathcal{V}\cong\mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}))$*, where* $\mathcal{V}$ *is transitive.* $\mathcal{V}=L[M_{\mathrm{sw}},\rho\mapsto\pi_{M_{\mathrm{sw}},\infty}(\rho)]$*, and* $\sigma\supset\pi_{M_{\mathrm{sw}},\infty}$*.*

Proof. By Lemma 2.12 (b) and by (9), it remains to be seen that

$$\sigma^{-1}((\rho\mapsto\rho^*)\restriction\delta_\infty)=\pi_{M_{\mathrm{sw}},\infty}\restriction\delta. \tag{33}$$

For $n<\omega$ let us write $s_n = \{\aleph_1^V, \ldots, \aleph_{n+1}^V\}$. Then for each $n<\omega$, $\pi_{M_{\rm sw},\infty} \restriction \gamma_{s_n}^{M_{\rm sw}} = \pi^{s_n}_{M_{\rm sw},\infty} \restriction \gamma_{s_n}^{M_{\rm sw}} \in M_{\rm sw}$ and $\sigma(\pi^{s_n}_{M_{\rm sw},\infty} \restriction \gamma_{s_n}^{M_{\rm sw}}) = \pi^{s_n}_{\mathcal{M}_\infty, \mathcal{M}_\infty^\infty}$, by the elementarity of σ and $\sigma(s_n) = s_n$, and the latter is equal to $\pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}$ which is hence in $\mathcal{M}_\infty$. But then $\sigma^{-1}((\rho \mapsto \rho^*) = \sigma^{-1}(\bigcup_{n<\omega} \pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}) = \bigcup_{n<\omega} \sigma^{-1}(\pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}) = \bigcup_{n<\omega} \pi^{s_n}_{M_{\rm sw},\infty} \restriction \gamma_{s_n}^{M_{\rm sw}} = \pi_{M_{\rm sw},\infty} \restriction \delta$, which shows (33). □ (Corollary 2.13)

Lemma 2.14 *Let* $\sigma\colon \mathcal{V} = L[M_{\rm sw}, \rho \mapsto \pi_{M_{\rm sw},\infty}(\rho)] \cong {\rm Hull}^{L[\mathcal{M}_\infty, \rho\mapsto\rho^*]}({\rm ran}(\pi_{M_{\rm sw},\infty}))$. *$\mathcal{V}$ is iterable via iteration trees which live on $M_{\rm sw}|\delta$.*

Proof. Implicitly, [17] contains a simplified version of the argument to follow, cf. [17, Lemma 3.46]. This was pointed out to the authors by Farmer Schlutzenberg who then independently arrived at a proof of Lemma 2.14.

We claim that Σ may serve as an iteration strategy for iteration trees on $\mathcal{V}$ which live on $M_{\rm sw}|\delta$. This makes sense by Claim 2.10 (b), Corollary 2.13, and the elementarity of σ.

Let $\mathcal{T}$ be a putative tree on $\mathcal{V}$ which lives on $M_{\rm sw}|\delta$ and is according to Σ. If $\mathcal{M}_\alpha^\mathcal{T}$ is a transitive proper class, $\alpha < {\rm lh}(\mathcal{T})$, then we may write $\mathcal{M}_\alpha^\mathcal{T} = L[M_\alpha, \pi_\alpha]$. The tree $\mathcal{T}$ induces a canonical tree, which we shall denote by $\bar{\mathcal{T}}$, on $M_{\rm sw}$ which is according to Σ.

Let us write Π for the set of all $\alpha < {\rm lh}(\mathcal{T})$ such that $\mathcal{M}_\alpha^\mathcal{T}$ is a proper class. If $\alpha \in {\rm lh}(\mathcal{T}) \setminus \Pi$, then $\mathcal{M}_\alpha^{\bar{\mathcal{T}}} = \mathcal{M}_\alpha^\mathcal{T}$. We claim that we may define a sequence

$$((M_\alpha, \pi_\alpha, M_\alpha^*, \pi_\alpha^*, \mathcal{V}_\alpha, \tilde{\pi}_\alpha) \colon \alpha \in \Pi)$$

such that

(a) $M_0 = M_{\rm sw}$, $\pi_0 = \pi_{M_{\rm sw},\infty}$, $M_0^* = \mathcal{M}_\infty$, $\pi_0^* = (\rho \mapsto \rho^*)$

and for all $\alpha \leq_\mathcal{T} \beta < {\rm lh}(\mathcal{T})$ with α, $\beta \in \Pi$:

(b) $M_\alpha = \mathcal{M}_\alpha^{\bar{\mathcal{T}}}$,

(c) $L[M_\alpha, \pi_\alpha \restriction {\rm OR}] = \mathcal{M}_\alpha^\mathcal{T}$,

(d) $\mathcal{V}_\alpha = L[M_\alpha^*, \pi_\alpha^*]$ is the Varsovian model derived from M_α,

(e) $\pi_\alpha \colon M_\alpha \to M_\alpha^*$ is an elementary embedding,

(f) $\tilde{\pi}_\alpha \colon L[M_\alpha, \pi_\alpha \restriction {\rm OR}] \to L[M_\alpha^*, \pi_\alpha^*]$ is an elementary embedding,

(g) $\tilde{\pi}_\beta \restriction {\rm lh}(E_\gamma) = \tilde{\pi}_\alpha \restriction {\rm lh}(E_\gamma)$ for $\alpha <_\mathcal{T} \gamma + 1 \leq_\mathcal{T} \beta$,

(h) $\tilde{\pi}_\alpha \supset \pi_\alpha$, and

(i) $\pi^{\mathcal{T}}_{\alpha,\beta} \supset \pi^{\bar{\mathcal{T}}}_{\alpha,\beta}$.

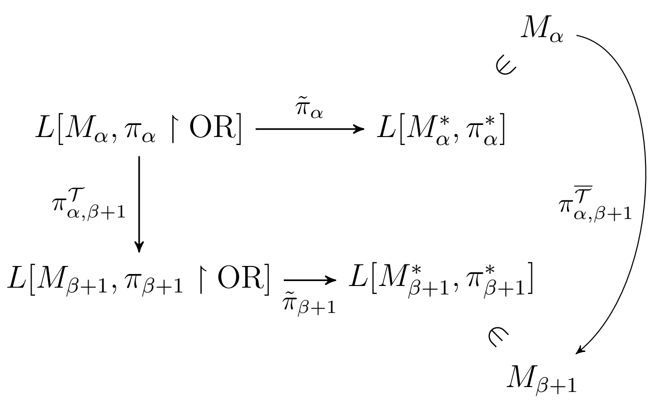
Let us present the successor steps of the construction, leaving the limit steps to the reader's discretion. Let $\alpha = \mathcal{T}$–prec$(\beta+1)$, where $\beta+1 \in \Pi$, and write $F = E^{\mathcal{T}}_\beta = E^{\bar{\mathcal{T}}}_\beta$.

We may define an elementary embedding

$$\tilde{\pi}_{\beta+1} \colon \mathrm{ult}(L[M_\alpha, \pi_\alpha \restriction \mathrm{OR}]; F) \to \mathcal{V}_{\beta+1}$$

by setting

$$\tilde{\pi}_{\beta+1}([a,f]^{\mathcal{M}^{\mathcal{T}}_\alpha}_F) = [a, u \mapsto \tilde{\pi}_\alpha(f)(\pi_\alpha(u))]^{M_\alpha}_F.$$



This is indeed well–defined and elementary, as we may use $(\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)}) \in M_\alpha$ and compute as follows. Let φ be a formula, let us assume for notational convenience that φ has only one free variable, and let $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f \colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to \mathcal{M}^{\mathcal{T}}_\alpha$, $f \in \mathcal{M}^{\mathcal{T}}_\alpha$.

$$
\begin{aligned}
& \mathcal{M}^{\mathcal{T}}_{\beta+1} \models \varphi([a,f]^{\mathcal{M}^{\mathcal{T}}_\alpha}) \\
\iff & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon \mathcal{M}^{\mathcal{T}}_\alpha \models \varphi(f(u))\} \in F_a \\
\iff & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)(\tilde{\pi}_\alpha(u)))\} \in F_a \\
\iff & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(u)))\} \in F_a \\
\iff & a \in \pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(u)))\}) \\
\iff & L[M^*_{\beta+1}, \pi^*_{\beta+1} \models \varphi(\pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\tilde{\pi}_\alpha(f))(((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(a))) \\
\iff & L[M^*_{\beta+1}, \pi^*_{\beta+1} \models \varphi(\pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\tilde{\pi}_\alpha(f))((\pi_\alpha(a))).
\end{aligned}
$$

Notice that $\tilde{\pi}_{\beta+1} \restriction \mathrm{lh}(F) = \tilde{\pi}_\alpha \restriction \mathrm{lh}(F)$, as required by (g).

The key point is now that

$$M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1}) \cong \mathcal{M}^{\bar{\mathcal{T}}}_{\beta+1}. \tag{34}$$

(34) is established by the argument which gave Schlutzenberg's Lemma 2.12. Let I denote the class of all M_{sw}–indiscernibles, and let us assume for notational convenience that all embeddings which we consider fix all the points in I.

In order to show (34), let $x \in M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1})$, say $x = \tilde{\pi}_{\beta+1}(\bar{x}) \in M^*_{\beta+1}$. We have that $\bar{x} \in \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_{\beta+1}}(\mathrm{lh}(F) \cup I)$, so that $x \in \mathrm{Hull}^{L[M^*_{\beta+1}, \pi^*_{\beta+1}]}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I) \cap M^*_{\beta+1}$. By the elementarity of $\pi^{\mathcal{T}}_{0,\beta+1}$, $L[M^*_{\beta+1}, \pi^*_{\beta+1}]$ is the Varsovian model derived from $M_{\beta+1}$ which in turn is equal to $\mathsf{HOD}^{P[h]}$ for all $P \in \mathcal{F}^{M_{\beta+1}}$ and all h which are $\mathrm{Col}(\omega, < \kappa^P)$–generic over P, cf. Claim 2.10 (a). We thus have $x \in \mathrm{Hull}^P(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I)$ for all $P \in \mathcal{F}^{M_{\beta+1}}$. By picking P sufficiently far out in the system, we thus get that

$$\pi^*_{\beta+1}(x) \in \mathrm{Hull}^{M_{\beta+1}}(\pi^*_{\beta+1} \circ \tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I). \tag{35}$$

However, for each ordinal ρ we may pick some $s \in [I]^{<\omega}$ such that $\rho \in \mathrm{dom}(\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\})$, i.e., $\pi^*_{\beta+1}(\rho) = (\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\}))(\rho)$, and then

$$\begin{aligned} \pi^*_{\beta+1}(\rho) &= (\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\}))(\rho) \\ &= \pi^{\mathcal{T}}_{0,\beta+1}(\pi^*_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}))(\rho) \\ &= \pi^{\mathcal{T}}_{0,\beta+1}(\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\})))(\rho). \end{aligned}$$

But $\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}) \in \mathrm{Hull}^{M_0}(I)$, hence $\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\})) \in \mathrm{Hull}^{M^*_0}(I)$, hence $\pi^{\mathcal{T}}_{0,\beta+1}(\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}))) \in \mathrm{Hull}^{M^*_{\beta+1}}(I)$. This shows that $\mathrm{Hull}^{M^*_{\beta+1}}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I)$ is closed under $\rho \mapsto \pi^*_{\beta+1}(\rho)$ as well as under $\rho \mapsto (\pi^*_{\beta+1})^{-1}(\rho)$, so that by $x \in M^*_{\beta+1}$, (35) is tantamount to saying that

$$x \in \mathrm{Hull}^{M^*_{\beta+1}}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I). \tag{36}$$

We have shown that $x \in M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1})$ implies (36). This gives (34).

By (34), we may let $\pi_{\beta+1} = \tilde{\pi}_{\beta+1} \restriction M_{\beta+1}$. It remains to be verified that

$$\pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha) = \tilde{\pi}_{\beta+1} \restriction \mathrm{OR}. \tag{37}$$

Let $\xi = \pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)$, where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f\colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to \mathrm{OR}$, $f \in \mathcal{M}^{\mathcal{T}}_\alpha$. Then

$$\begin{aligned}
\pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha)(\xi) &= \pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha)(\pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)) \\
&= \pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha \circ f)(\pi^{\mathcal{T}}_{\alpha,\beta+1}(a)) \\
&= \pi^{\mathcal{T}}_{\alpha,\beta+1}(u \mapsto \tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{<\omega})(u))(a) \\
&= \tilde{\pi}_{\beta+1}(\pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)) \\
&= \tilde{\pi}_{\beta+1}(\xi).
\end{aligned}$$

□ (Theorem 2.14)

The proof of Theorem 2.16 makes use of the following result. We know that $\mathcal{M}_\infty$ is an iterate of M_{sw} via an ω–stack of normal trees, $(\mathcal{T}_n\colon n < \omega)$. The normalizing procedure which is developed in the papers [12], [13], and [16] produces a normal iteration tree $X(\mathcal{T}_n\colon n < \omega)$ on M_{sw} with last model $\mathcal{M}_\infty$.

Theorem 2.15 (F. Schlutzenberg, J. Steel) ([12], [13], [16]) $\mathcal{M}_\infty$ is a Σ–iterate of M_{sw} via a *normal* iteration tree on M_{sw} which lives on $M_{\mathrm{sw}}|\delta$ and with iteration map $\pi_{M_{\mathrm{sw}},\infty}$.

Theorem 2.16 *δ is a Woodin cardinal inside $L[M_{\mathrm{sw}}, \rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)]$.*

Proof. The proof we are about to present was also found independently by Farmer Schlutzenberg following a hint by John Steel.

Let $\mathcal{T}$ be the (unique) tree on M_{sw} which witnesses the statement of Theorem 2.15. By Corollary 2.13 (b), we may construe $\mathcal{T}$ as a tree on $L[M_{\mathrm{sw}}, \rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)]$, and we may lift the iteration map $\pi_{M_{\mathrm{sw}},\infty}$ to an iteration map

$$\tilde{\pi}\colon L[M_{\mathrm{sw}}, \rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)] \to L[\mathcal{M}_\infty, \sigma],$$

where σ is the image of $\rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)$ under $\tilde{\pi}$. However, the same argument as in the proof of Corollary 2.13 (a) shows that

$$\pi_{M_{\mathrm{sw}},\infty}(\pi_{M_{\mathrm{sw}},\infty} \restriction \delta) = \pi^\infty_{0,\infty} \restriction \delta_\infty. \tag{38}$$

This is true because if again $s_n = \{\aleph_1, \ldots, \aleph_{n+1}\}$ for $n < \omega$, then $\pi_{M_{\mathrm{sw}},\infty}(\pi_{M_{\mathrm{sw}},\infty} \restriction \delta) = \pi_{M_{\mathrm{sw}},\infty}(\bigcup_{n<\omega} \pi^{s_n}_{M_{\mathrm{sw}},\infty} \restriction \gamma^{M_{\mathrm{sw}}}_{s_n}) = \bigcup_{n<\omega} \pi_{M_{\mathrm{sw}},\infty}(\pi^{s_n}_{M_{\mathrm{sw}},\infty} \restriction \gamma^{M_{\mathrm{sw}}}_{s_n}) = \bigcup_{n<\omega} \pi^\infty_{0,\infty} \restriction \gamma^{\mathcal{M}_\infty}_{s_n} = \pi^\infty_{0,\infty} \restriction \delta_\infty$.

We therefore have that

$$\tilde{\pi}\colon L[M_{\rm sw}, \rho \mapsto \pi_{M_{\rm sw},\infty}(\rho)] \to L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$$

is given by the normal iteration tree $\mathcal{T}$.

Let us now suppose that δ is not a Woodin cardinal in $L[M_{\rm sw}, \rho \mapsto \pi_{M_{\rm sw},\infty}(\rho)]$ which implies that δ_∞ is not a Woodin cardinal in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. Notice that $\mathcal{T}$ must have length $\delta_\infty + 1 = \kappa^{+M_{\rm sw}} + 1$, and $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$ is guided by ¶–small $\mathcal{Q}$–structures, so that $\mathcal{T} \restriction \kappa^{+M_{\rm sw}} \in M_{\rm sw}$.

Write $\lambda = \kappa^{++M_{\rm sw}}$, and $\mathcal{V} = L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. Let $g \in V$ be ${\rm Col}(\omega, \lambda)$–generic over $M_{\rm sw}$. Inside $M_{\rm sw}[g]$, let T be a tree of height ω searching for a $\mathcal{Q}$ and b such that

(α) $\mathcal{Q}$ is a transitive model of ZFC^- of height λ such that δ is a cardinal in $\mathcal{Q}$ and $H_\delta^{\mathcal{Q}} = M_{\rm sw}|\delta$,

(β) b is a cofinal branch through $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$ such that when $\mathcal{T}'$ is $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$, being construed as a tree on $\mathcal{Q}$,[10] then all the models $\mathcal{M}_\alpha^{\mathcal{T}'}$, $\alpha < \kappa^{+M_{\rm sw}}$, are well–founded, and

$$\pi_{0,b}^{\mathcal{T}'}\colon \mathcal{Q} \to H_\lambda^{\mathcal{V}}.$$

T is ill–founded in V, as we may set $\mathcal{Q} = H_\lambda^{L[M_{\rm sw}, \pi_{M_{\rm sw},\infty} \restriction {\rm OR}]}$ and $b = [0, \kappa^{+M_{\rm sw}})_{\mathcal{T}}$. Therefore, T is ill–founded in $M_{\rm sw}[g] \subset V$ as well. Let $\mathcal{Q}$ and b in $M_{\rm sw}[g]$ be given by a branch through T. Suppose that $b \neq [0, \kappa^{+M_{\rm sw}})_{\mathcal{T}}$. As $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$ is normal, the "zipper argument," cf. e.g. [15, p. 1645f.], then shows that $\delta(\mathcal{T} \restriction \kappa^{+M_{\rm sw}}) = \delta_\infty$ must be Woodin in $H_\lambda^{\mathcal{V}}$ which is against our current hypothesis.

Therefore, $[0, \kappa^{+M_{\rm sw}})_{\mathcal{T}} = b \in M_{\rm sw}[g]$. As this was shown to be true for *any* b such that $\mathcal{Q}$ and b come from a branch through T for some $\mathcal{Q}$, we must have that $[0, \kappa^{+M_{\rm sw}})_{\mathcal{T}} \in M_{\rm sw}$ by the homogeneity of ${\rm Col}(\omega, \lambda)$. But this gives that

$$\pi_{M_{\rm sw},\infty} \restriction \delta = \pi_{0,[0,\kappa^{+M_{\rm sw}})_{\mathcal{T}}}^{\mathcal{T} \restriction \kappa^{+M_{\rm sw}}} \in M_{\rm sw},$$

which is a map which sends $\delta < \kappa$ cofinally into $\delta_\infty = \kappa^{+M_{\rm sw}}$. Hence $\kappa^{+M_{\rm sw}}$ is singular in $M_{\rm sw}$. Contradiction! □ (Theorem 2.16)

J. Steel observed that if g is ${\rm Col}(\omega, <\kappa)$–generic over $M_{\rm sw}$, then $M_{\rm sw}[g]$ is *not* a model of "every OD–set of reals is determined," so that one canot use [5] to deduce the conclusion of Lemma 2.16.

[10] This is possible by item (α).

Lemma 2.17 *$L[\mathcal{M}_\infty, \rho \mapsto \rho^*] = L[\mathcal{M}_\infty|\delta_\infty, \bar{\Sigma}]$, where $\bar{\Sigma}$ is the iteration strategy for $\mathcal{M}_\infty|\delta_\infty$.*

Proof sketch. Write $W = L[\mathcal{M}_\infty|\delta_\infty, \bar{\Sigma}]$. We have an elementary embedding $j : \mathcal{M}_\infty \to (K(\mathcal{M}_\infty|\delta_\infty))^W$. Suppose $j \neq \mathrm{id}$. We may then reconstruct $j \restriction \mathcal{M}_\infty|\mathrm{crit}(j)^+$ inside W. Contradiction! □ (Lemma 2.17)

In a sequel to this paper, cf. [7], we will study Varsovian models in more generality.

The attentive reader will notice that the preceding arguments actually produced the following statement.

Theorem 2.18 *For a cone of reals x, $M_s(x)$ has a 2–small core model $K = K^{M_s(x)}$ which in V is an iterate of M_{sw}, and the mantle of $M_s(x)$ is the Varsovian model $L[K, \Sigma_K]$, where Σ_K is the tail of Σ.*

3 Appendix: Bukovský's theorem.

Definition 3.1 *Let W be an inner model of V. Let λ be an infinite cardinal. We say that W* uniformly λ–covers *V iff for all functions $f \in V$ with $\mathrm{dom}(f) \in W$ and $\mathrm{ran}(f) \subset W$ there is some function $g \in W$ with $\mathrm{dom}(g) = \mathrm{dom}(f)$ such that $f(x) \in g(x)$ and $\mathrm{Card}(g(x)) < \lambda$ for all $x \in \mathrm{dom}(g)$.*

If there is some poset $\mathbb{P} \in W$ having the λ–c.c. in W and some g which is $\mathbb{P}$–generic over W such that $V = W[g]$, then W uniformly λ–covers V. Bukovský's Theorem 3.5 will say that the converse is true also.

The following is probably part of the folklore.

Theorem 3.2 *Let W be an inner model of V, and let λ be an infinite regular cardinal. Assume that W uniformly λ–covers V, and assume also that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$. Then $W = V$.*

Proof. Let us call any set Γ of functions an *antichain* iff for all a, $b \in \Gamma$ with $a \neq b$ there is some $i \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$ with $a(i) \neq b(i)$.

It is easily seen that the hypotheses on W give that

$$^{2^{<\lambda}}W \subset W. \tag{39}$$

To verify (39), notice first that by $\mathcal{P}(2^{<\lambda}) \cap V \subset W$, W computes the cardinal successor of $2^{<\lambda}$ correctly and for every $\gamma < (2^{<\lambda})^+$, $\mathcal{P}(\gamma) \cap V \subset W$.

Now let $f\colon 2^{<\lambda} \to \mathrm{OR}$, $f \in V$. Using the fact that W uniformly λ–covers V, let $g \in W$ be a function with $\mathrm{dom}(g) = 2^{<\lambda}$ such that $g(\xi)$ is a set of ordinals, $f(\xi) \in g(\xi)$, and $\mathrm{Card}(g(\xi)) < \lambda$ for all $\xi < 2^{<\lambda}$. Let $e\colon \gamma \cong \bigcup \mathrm{ran}(g)$ be the (inverse of the) transitive collapse of $\bigcup \mathrm{ran}(g)$, so that $e \in W$ and $\gamma < (2^{<\lambda})^+$. As $\mathcal{P}(\gamma) \cap V \subset W$, the function $e^{-1} \circ f\colon 2^{<\lambda} \to \gamma$ is in W, which gives that $f = e \circ (e^{-1} \circ f) \in W$. We showed (39).

Assume that $A\colon \alpha \to 2$, for some ordinal α, is such that $A \in V \setminus W$. Let us write $\mathcal{F}$ for the collection of all functions a such that there is some $x \subset \alpha$ of size $< \lambda$ such that $a\colon x \to 2$. Using again the fact that W uniformly λ–covers V,[11] we may pick a function g in W such that if $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, then

(i) $g(\Gamma) \in W$ is a subset of Γ of size $< \lambda$, and

(ii) if there is some (unique!) $a \in \Gamma$ with $a = A \restriction \mathrm{dom}(a)$, then $a \in g(\Gamma)$.

We call $a \in \mathcal{F}$ *legal* iff for no antichain $\Gamma \in W$, $a \in \Gamma \setminus g(\Gamma)$. Notice that being legal is defined inside W (from the parameter $g \in W$).

Every $A \restriction x$, where $x \subset \alpha$ has size $< \lambda$, is legal.

If $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, and if every $a \in \Gamma$ is legal, then we must have $g(\Gamma) = \Gamma$, from which it follows that Γ has size $< \lambda$.

Let $\theta \gg \alpha$ be such that $\theta^{<\lambda} = \theta$. Let

$$X \prec (H_\theta; \in, \{A\}, \mathcal{F}, g, H_\theta \cap W)$$

be such that ${}^{<\lambda}X \subset X$ and $\mathrm{Card}(X) = 2^{<\lambda}$. By (39), $X \cap W \in W$, and of course

$$X \cap W \prec (H_\theta \cap W; \in, \mathcal{F}, g) \in W. \tag{40}$$

Write $\sigma\colon \bar{W} \cong X \cap W$ for the (inverse of the) transitive collapse of $X \cap W$, so that $\sigma \in W$. σ extends to $\tilde{\sigma}\colon H \cong X$, the (inverse of the) transitive collapse of X.

Notice that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$ gives that $\bar{A} = \tilde{\sigma}^{-1}(A) \in W$, which in turn yields that

$$A \restriction (X \cap \alpha) = \sigma\text{"}\bar{A} \in W. \tag{41}$$

We are now going to derive a contradiction from (41).

Using (41), we may work inside W and define a sequence $(a_i\colon i < \lambda)$ of elements of $\mathcal{F}$ such that $a_i \in X$ and $\mathrm{dom}(a_i) \supset \mathrm{dom}(a_j)$ for all $j < i < \lambda$ as follows. Assume $(a_j\colon j < i)$ has already been chosen. Notice that $(a_j\colon j < i) \in X$ by

[11] This use is now substantial, in contrast to the previous one.

${}^{<\lambda}X \subset X$. Write $x = \bigcup_{j<i} \text{dom}(a_j)$, so that $x \in X$. Clearly, for every $\xi < \alpha$ there is some legal $a \in \mathcal{F}$ such that $x \cup \{\xi\} \subset \text{dom}(a)$ and $a = A \upharpoonright \text{dom}(a)$ (just pick $A \upharpoonright (x \cup \{\xi\})$). There must then be some $\xi < \alpha$ such that there are legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \text{dom}(a) \cap \text{dom}(b)$ and $a(\xi) \neq b(\xi)$, as otherwise A would be the union of all legal $a \in \mathcal{F}$ with $a \supset A \upharpoonright x$ and thus A would be in W.

By (40) we must then have inside X some $\xi < \alpha$ and some legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \text{dom}(a) \cap \text{dom}(b)$ and $a(\xi) \neq b(\xi)$. By (41), we may then choose in W some $\xi \in \alpha \cap X$ and some $a \in \mathcal{F} \cap X$ such that $x \cup \{\xi\} \subset \text{dom}(a)$, $a \upharpoonright x = (A \upharpoonright (X \cap \alpha)) \upharpoonright x$ $(= A \upharpoonright x)$, and $a(\xi) \neq (A \upharpoonright (X \cap \alpha))(\xi)$ $(= A(\xi))$. Let $a_i = a$.

Writing $\Gamma = \{a_i \colon i < \lambda\}$, $\Gamma \in W$, and Γ is an antichain consisting of legal functions. But this is a contradiction! □ (Theorem 3.2)

Let us fix $W \subset V$, an inner model, and let λ and μ be infinite cardinals, $\lambda \leq \mu$. We aim to define a poset in W which will be a candidate for generically adding a given subset of μ.

Working in W, let $\mathcal{L}$ be the infinitary language with atomic fomulae "$\check{\xi} \in \dot{a}$," for $\xi < \mu$, and such that the set of formulae is closed under negation and infinite disjunctions of the form $\bigvee\!\!\!\bigvee \Gamma$ for all well–ordered sets Γ of fomulae with $\text{Card}(\Gamma) < \lambda$. Writing $\mu^{<\lambda} = (\mu^{<\lambda})^W$, $\mathcal{L}$ has size $\mu^{<\lambda}$.

For $A \subset \mu$, $A \in V^{\text{Col}(\omega,\mu^{<\lambda})}$, and $\varphi \in \mathcal{L}$, we may define the meaning of "$A \models \varphi$" in the obvious recursive fashion: $A \models$ "$\check{\xi} \in \dot{a}$" iff $\xi \in A$, $A \models \neg\varphi$ iff $A \not\models \varphi$, and $A \models \bigvee\!\!\!\bigvee \Gamma$ iff $A \models \varphi$ for some $\varphi \in \Gamma$. Inside $V^{\text{Col}(\omega,\mu^{<\lambda})}$, the relation "$A \models \varphi$" is Borel in the codes. For $\Gamma \subset \mathcal{L}$, $A \models \Gamma$ means $A \models \varphi$ for all $\varphi \in \Gamma$. For $\Gamma \cup \{\varphi\} \in \mathcal{P}(\mathcal{L}) \cap W$, we write

$$\Gamma \vdash \varphi \tag{42}$$

iff in $W^{\text{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \models \Gamma$, then $A \models \varphi$. (42) is thus defined over W, and inside $W^{\text{Col}(\omega,\mu^{<\lambda})}$, (42) is Π^1_1 in the codes By absoluteness, (42) is thus equivalent with the fact that in $V^{\text{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \models \Gamma$, then $A \models \varphi$. For $\Gamma \in \mathcal{P}(\mathcal{L}) \cap W$, Γ is called *consistent* iff there is no $\varphi \in \mathcal{L}$ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, which in turn is easily seen to be equivalent with the fact that in $W^{\text{Col}(\omega,\mu^{<\lambda})}$ (equivalently, in $V^{\text{Col}(\omega,\mu^{<\lambda})}$) there is some $A \subset \mu$ with $A \models \Gamma$.

Now let

$$g \colon [\mathcal{L}]^{\lambda} \cap W \to [\mathcal{L}]^{<\lambda} \cap W,\ g \in W$$

be a function such that

(i) $g(\Gamma) \subset \Gamma$, and

(ii) $\text{Card}(g(\Gamma)) < \lambda$

for all $\Gamma \in [\mathcal{L}]^\lambda \cap W$. Let us call $\varphi \in \mathcal{L}$ *illegal* iff there is some $\Gamma \in [\mathcal{L}]^\lambda \cap W$ such that $\varphi \in \Gamma \setminus g(\Gamma)$, and let us write T^g for the set of all formulae of the form[12]

$$\varphi \to \bigvee\!\!\!\bigvee g(\Gamma), \tag{43}$$

where φ is illegal, $\Gamma \in [\mathcal{L}]^\lambda \cap W$, and $\varphi \in \Gamma \setminus g(\Gamma)$.

Let us write $\mathbb{P}^g$ for the set of all $\varphi \in \mathcal{L}$ such that $T^g \cup \{\varphi\}$ is consistent. We also write

$$\varphi \leq_{\mathbb{P}^g} \varphi' \tag{44}$$

for $T^g \cup \{\varphi\} \vdash \varphi'$.

Claim 3.3 *$\mathbb{P}^g$ has the λ–c.c. inside W.*

Proof. Let $\Gamma \in [\mathbb{P}^g]^\lambda \cap W$. Let $\varphi \in \Gamma \setminus g(\Gamma)$. By (43), $\varphi \leq_{\mathbb{P}^g} \bigvee\!\!\!\bigvee g(\Gamma)$, so that Γ cannot be an antichain. □ (Claim 3.3)

For an arbitrary choice of g, we might have that $\mathbb{P}^g$ is quite trivial, or even $\mathbb{P}^g = \emptyset$. Let $A \subset \mu$, $A \in V$. We set

$$G_A = \{\varphi \in \mathbb{P}^g \colon A \models \varphi\}.$$

Claim 3.4 *Assume that $A \models T^g$. Then $G_A \subset \mathbb{P}^g$ is a $\mathbb{P}^g$–generic filter over W and*

$$A = \{\xi < \mu \colon \text{ “}\check{\xi} \in \dot{a}\text{”} \in G_A\} \in W[G_A].$$

Proof. If φ, $\varphi' \in \mathbb{P}^g$, $A \models \varphi$, and $\varphi \leq_{\mathbb{P}^g} \varphi'$, then $A \models \varphi'$ using absoluteness. If φ, $\varphi' \in \mathbb{P}^g$, $A \models \varphi$, and $A \models \varphi'$, then $A \models \varphi \wedge \varphi'$,[13] $\varphi \wedge \varphi' \in \mathbb{P}^g$ by $A \models T^g$, and clearly $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi$ and $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi'$. Hence G_A is a filter.

Now let $\Gamma \in W$ be a maximal antichain in $\mathbb{P}^g$. By Claim 3.3, $\Gamma \in [\mathbb{P}^g]^{<\lambda}$. If $G_A \cap \Gamma = \emptyset$, then $A \models \neg \bigvee\!\!\!\bigvee \Gamma$. By $A \models T^g$, $\neg \bigvee\!\!\!\bigvee \Gamma \in \mathbb{P}^g$, and

$$\Gamma \cup \{\neg \bigvee\!\!\!\bigvee \Gamma\} \supsetneq \Gamma$$

is an antichain. Contradiction!

The rest is easy. □ (Claim 3.4)

[12] $\varphi \to \varphi'$ is short for $\bigvee\!\!\!\bigvee\{\neg\varphi, \varphi'\}$.
[13] $\varphi \wedge \varphi'$ is short for $\neg \bigvee\!\!\!\bigvee\{\neg\varphi, \neg\varphi'\}$.

Theorem 3.5 (Lev Bukovský) *Let $W \subset V$ be an inner model, and let λ be an infinite regular cardinal such that W uniformly λ–covers V. Let $e\colon 2^{2^{<\lambda}} \to \mathcal{P}(2^{<\lambda})$ be a bijection, and let*

$$A = \{2^{<\lambda} \cdot \eta + \xi \colon \eta < 2^{2^{<\lambda}} \wedge \xi \in e(\eta)\}.$$

There is then some poset $\mathbb{P} \in W$ such that

(a) $\mathbb{P}$ *has the λ–c.c. in W,*

(b) $\mathbb{P}$ *has size $2^{2^{<\lambda}}$ in W,*

(c) A *is $\mathbb{P}$–generic over W, and*

(d) $V = W[A]$.

Proof. Let us write

$$\mu = 2^{2^{<\lambda}},$$

as being computed in V.

By the fact that W uniformly λ–covers V, we may find a function

$$g\colon [\mathcal{L}]^{\lambda} \to [\mathcal{L}]^{<\lambda},\ g \in W$$

such that for all $\Gamma \in [\mathcal{L}]^{\lambda} \cap W$,

(i) $g(\Gamma) \subset \Gamma$,

(ii) $\mathrm{Card}(g(\Gamma)) < \lambda$, and

(iii) if $A \vDash \varphi$ for *some* $\varphi \in \Gamma$, then $A \vDash \bigvee\!\!\!\bigvee g(\Gamma)$.

For this choice of g, $A \vDash T^g$. Hence by Claim 3.4, G_A is $\mathbb{P}^g$–generic over W, and $A \in W[G_A]$. This gives (a), (b), and (c). Clearly, $W[G_A]$ inherits from W the fact that it uniformly λ–covers V, so that (d) is given by Theorem 3.2. □ (Theorem 3.5)

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