Negative results on precipitous ideals on ω_1 *†‡

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Abstract

We show that in extender models there are no generic embeddings with critical point ω_1 that resemble the stationary tower.

Given an ideal $\mathcal{I}$ on a cardinal κ, let $\mathbb{P}_{\mathcal{I}} = \wp(\kappa)/\mathcal{I}$. It is well known that forcing with $\mathbb{P}_{\mathcal{I}}$ adds a V-ultrafilter on κ. An ideal $\mathcal{I}$ on κ is called *precipitous* if whenever $G \subseteq \wp(\kappa)$ is a $\mathbb{P}_{\mathcal{I}}$-generic ultrafilter, $Ult(V, G)$ is well-founded. $\mathcal{I}$ is λ-complete if for any $\gamma < \kappa$ and $(A_\alpha : \alpha < \gamma) \subseteq \mathcal{I}$, $\cup_{\alpha<\gamma} A_\alpha \in \mathcal{I}$. If $\mathcal{I}$ is a λ-complete precipitous ideal on κ such that $\lambda \leq \kappa$ then the generic embedding produced by $\mathcal{I}$ has critical point $\geq \lambda$.

It is mentioned in [2] that Jech asked whether supercompact cardinals imply that the non-stationary ideal on ω_1 or on any cardinal κ is precipitous. Theorem 33 of [2] shows that this is not the case, as any normal precipitous ideal can be destroyed in a forcing extension. However, the following question remained open.

Question 0.1 *Do large cardinals imply that there exists a precipitous ideal on ω_1 or on other regular cardinals?*

It is in fact not hard to show that sufficiently nice extender models do not carry precipitous ideals on ω_1. Theorem 0.2 was independently discovered by many inner model theorists. The proof generalizes to obtain stronger results on non-existence of precipitous ideals. To the author's best knowledge, these results are unpublished and not due to the author. Because of this we will not dwell on them and will just give the prototypical argument.

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Theorem 0.2 *Suppose $\mathcal{M}$ is a countable ω_1+1-iterable mouse and κ is a successor cardinal of $\mathcal{M}$ such that $\mathcal{M} \models$ "κ^{+2} exists". Then $\mathcal{M}$ doesn't carry a κ-complete precipitous ideals.*

Proof. Towards a contradiction assume that $\mathcal{I}$ is a κ-complete precipitous ideal on κ. Let $j : \mathcal{M} \to \mathcal{N} \subseteq \mathcal{M}[g]$ be a generic embedding given by $\mathcal{I}$. We have that $(\kappa^{++})^{\mathcal{M}}$ is a cardinal of $\mathcal{M}[g]$ as $|\mathcal{I}|^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}}$. Let $\lambda = (\kappa^{++})^{\mathcal{M}}$ and η be the predecessor of κ. Because λ is a cardinal of $\mathcal{M}[g]$ and $j(\kappa) = (\eta^+)^{\mathcal{N}}$, we have that $\mathcal{M}|\lambda \ntrianglelefteq j(\mathcal{M}|\lambda)$.

Notice that in V there is a real x that codes a premouse $\mathcal{Q}$ and an elementary embedding $\pi : \mathcal{Q} \to j(\mathcal{M}|\lambda)$ such that

1. $\mathcal{M}|\eta \trianglelefteq \mathcal{Q}$,
2. $\pi \restriction (\eta + 1) = id$ and
3. $\mathcal{Q} \ntrianglelefteq j(\mathcal{M}|\lambda)$.

Take for instance a real that codes $(\mathcal{M}|\lambda, j \restriction \mathcal{M}|\lambda)$. It follows by absoluteness that there is such a real in $\mathcal{N}^{Coll(\omega, j(\lambda))}$.

Let $h \subseteq Coll(\omega, \lambda)$ be $\mathcal{M}$-generic. It follows from elementarity that there is a pair $(\mathcal{R}, \sigma) \in \mathcal{M}[h]$ such that

1. $\mathcal{M}|\eta \trianglelefteq \mathcal{R}$,
2. $\sigma : \mathcal{R} \to \mathcal{M}|\lambda$ is elementary,
3. $\sigma \restriction (\eta + 1) = id$ and
4. $\mathcal{R} \ntrianglelefteq \mathcal{M}|\lambda$.

Clause 2 above implies that $(\eta^+)^{\mathcal{R}}$ is the largest cardinal of $\mathcal{R}$. Because $\mathcal{M}$ is ω_1+1-iterable, condensation implies that $\mathcal{R} \trianglelefteq \mathcal{M}$. $\square$

Woodin showed that *strong condensation*, an axiom that he formulated, implies the non-existence of precipitous ideals on ω_1 and cardinals below the least inaccessible cardinal. The proof is very similar to the one we gave above (see [16, Definition 8.5] and [16, Corollary 8.9]). The authors of [8] say that Steel showed that in some extender models, κ carries a precipitous ideal if and only if it is measurable. The authors of [8] showed that in the minimal extender model with Woodin cardinal that is itself a limit of Woodin cardinals ω_1 does not carry precipitous ideal (see [8, Corollary 4]). The authors of [1] showed that if the extender model is a model of

$V = K$ then κ carries a precipitous ideal if and only if it is a measurable cardinal (see [1, Theorem 0.3]).

The proof of our main theorem, Theorem 0.5, uses a different type of argument that is not based on condensation. It is not clear to us how to prove Theorem 0.5 via condensation-like arguments or arguments based on the core model.

It is a well-known result of Woodin that if there is a Woodin cardinal δ then letting $\mathbb{Q}_\delta$ be the countable stationary tower forcing associated to δ (see [3]), there is $G \subseteq \mathbb{Q}_\delta$ and an embedding $j : V \to M \subseteq V[G]$ definable in $V[G]$ such that

1. $\mathrm{crit}(j) = \omega_1$ and
2. $V[G] \models M^\omega \subseteq M$.

The question on the existence of precipitous ideals on ω_1 can be interpreted in at least two ways. One, of course, is the most direct interpretation. However, it can also be perceived as a question on the existence of generic embeddings that resemble the stationary tower embedding but are produced via small forcing, smaller than the size of the least Woodin cardinal.

In this paper, we investigate this interpretation of the question.

Definition 0.3 *Suppose δ is a Woodin cardinal which is not a limit of Woodin cardinals. Let μ be the supremum of Woodin cardinals $< \delta$. We say there is a stationary-tower like embedding (st-like-embedding) below δ if there is a partial ordering $\mathbb{P}$ such that whenever $g \subseteq \mathbb{P}$ is generic,*

1. $\mu < |\mathbb{P}| < \delta$,
2. $(\mu^+)^V < \omega_1^{V[g]}$,
3. *there is an elementary embedding $j : V \to M \subseteq V[g]$ with the property that $\mathrm{crit}(j) = \omega_1$ and $\mathbb{R}^{V[g]} \subseteq M$.*

The main question we deal with in this paper is the following.

Question 0.4 *Assume there is a Woodin cardinal δ. Is there an st-like-embedding below δ?*

We will show that the answer is negative in all known sufficiently nice extender models. In particular, we show that.

Theorem 0.5 *Let $\mathcal{M}$ be the minimal mouse with a Woodin cardinal that is a limit of Woodin cardinals. Let δ be the second Woodin cardinal of $\mathcal{M}$. Then there is no st-like-embedding below δ.*

Upon seeing the results of this paper, Woodin informed us that he already knew that in extender models there is no st-embedding below the first Woodin cardinal (in fact condensation style arguments give this). He also informed us that the answer was not known for the second Woodin cardinal and beyond. We could have chosen any Woodin cardinal δ such that the least cardinal κ that is $< \delta$-strong is not a limit of Woodin cardinals. Our proof has all the main ideas, and this is not a vanity contest. Thus, we chose to work with the second Woodin cardinal.

We have not tried to prove results for overlapped Woodins, and believe that this is an interesting project. The methods of [6] are probably relevant to this project.

Our methods are methods developed by inner model theorists for the last 60 years or so. We rely heavily on the writings of Mitchell and Steel. Readers familiar with the papers [4] and [14] can see their influence on the current paper.

We started thinking about generic embeddings in extender models because of Mathew Foreman. He informed us that it is not known if large cardinals imply the existence of precipitous ideals on ω_1. We thank him for asking us this question.

Our motivation was just to show that inner model theory is a subject relevant to combinatorial set theory in a sense that a great deal of combinatorics beyond principles such as $\diamondsuit$ and $\square$ can be investigated and understood inside inner models. One only needs to try.

Nevertheless, we do agree with the view that the internal combinatorial structure of extender models have not been very extensively studied beyond [9]. However, there are several papers in print that do investigate the internal structure of mice in different ways than [9] does. For instance, [12] characterizes homogeneously Suslin sets in extender models, and [7] investigates grounds of certain types of extender models.

1 On S-reconstructible operators

Here we discuss some facts that describe the internal structure of a large class of mice. Suppose that

1. $\mathcal{M}$ is a class size mouse over some set x satisfying a sentence ϕ,
2. there is no active level $\mathcal{R} \trianglelefteq \mathcal{M}$ such that if E is the last extender of $\mathcal{R}$ then $\mathcal{R}|\text{crit}(E) \vDash \phi$ and

3. there is an active mouse $\mathcal{R}$ such that if E is the last extender of $\mathcal{R}$ then $\mathcal{R}|\text{crit}(E) \vDash \phi$.

Clause 3 above implies that $\mathcal{M}$ has a club of indiscernibles. We then say $\mathcal{M}$ is the *minimal class size* x-mouse satisfying ϕ if $\mathcal{M}$ is the hull of a club of indiscernibles. It is a well-known fact that there is a unique minimal x-mouse satisfying ϕ. This can be shown via a standard comparison argument. Just notice that if $\mathcal{M}$ and $\mathcal{N}$ are both minimal x-mice satisfying ϕ then their comparison has a club of fixed points all of which are indiscernbles.

We say $\mathbb{M} : V \to V$ is a mouse operator if for some formula ϕ,

1. $\text{dom}(\mathbb{M}) = \{x : L_\omega[x] \vDash \text{“}x \text{ is wellordered”}\} \cap \{x :$there is a minimal class size x-mouse satisfying $\phi\}$,

2. for each $x \in dom(\mathbb{M})$, $\mathbb{M}(x)$ is the minimal class size mouse satisfying ϕ.

We also say that $\mathbb{M}$ is determined by ϕ and denote it by $\mathbb{M}_\phi$. When ϕ is clear from context we drop it from our notation, and for $x \in dom(\mathbb{M})$, we let $\mathcal{M}(x) = \mathbb{M}(x)$. We say $\mathbb{M}$ is total on a set X if $\mathcal{M}(x)$ is defined for every $x \in X \cap \{x : L_\omega[x] \vDash \text{“}x$ is wellordered”$\}$.

Definition 1.1 *We say $\mathbb{M}_\phi$ is an $\mathcal{S}$-reconstructible mouse operator if*

1. *$dom(\mathbb{M}_\phi) = \{a \in HC : L_\omega[a] \vDash \text{“}a$ is well-ordered”$\}$,*

2. *for each $a \in dom(\mathbb{M}_\phi)$, $\mathcal{M}_\phi(a)$ has infinitely many Woodin cardinals the ω of which are $(\delta_{a,i} : i \in \omega)$,*

3. *for each $a \in dom(\mathbb{M}_\phi)$, for each $i \in \omega$, for each $\mathcal{M}_\phi(a)$-generic g for a poset of size $< \delta_{a,i}$, for each $x \in (\mathcal{M}_\phi(a)|\delta_{a,i})[g]$, and for each $\eta < \delta_{a,i}$ such that $x \in (\mathcal{M}_\phi(a)|\eta)[g]$, letting*

   (a) $\mathcal{P}$ be the output of the fully backgrounded construction of $(\mathcal{M}_\phi(a)|\delta_a)[g]$ done over x using extenders with critical points greater than η and

   (b) $\mathcal{N}$ be the result of an S-construction that translates $\mathcal{M}$ into an x-mouse over $\mathcal{P}$,

   $\mathcal{N} \vDash \phi$.

S constructions are standard constructions in inner model theory. They were first considered by John Steel (hence the “S”). The first known full treatment of

S constructions was presented in [10], where, for some truly unfortunate though fully understandable reasons[1], they were called P constructions where P stands for nothing in particular. The reader can also consult [5, Chapter 3.8].

Our goal is to consider two particular kinds of mice, $\mathcal{M}_\omega$ and $\mathcal{M}_{wlw}$. The first is the minimal class size mouse with ω Woodin cardinals, and the second is the minimal class size mouse with a Woodin cardinal δ that is a limit of Woodin cardinals. We will prove our theorems for S-reconstructible mice that have the *internal covering property* (see Definition 2.1). It is straightforward to check that both $\mathcal{M}_\omega$ and $\mathcal{M}_{wlw}$ satisfy our definition of S-reconstructible. Later we will show that they also satisfy the internal covering property (see Theorem 2.2).

Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator. Given $a \in dom(\mathbb{M}_\phi)$, we let $\mathcal{W}(a) = \mathcal{M}_\phi(a)|\omega_1^{\mathcal{M}_\phi(a)}$. We think of $\mathcal{W}$ as a function whose domain is $dom(\mathbb{M}_\phi)$. Given a transitive set N, let $\mathcal{W}^N = \mathcal{W} \restriction N$. The following is a corollary to our definition.

Corollary 1.2 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator. Fix $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$, and set $\mathcal{M} = \mathcal{M}_\phi(a)$ and $\delta = \delta_{a,i}$. Then $\mathcal{W}^{\mathcal{M}|\delta}$ is uniformly definable over $\mathcal{M}|\delta$. More precisely, there is a formula ψ with the property that for any poset $\mathbb{P} \in \mathcal{M}|\delta$, for any $\mathcal{M}$-generic $g \subseteq \mathbb{P}$, for any $x \in HC^{\mathcal{M}[g]}$ and for any $\mathcal{R}$,*

$$\mathcal{R} \trianglelefteq \mathcal{W}(x) \text{ if and only if } \mathcal{R} \in \mathcal{M}[g] \text{ and } \mathcal{M}[g] \vDash \psi[\mathcal{R}].$$

It is clear what ψ must be, it is just the formula defining the fully backgrounded constructions. Note that the language of $\mathcal{M}$ has a symbol for the extender sequence of $\mathcal{M}$, and so ψ may or may not mention the extender sequence of $\mathcal{M}|\delta$. Results of Schlutzenberg suggest that $\mathcal{W}$ maybe even definable over the universe of $\mathcal{M}|\delta$ (see [13]). However, we do not need such fine calculations.

Suppose $(\mathbb{P}, g, x, \mathcal{R})$ are as in the hypothesis of Corollary 1.2. Then

$\psi[\mathcal{R}]$: there is $\lambda < \delta$ such that for every $\eta \in (\lambda, \delta)$, letting $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over x using extenders with critical points $> \eta$, $\mathcal{R} \trianglelefteq \mathcal{P}$.

The next results show that $\mathcal{M}_a$ in fact knows some fragments of its own strategy. The first lemma is a useful and easy lemma. We let $\delta_{a,-1} = 0$.

Lemma 1.3 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator. Fix $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$, and set $\mathcal{M} = \mathcal{M}_\phi(a)$ and $\delta = \delta_{a,i}$. Let $\mathbb{P} \in \mathcal{M}|\delta$ and suppose $g \subseteq \mathbb{P}$ is*

[1] Notice the "S" in the last names of all the people involved in this bustiness.

$\mathcal{M}$-generic. Let $x \in \mathcal{M}|\delta[g]$ and $\lambda \in (\max(\delta_{i-1}, |\mathbb{P}|^{\mathcal{M}}), \delta)$ be such that $x \in \mathcal{M}|\lambda[g]$. Let $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over x using extenders with critical points $> \lambda$. Then $\mathcal{P} \vDash$ "there are no Woodin cardinals".

Proof. Towards a contradiction, assume not. Let η be a Woodin cardinal of $\mathcal{P}$. Then $\mathcal{M}|\eta$ is generic over $\mathcal{P}$ for the extender algebra at η that uses η-generators. We claim that $\mathcal{P}[\mathcal{M}|\eta] \vDash$ "η is a Woodin cardinal".

To see this, let $f : \eta \to \eta$ be a function in $\mathcal{P}[\mathcal{M}|\eta]$. Because the forcing that adds $\mathcal{M}|\eta$ has the η-cc, there is $h : \eta \to \eta$ in $\mathcal{P}$ such that for every $\alpha < \eta$, $f(\alpha) < h(\alpha)$. Let $E \in \vec{E}^{\mathcal{P}}$ be an extender such that $\nu =_{def} \nu_E$ is a $\mathcal{P}$-cardinal such that letting $\kappa = \mathrm{crit}(E)$,

$$\pi_E^{\mathcal{P}}(h)(\kappa) < \nu.$$

Let $F \in \mathcal{M}$ be the resurrection of E. Let $\mathcal{S}$ be the model appearing in the construction producing $\mathcal{P}$ such that F is added to $\mathcal{S}$. We have that no further model appearing in the construction projects below ν (as $\mathcal{P}|\nu$ is an initial segment of the final model of the construction). It follows that the canonical factor map $k : Ult(\mathcal{P}, E) \to \pi_F^{\mathcal{M}}(\mathcal{P})$ has a critical point $\geq \nu$. Hence, $k(\pi_E^{\mathcal{P}}(h)(\kappa)) = \pi_E^{\mathcal{P}}(h)(\kappa)$. It follows that

$$\pi_F^{\mathcal{M}}(f)(\kappa) < \nu \leq \nu_F.$$

If $F \notin \vec{E}^{\mathcal{M}|\eta}$ then for some $\mathcal{M}$-inaccessible $\xi \in (\nu, \eta)$, $F \restriction \xi \in \vec{E}^{\mathcal{M}|\eta}$. It follows that $F \restriction \xi$ witnesses Woodiness for f in $\mathcal{P}[\mathcal{M}|\eta]$. □

Our first proposition shows that some fragments of the iteration strategy are universally Baire inside the mouse operators.

Proposition 1.4 *Suppose $\mathbb{M}_\phi$ is an $\mathcal{S}$-reconstructible mouse operator and $a \in dom(\mathbb{M}_\phi)$. Fix $i \in \omega$ and set $\delta = \delta_{a,i}$ and $\mathcal{M} = \mathcal{M}_a$. Let Σ be the unique iteration strategy of $\mathcal{M}$. Suppose $\kappa \in (\delta_{a,i-1}, \delta)$ and Λ is the fragment of Σ that acts on non-dropping trees on $\mathcal{M}|\kappa$ that are above $\delta_{a,i-1}$. Then $\Lambda \restriction (\mathcal{M}|\delta) \in \mathcal{M}$ and whenever $g \subseteq Coll(\omega, \kappa)$ is $\mathcal{M}$-generic, $\Lambda \restriction HC^{\mathcal{M}[g]} \in \mathcal{M}[g]$ and*

$$\mathcal{M}[g] \vDash \text{"}\Lambda \restriction HC^{\mathcal{M}[g]} \text{ is } \delta\text{-}uB\text{"}.$$

Furthermore, whenever h is $\mathcal{M}[g]$-generic for a poset of size $< \delta$, $\Lambda \restriction HC^{\mathcal{M}[g*h]}$ is the canonical extension of $\Lambda \restriction HC^{\mathcal{M}[g]}$.

Proof. The representative case is when $i = 0$. When $i > 0$ we need to work over $\mathcal{M}|\delta_{a,i-1}$. Here we assume $i = 0$.

The fact that $\Lambda \restriction HC^{\mathcal{M}[g]} \in \mathcal{M}[g]$ follows from Corollary 1.2. Indeed, notice that given a tree $\mathcal{T}$ on $\mathcal{M}|\kappa$ of limit length and according to Λ, $\Lambda(\mathcal{T})$ is the unique branch b such that $\mathcal{Q}(\mathcal{T}, b)$ exists and $\mathcal{Q}(\mathcal{T}, b) \trianglelefteq \mathcal{W}(\mathcal{M}(\mathcal{T}))$. Thus, to define Λ in generic extensions of $\mathcal{M}$, it is enough to know that the function $\mathcal{T} \to \mathcal{W}(\mathcal{M}(\mathcal{T}))$ is definable on the domain of Λ. This follows from Corollary 1.2. For the rest of the argument we assume that κ is a regular cardinal of $\mathcal{M}$. This assumption doesn't cause loss of generality, since if κ is singular then the conclusion of the proposition can be reached by using the conclusion of the proposition for $(\kappa^+)^{\mathcal{M}}$.

Next we show that Λ is δ-uB in $\mathcal{M}[g]$. Our generically absolute definition of Λ will also show the "furthermore" clause of the proposition. Let $\lambda \in (\kappa, \delta)$ be a cardinal, and let $\vec{C} = (\mathcal{S}_\xi, \mathcal{R}_\xi, F_\xi : \xi < \delta)$ be the models of fully backgrounded construction of $\mathcal{M}|\delta$ (or $\mathcal{M}|\delta[g]$) done over a in which extenders used have critical points $> \lambda$. We claim that

Claim 1. for some ξ, $\mathcal{R}_\xi$ is an iterate of $\mathcal{M}|\kappa$ via a tree $\mathcal{W}$ such that $\pi^{\mathcal{W}}$ exists.

Proof. Notice that as $\mathcal{M}|\kappa$ has no Woodin cardinals, if there was such a tree $\mathcal{W}$ then $\mathcal{W} \in \mathcal{M}$. Now towards a contradiction assume our claim is false. We now compare $\mathcal{M}|\kappa$ with the construction $(\mathcal{S}_\xi, \mathcal{R}_\xi, F_\xi : \xi < \delta)$[2]. We use $\Sigma_{\mathcal{M}|\kappa}$ on the $\mathcal{M}|\kappa$-side and Σ on the $\mathcal{M}$-side. The comparison produces a tree $\mathcal{T}$ on $\mathcal{M}|\kappa$ according to Λ with last model $\mathcal{N}$ and a non-dropping tree $\mathcal{U}$ on $\mathcal{M}$ according to Σ with last model $\mathcal{M}_1$ such that $\pi^{\mathcal{U}}(\mathcal{R}_\delta) \trianglelefteq \mathcal{N}$.

Indeed, if $\mathcal{M}|\kappa$-side lost then the comparison would have stopped before reaching stage $\pi^{\mathcal{U}}(\delta)$, and so there would be some ξ such that the second model of $\pi^{\mathcal{U}}(\vec{C})(\xi)$ was an iterate of $\mathcal{M}|\kappa$. This fact would be witnessed inside $\mathcal{M}_1$, and hence by elementarity our claim would be true in $\mathcal{M}$.

Because $\mathcal{M}|\kappa$ has no Woodin cardinals, it must be that $\mathcal{T}$ drops and $rud(\mathcal{N}) \vDash$ "$\pi^{\mathcal{U}}(\delta)$ is not a Woodin cardinal". Because all initial segments of $\mathcal{M}|\kappa$ are ϕ-small, we have that if $\mathcal{R}$ is the result of S-construction that translates $\mathcal{M}_1$ into a mouse over $\pi^{\mathcal{U}}(\mathcal{R}_\delta)$ then $\mathcal{N} \trianglelefteq \mathcal{R}$. However, because $\pi^{\mathcal{U}}(\delta)$ is a Woodin cardinal of $\mathcal{M}_1$, we have that $\mathcal{R} \vDash$ "$\pi^{\mathcal{U}}(\delta)$ is a Woodin cardinal", contradiction. □

We now use branch condensation of Λ to get a generically absolute definition of Λ. Let $g \subseteq Coll(\omega, \kappa)$ be $\mathcal{M}$-generic. For each $\lambda \in (\kappa, \delta)$, let $\vec{C}_\lambda = (\mathcal{S}_\xi, \mathcal{R}_\xi, F_\xi : \xi < \delta)$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over a in which

[2] Such comparison arguments were studied in [11].

extenders used have critical points $> \lambda$. Let ξ_λ be such that $\mathcal{R}_{\xi_\lambda}$ is an iterate of $\mathcal{M}|\kappa$. This iteration must be according to $\Sigma_{\mathcal{M}|\kappa}$. Let $\pi_\lambda : \mathcal{M}|\kappa \to \mathcal{R}_{\xi_\lambda}$.

Suppose now h is any $\mathcal{M}[g]$-generic for a poset of size $< \delta$. Then given a non-dropping tree $\mathcal{T} \in \mathcal{M}|\delta[g][h]$ on $\mathcal{M}|\kappa$ we say $\mathcal{T}$ is *correct* if for all limit $\alpha < lh(\mathcal{T})$, for some η such that $\mathcal{T} \in \mathcal{M}|\eta[g][h]$ for all $\lambda \in (\eta, \delta)$, there is an embedding $\sigma : \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{R}_{\xi_\lambda}$ such that

$$\pi_\lambda = \sigma \circ \pi^{\mathcal{T}}_{0,\alpha}.$$

Given a correct tree $\mathcal{T} \in \mathcal{M}[g * h]$, we let $\phi[\mathcal{T}, b, \mathcal{Q}]$ be the statement that for some $\eta < \delta$ for all $\lambda \in (\eta, \delta)$

1. b is a cofinal well-founded branch of $\mathcal{T}$ such that $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ and
2. there is an embedding $\sigma : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{R}_{\xi_\lambda}$ such that $\pi_\lambda = \sigma \circ \pi^{\mathcal{T}}_b$.

Let $\psi[\mathcal{T}, b, \mathcal{Q}]$ be the statement that $\mathcal{T}$ is correct and $\phi[\mathcal{T}, b, \mathcal{Q}]$ holds. Notice that

(1) in $\mathcal{M}[g]$, whenever $\mathbb{P}$ is a poset of size $< \delta$, $\mathbb{P}$ forces that for any correct tree $\mathcal{T}$ there is $b, \mathcal{Q}$ such that $\phi[\mathcal{T}, b, \mathcal{Q}]$.

The branch condensation of Λ implies that such a pair $(b, \mathcal{Q})$ must be unique. We then get that ψ is a generically correct definition of Λ.

Claim 2. For a club of countable $X \prec \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$, letting $\pi_X : \mathcal{N}_X \to \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$ be the transitive collapse of X, and letting $h \in \mathcal{M}[g]$ be $\mathcal{N}_X$-generic for a poset of size $< \pi_X^{-1}(\delta)$, for any $(\mathcal{T}, b, \mathcal{Q}) \in \mathcal{N}_X[h]$,

$$\mathcal{N}_X[h] \vDash \psi[\mathcal{T}, b, \mathcal{Q}] \text{ if and only } \mathcal{M}[g] \vDash \psi[\mathcal{T}, b, \mathcal{Q}].$$

Proof. Left to right direction is easy and we leave it to the reader. For the other direction, suppose that $(\mathcal{T}, b, \mathcal{Q}) \in \mathcal{N}_X[h]$ and $\mathcal{M}[g] \vDash \psi[\mathcal{T}, b, \mathcal{Q}]$. First we claim that $\mathcal{N}_X[h] \vDash$ "$\mathcal{T}$ is correct". Suppose otherwise. Then there is a limit $\alpha < lh(\mathcal{T})$ such that $\mathcal{N}_X[h] \vDash$ "$\mathcal{T} \restriction \alpha$ is correct and $\mathcal{T} \restriction \alpha + 1$ is not correct". It follows from (1) that there is $c, \mathcal{Q} \in \mathcal{N}_X$ such that c is not the branch of $\mathcal{T} \restriction \alpha$ in $\mathcal{T}$ and $\mathcal{N}_X \vDash \psi[\mathcal{T} \restriction \alpha, c, \mathcal{Q}]$. It follows that $\mathcal{M}[g] \vDash \psi[\mathcal{T} \restriction \alpha, c, \mathcal{Q}]$ implying that c is the branch of $\mathcal{T} \restriction \alpha$ in $\mathcal{T}$. A similar argument shows that in fact $\mathcal{N}_X \vDash \psi[\mathcal{T}, b, \mathcal{Q}]$. □

□

We now show that countable submodels also have universally Baire strategies.

Proposition 1.5 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible operator, $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$. Set $\mathcal{M} =_{def} \mathcal{M}_\phi(a)$ and $\delta = \delta_{a,i}$. Let $\pi : \mathcal{N} \to \mathcal{M}|(\delta^+)^{\mathcal{M}}$ be a countable hull inside $\mathcal{M}$. Then $\mathcal{M} \vDash$ "$\mathcal{N}$ has a δ-uB iteration strategy that acts on trees above $\pi^{-1}(\delta_{a,i-1})$".*

Proof. Again, we only do the proof of the representative case $i = 0$. Let Σ be the unique iteration strategy of $\mathcal{M}$, and let $(\mathcal{S}_\xi, \mathcal{P}_\xi, F_\xi : \xi < \delta)$ be the models of the fully backgrounded constructions of $\mathcal{M}|\delta$ over a. We claim that for some $\xi < \delta$ there is an embedding $\sigma : \mathcal{N} \to \mathcal{P}_\xi$. To build such an embedding, we compare $\mathcal{N}$ with the aforementioned construction of $\mathcal{M}|\delta$. We use the π-pullback of Σ to iterate $\mathcal{N}$. We claim that the construction side wins the comparisons.

To see this, assume not. We then get a tree $\mathcal{T}$ on $\mathcal{N}$ and a tree $\mathcal{U}$ on $\mathcal{M}|\delta$ with last models $\mathcal{N}_1$ and $\mathcal{M}_1$ respectively such that $\pi^{\mathcal{U}}$ exists and $\pi^{\mathcal{U}}(\mathcal{P}_\delta) \trianglelefteq \mathcal{N}_1$. As there are no Woodin cardinals in $\mathcal{P}_\delta$ (see Lemma 1.3), $(\mathcal{T} \restriction lh(\mathcal{T}) - 1) \in \mathcal{M}_1$. It follows that there is a tree $\mathcal{W} \in \mathcal{M}$ on $\mathcal{N}$ such that $\mathcal{M}(\mathcal{W}) = \mathcal{P}_\delta$. It follows that $\mathcal{M}|(\omega_1)^{\mathcal{M}} \trianglelefteq \mathcal{N}$, contradicting the fact that $\mathcal{N}$ is countable in $\mathcal{M}$. This contradiction shows that there is $\sigma : \mathcal{N} \to \mathcal{R}_\xi$ for some $\xi < \delta$. The rest follows from Proposition 1.4. It is not hard to show that the σ-pullback of the strategy of $\mathcal{R}_\xi$ induced by Σ is δ-uB in $\mathcal{M}$. □

We state, without a proof, a somewhat stronger version of Proposition 1.5.

Proposition 1.6 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator, $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$. Set $\delta = \delta_{a,i}$ and $\mathcal{M} = \mathcal{M}_{a,i}$. Let Σ be the unique iteration strategy of $\mathcal{M}$. Suppose g is $\mathcal{M}$-generic for a poset of size $< \delta$, and let $\pi : \mathcal{N}[\bar{g}] \to \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$ be a countable hull in $\mathcal{M}[g]$. Then $\mathcal{M}[g] \vDash$ "$\mathcal{N}$ has a δ-uB iteration strategy acting on trees that are above $\pi^{-1}(\delta_{a,i-1})$".*

The next lemma shows that for any x, proper initial segments of $\mathcal{W}(x)$ have universally Baire iterations strategies. However, the function $x \to \mathcal{W}(x)$ cannot be universally Baire. For this we need to collapse the first strong cardinal of $\mathcal{M}_a$. The reason is that $\mathcal{W}(x)$ is the set of all OD subsets of x in the derived model of $\mathcal{M}_a$ computed at $\delta_{a,\omega}$, and this derived model is a model in which all sets are ordinal definable from a real. For more on this we refer the reader to [14].

Proposition 1.7 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator. Let $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$. Set $\mathcal{M} = \mathcal{M}_\phi(a)$ and $\delta = \delta_{a,i}$. Let g be $\mathcal{M}$-generic for a poset of size $< \delta$, $x \in \mathcal{M}|\delta[g] \cap dom(\mathcal{W})$ and $\mathcal{Q} \trianglelefteq \mathcal{W}(x)$ be such that $\rho(\mathcal{Q}) = \omega$. Let Λ be the unique strategy of $\mathcal{Q}$. Then $\Lambda \restriction HC^{\mathcal{M}[g]} \in \mathcal{M}|\delta[g[$ and is δ-uB in $\mathcal{M}$ in the stronger sense that for any $\mathcal{M}[g]$-generic h, $\Lambda \restriction HC^{\mathcal{M}[g*h]}$ is the canonical extension of $\Lambda \restriction HC^{\mathcal{M}[g]}$.*

Proof. We again do the proof in the representative case of $i = 0$. To prove the claim fix $g, x, \mathcal{Q}$ as in the statement of the proposition. Let $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over x using extenders with critical points $> \lambda$ where λ is some cardinal $< \delta$ bigger than the size of the poset. We have that $\mathcal{Q} \trianglelefteq \mathcal{P}$. Thus, again, the iterability of $\mathcal{Q}$ reduces to the iterability of some $\mathcal{M}|\kappa$ where $\kappa > \lambda$ is a regular cardinal of $\mathcal{M}$. The rest of the claim follows from Proposition 1.4 and Proposition 1.5. □

2 The Internal Covering Property

We will need to deal with S-reconstructible operators with a stronger property. Recall $\mathcal{W}(x)$ function given by $\mathcal{W}(x) = \mathcal{M}_x|\omega_1^{\mathcal{M}_x}$.

Definition 2.1 *Suppose $\mathbb{M}_\phi$ is an S-reconstructible mouse operator. We say $\mathbb{M}_\phi$ has the internal covering property if for any $a \in dom(\mathbb{M}_\phi)$ and $i \in \omega$, letting $\mathcal{M} = \mathcal{M}_a$ and $\delta = \delta_{a,i}$, for any $\mathbb{P} \in \mathcal{M}|\delta$, $\mathcal{M}$-generic $g \subseteq \mathbb{P}$, $x \in \mathcal{M}|\delta[g]$, and $\lambda \in (\delta_{a,i-1}, \delta)$ such that*

1. *$a \in L_\omega[x]$,*
2. *$L_\omega[x] \vDash$ "x is well-ordered",*
3. *$x \in \mathcal{M}|\lambda[g]$,*
4. *$\mathcal{M}|\delta_{a,i-1}$ is generic over $\mathcal{W}(x)$,*

letting $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over the tuple (x, a) using extenders with critical points greater than λ, for unboundedly many $\kappa < \delta$, $(\kappa^+)^{\mathcal{P}} = (\kappa^+)^{\mathcal{M}}$.

Let $\mathbb{M}$ be either $x \to \mathcal{M}_\omega(x)$ or $x \to \mathcal{M}_{wlw}(x)$. Both of these operators are S-reconstructible. Here we show that they also have the internal covering property.

Theorem 2.2 $\mathbb{M}$ *has the internal covering property.*

Proof. We show that $\mathcal{M} =_{def} \mathcal{M}(\emptyset)$ satisfies the internal covering property. Here the representative case is $i = 1$, so we assume $i = 1$. Let $\delta_0 = \delta_{\emptyset,0}$ and $\delta = \delta_{\emptyset,1}$.

Let ξ be the sup of the Woodin cardinals of $\mathcal{M}$ and $g \subseteq Coll(\omega, < \xi)$ be generic over $\mathcal{M}$. Let W be the derived model of $\mathcal{M}$ as computed in $\mathcal{M}[g]$. More precisely,

$W = L(\Gamma, \mathbb{R}^*)$ where $\mathbb{R}^* = \cup_{\kappa<\xi}\mathbb{R}^{\mathcal{M}[g\cap Coll(\omega,<\kappa)]}$ and Γ is the collection of all those sets of reals A of $\mathcal{M}(\mathbb{R}^*)$ such that $L(A, \mathbb{R}^*) \vDash AD^+$. Woodin's celebrated *derived model theorem* says that $L(\Gamma, \mathbb{R}^*) \vDash AD^+$ and in $\mathcal{M}(\mathbb{R}^*)$, $\wp(\mathbb{R}^*) \cap W = \Gamma$. In the case of $\mathcal{M} = \mathcal{M}_\omega$, W is just $L(\mathbb{R}^*)$.

Working in W, let ν be the supremum of OD^W prewellorderings of $\mathbb{R}$. Below we collect some facts that can be proved using HOD-analysis done inside W. The reader should consult [15]. Giving the complete proofs of these facts is beyond this paper. Let $\mathcal{H} = \mathrm{HOD}^W_{\mathcal{M}|\delta_0}$ and let Σ be the unique iteration strategy of $\mathcal{M}$.

1. $V^{\mathcal{H}}_\nu$ can be represented[3] as a Σ-iterate of $\mathcal{M}|\delta$ via an iteration that is above δ_0.

2. $\mathcal{H} \vDash$ "ν is a Woodin cardinal".

3. $\mathcal{H} \vDash$ "$\mathcal{H}|\nu$ is $\nu+1$-iterable for trees that are in $L[\mathcal{H}|\nu]$".

4. Let $\mathcal{S}$ be the iterate of $\mathcal{M}$ such that $\mathcal{H}|\nu \trianglelefteq \mathcal{S}$ and if $i : \mathcal{M} \to \mathcal{S}$ is the iteration embedding then the generators of i are contained inside ν. Then the aforementioned strategy of $\mathcal{H}|\nu$ is the relevant fragment of $\Sigma_{\mathcal{S}}$.

Let now $\mathcal{S}$ be as in clause 4 above. We want to prove now that $\mathcal{S}$ satisfies the internal covering. We have that $\mathcal{S}|\nu = \mathcal{H}|\nu$.

Fix $\lambda \in (\delta_0, \nu)$ and let $\mathbb{P} \in \mathcal{S}|\lambda$ be a poset. Let $g \subseteq \mathbb{P}$ be $\mathcal{S}$-generic and $x \in \mathcal{M}|\lambda[g]$ be such that $\mathcal{M}|\delta_0$ is generic over x and $L_\omega[x] \vDash$ "x is well-ordered". Let $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over x using extenders with critical points greater than λ. Let $\mathcal{W}$ be the output of the fully backgrounded construction of $\mathcal{P}[\mathcal{M}|\delta_0]$ done over x in which extenders used have critical points $> \lambda$. It is enough to show that in $\mathcal{S}$, $\mathcal{W}$ computes unboundedly many successors correctly.

We now compare $\mathcal{H}|\nu$ with $\mathcal{W}$. On $\mathcal{H}|\nu$ side we use the $\nu+1$-strategy in $\mathcal{H}$ that acts on tree is $L[\mathcal{H}|\nu]$. Let Λ be this strategy (which is a fragment of $\Sigma_{\mathcal{S}|\nu}$). Notice that Λ induces a strategy for $\mathcal{W}$. Let then Ψ be the strategy of $\mathcal{W}$ induced by Λ. Both Λ and Ψ act on trees of length $\leq \nu$ that are in $L[\mathcal{H}|\nu]$.

The aforementioned comparison process lasts at most $\nu + 1$ steps. Suppose first that the comparison process stops in $< \nu$-steps. Let $\mathcal{T}$ and $\mathcal{U}$ be the trees on $\mathcal{H}|\nu$ and $\mathcal{W}$ respectively with last models $\mathcal{H}_1$ and $\mathcal{W}_1$ respectively. We must have that both $\pi^{\mathcal{T}}$ and $\pi^{\mathcal{U}}$ exist. It follows that there is a club of ξ such that $\pi^{\mathcal{T}}(\xi) = \xi = \pi^{\mathcal{U}}(\xi)$. For any such ξ we have that

[3]We say "represented" rather than "is" because $\mathcal{M}$ is a structure in a different language. In particular, $\mathcal{M}|\delta$ has the extender sequence of $\mathcal{M}|\delta$ as a predicate.

$$(\xi^+)^{\mathcal{H}} = (\xi^+)^{\mathcal{H}_1} = (\xi^+)^{\mathcal{W}_1} = (\xi^+)^{\mathcal{W}},$$

which is what we wanted to show.

Suppose next that the comparison process lasts ν-steps. Let $\mathcal{T}$ and $\mathcal{U}$ be the trees on $\mathcal{H}|\nu$ and $\mathcal{W}$ respectively. In order to apply Λ and Ψ we must first show that both $\mathcal{T},\mathcal{U} \in L[\mathcal{H}|\nu]$. Notice that for any limit $\alpha < \nu$, the branch chosen by $\mathcal{T}$ and $\mathcal{U}$ at stage α is determined by the corresponding $\mathcal{Q}$-structures. More precisely, if $b = \Lambda(\mathcal{T} \restriction \alpha)$ and $c = \Psi(\mathcal{U} \restriction \alpha)$ then both $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ and $\mathcal{Q}(c, \mathcal{U} \restriction \alpha)$ exist and are equal to respectively $\mathcal{Q}(\mathcal{T} \restriction \alpha)$ and $\mathcal{Q}(\mathcal{U} \restriction \alpha)$. However, since $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{U})$, we must have that $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\mathcal{U})$. Thus, the comparison process is definable over $\mathcal{H}|\nu$. Indeed, the branches chosen are the ones with $< \nu$-iterable $\mathcal{Q}$-structures (for instance see Corollary 1.4). It follows that indeed $\mathcal{T},\mathcal{U} \in L[\mathcal{H}|\nu]$. Let $b = \Lambda(\mathcal{T})$ and $c = \Psi(\mathcal{U})$.

Suppose first that $\mathcal{M}_b^{\mathcal{T}} \neq \mathcal{M}_c^{\mathcal{U}}$. Because ν is inaccessible in $\mathcal{H}$, we have that either $\mathcal{M}_b^{\mathcal{T}} \trianglelefteq \mathcal{M}_c^{\mathcal{U}}$ or $\mathcal{M}_c^{\mathcal{U}} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$[4]. Because both cases are symmetric let us deal with the case $\mathcal{M}_c^{\mathcal{U}} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ and leave the other case (which actually is easier as $\mathcal{T}$ is a tree on the universe itself) to the reader. Because $\mathcal{M}_b^{\mathcal{T}} \neq \mathcal{M}_c^{\mathcal{U}}$, we must have that $\mathcal{R} =_{def} \mathcal{Q}(\mathcal{U})$ exists and $\mathcal{Q}(\mathcal{U}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

Let $\mathcal{N}$ be the result of S-construction that translates $\mathcal{S}$ into a mouse over $\mathcal{W}$. We have that $\mathcal{N} \vDash \phi$ where ϕ is the formula that defines $\mathbb{M}$. Because $\mathcal{U}$ and c are according to the strategy induced by the strategy of $\mathcal{S}$, we have that $\mathcal{U}$ and c can be applied to $\mathcal{N}$. Indeed, letting $\mathcal{W}_1 = \mathcal{M}_c^{\mathcal{U}}$ where we apply $\mathcal{U}$ and c to $\mathcal{W}$ and $\mathcal{N}_1 = \mathcal{M}_c^{\mathcal{U}}$ where we apply $\mathcal{U}$ and c to $\mathcal{N}$, we have and embedding $k : \mathcal{W}_1 \to \pi_b^{\mathcal{T}}(\mathcal{W})$ such that

$$\pi_b^{\mathcal{T}} \restriction \mathcal{W} = k \circ \pi_c^{\mathcal{U}} \text{ and } k \text{ extends to } k^+ : \mathcal{N}_1 \to \pi_b^{\mathcal{T}}(\mathcal{N}).$$

We now have that $\mathcal{N}_1 \vDash \phi$ and $\mathcal{R}$ is a sound mouse over $\mathcal{W}_1$ such that $\rho(\mathcal{R}) = \nu$ and none of its active levels satisfy ϕ. It follows that $\mathcal{R} \trianglelefteq \mathcal{N}_1$. Therefore, $\mathcal{N}_1 \vDash$ "ν is not a Woodin cardinal". It follows that $\mathcal{N} \vDash$ "ν is not a Woodin cardinal", contradiction (see for instance the proof of Lemma 1.3 where it is shown that η remains Woodin inside S-construction).

We now have that $\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}_c^{\mathcal{U}}$. It follows that $\pi_b^{\mathcal{T}}(\nu) = \pi_c^{\mathcal{U}}(\nu)$. We then have that there is, in $\mathcal{H}$, an ω-club of $\xi < \nu$ such that $\pi_b^{\mathcal{T}}(\xi) = \xi$. However, considering the above embedding k, we also have that $\pi_c^{\mathcal{U}}(\xi) = \xi$. We then again have that

$$(\xi^+)^{\mathcal{H}} = (\xi^+)^{\mathcal{M}_b^{\mathcal{T}}} = (\xi^+)^{\mathcal{M}_c^{\mathcal{U}}} = (\xi^+)^{\mathcal{W}},$$

which is what we wanted to prove. □

The above proof shows that in fact covering holds on a stationary set.

[4]This follows from the usual comparison argument for weasels.

3 No towers resembling the stationary tower

Theorem 3.1 *Suppose* $\mathbb{M}_\phi$ *is an* S*-reconstructible mouse operator with the internal covering property. Let* $\alpha \in dom(\mathbb{M})$ *and set* $\mathcal{M} = \mathcal{M}_a$ *and* $\delta = \delta_{a,1}$*. Then there is no st-like-embedding below* δ*.*

Proof. Towards a contradiction suppose $\mathbb{P} \in \mathcal{M}|\delta$ is such that $\delta_{a,0} < |\mathbb{P}| < \delta$ and whenever $g \subseteq \mathbb{P}$ is generic, there is an elementary embedding $j : \mathcal{M} \to \mathcal{N} \subseteq \mathcal{M}[g]$ in $\mathcal{M}[g]$ with the property that $\mathrm{crit}(j) = \omega_1$ and $\mathbb{R}^{\mathcal{M}[g]} \subseteq \mathcal{N}$. Fix such a tuple $(g, \mathcal{N}, j)$. Let $\kappa = \omega_1^{\mathcal{M}}$ and let $\lambda = \omega_1^{\mathcal{M}[g]}$. Let Σ be the unique iteration strategy of $\mathcal{M}$. Let $\mathcal{W}(x) = \mathcal{M}_x|\omega_1^{\mathcal{M}_x}$.

Let $\mathcal{R}^* \trianglelefteq \mathcal{N}$ be the least such that $\rho(\mathcal{R}^*) = \omega$ and $\mathcal{M}|\kappa \triangleleft \mathcal{R}^*$. Let Φ be the δ-strategy of $\mathcal{R}^*$ in $\mathcal{N}$. Corollary 1.2 implies that $\mathcal{W} \restriction \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$. It then makes sense to write $\mathcal{W}^{\mathcal{N}}$ for the function given by the same definition over $\mathcal{N}|\delta$.

Claim 1. Suppose $x \in HC^{\mathcal{M}|\delta[g]}$ is a transitive set such that $L_\omega[x] \vDash$ "x is well-ordered". Then

$$\mathcal{W}(x) \trianglelefteq \mathcal{W}^{\mathcal{N}}(x)$$

Proof. Let $\mathcal{W}^* \trianglelefteq \mathcal{W}(x)$ be such that $\rho(\mathcal{W}) = Ord \cap x$. The proof of Proposition 1.7 shows that for some $\kappa \in (|\mathbb{P}|, \delta)$, $\mathcal{W}^*$ appear as a model of the fully backgrounded construction of $\mathcal{M}|(\kappa^+)^{\mathcal{M}}$ done using extenders with critical points $> |\mathbb{P}|$. Working in $\mathcal{M}[g]$, let $\pi : \bar{\mathcal{M}}[\bar{g}] \to (\mathcal{M}|(\delta^+)^{\mathcal{M}}[g])$ be such that $x \in \bar{\mathcal{M}}[\bar{g}]$ and $\kappa \in rng(\pi)$. Let $\bar{\kappa} = \pi^{-1}(\kappa)$. We then have that the iterability of $\mathcal{W}^*$ reduces to the iterability $\bar{\mathcal{M}}|(\bar{\kappa}^+)^{\bar{\mathcal{M}}}$ for non-dropping trees that are above $|\pi^{-1}(\mathbb{P})|^{\bar{\mathcal{M}}}$. By absoluteness we have $\sigma : \bar{\mathcal{M}} \to j(\mathcal{M}|(\kappa^+)^{\mathcal{M}})$ in $\mathcal{N}$. It follows from Proposition 1.4 that $j(\mathcal{M}|(\kappa^+)^{\mathcal{M}})$ is δ-iterable in $\mathcal{N}$, and hence $\mathcal{W}^*$ is δ-iterable in $\mathcal{N}$. Therefore, $\mathcal{W}^* \trianglelefteq \mathcal{W}^{\mathcal{N}}(x)$. □

The proof of Claim 1 is a prototypical argument that we will use again below.

Claim 2. There is a premouse $\mathcal{X} \in HC^{\mathcal{M}[g]}$ such that

1. $\mathcal{M}|(\delta_{a,0}^+)^{\mathcal{M}}$ is generic over $\mathcal{X}$,
2. $\mathcal{X}$ is Σ-iterate of $\mathcal{M}|(\delta_{a,0}^+)^{\mathcal{M}}$ such that the iteration embedding $k : \mathcal{M}|(\delta_{a,0}^+)^{\mathcal{M}} \to \mathcal{X}$ exists,
3. there is a sound $\mathcal{X}$-premouse $\mathcal{R} \in \mathcal{M}[g]$ such that $\rho(\mathcal{R}) = Ord \cap \mathcal{X}$ and $\mathcal{M}[g] \vDash$ "$\mathcal{R}$ is not δ-iterable above $Ord \cap \mathcal{X}$",

4. $\mathcal{R} \vDash$ "$k(\delta_{a,0})$ is a Woodin cardinal",

5. $rud(\mathcal{R}) \vDash$ "$(k(\delta_{a,0})^+)^{\mathcal{X}}$ is not a cardinal" and

6. $\mathcal{X}$ and $\mathcal{R}$ are countable in $\mathcal{M}[g]$.

Proof. Let $\mathcal{Y} = \mathcal{M}|(\delta_{a,0}^+)^{\mathcal{M}}$. Working inside $\mathcal{M}[g]$, we compare $\mathcal{Y}$ with $\mathcal{R}^*$. $\mathcal{Y}$ is not δ-iterable inside $\mathcal{M}[g]$. However, it follows from Corollary 1.2 that the fragment of $\Sigma_{\mathcal{Y}} \restriction (\mathcal{M}[g])$ that acts on *short trees*, i.e. trees $\mathcal{T}$ for which $\mathcal{Q}(\mathcal{T})$ exists, is in $\mathcal{M}[g]$. We then want to use the aforementioned fragment of $\Sigma_{\mathcal{Y}}$ for the comparison that we would like to perform. Finally, we would like to incorporate $\mathcal{Y}$-genericity iteration into the above mentioned comparison. More precisely, working inside $\mathcal{M}[g]$ we first iterate the least $\mathcal{Y}$-measurable cardinal $\delta_{a,0}+1$-times and get $\mathcal{Y}_1$ and then construct iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{Y}_1$ and $\mathcal{R}^*$ respectively such that

1. $\mathcal{T}$ is according to the short fragment of $\Sigma_{\mathcal{Y}} \restriction (\mathcal{M}[g])$,

2. $\mathcal{U}$ is according to Φ (recall that our hypothesis is that $\mathbb{R}^{\mathcal{M}[g]} = \mathbb{R}^{\mathcal{N}}$),

3. for $\alpha < \lambda$, given $\mathcal{T} \restriction \alpha+1$ and $\mathcal{U} \restriction \alpha+1$ we let

   (a) $E^{\mathcal{T}}_{\alpha,0}$ be the least extender, if it exists, on the sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ that violates an identity in the relevant extender algebra,

   (b) $E^{\mathcal{T}}_{\alpha,1}$ be the least extender, if it exists, that causes disagreement between $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{M}^{\mathcal{U}}_\alpha$,

4. $E^{\mathcal{T}}_\alpha$ is defined if either $E^{\mathcal{T}}_{\alpha,0}$ or $E^{\mathcal{T}}_{\alpha,1}$ is defined and

$$E^{\mathcal{T}}_\alpha = \begin{cases} E^{\mathcal{T}}_{\alpha,0} & : lh(E^{\mathcal{T}}_{\alpha,0}) \leq lh(E^{\mathcal{T}}_{\alpha,1}) \\ E^{\mathcal{T}}_{\alpha,1} & : lh(E^{\mathcal{T}}_{\alpha,1}) \leq lh(E^{\mathcal{T}}_{\alpha,0}) \end{cases}$$

5. $E^{\mathcal{U}}_\alpha$ is defined if $E^{\mathcal{T}}_\alpha = E^{\mathcal{T}}_{\alpha,1}$ in which case $E^{\mathcal{U}}_\alpha = \vec{E}^{\mathcal{M}^{\mathcal{U}}_\alpha}(lh(E^{\mathcal{T}}_\alpha))$.

Because $\mathcal{R}^* \ntrianglelefteq \mathcal{M}$ we must have that $\mathcal{R}^*$-side must win any successful coiteration with $\mathcal{Y}$. Notice that Claim 1 implies that the construction of $(\mathcal{T},\mathcal{U})$ can be carried out inside $\mathcal{N}$. It follows that our construction of $\mathcal{T}$ and $\mathcal{U}$ can last at most $\omega_1^{\mathcal{N}}$ many steps producing trees $\mathcal{T}$ and $\mathcal{U}$. If now $\mathcal{U}$ is of limit length then because Φ acts on $\mathcal{U}$, we can let $c = \Phi(\mathcal{U})$. Let then $\mathcal{R}_1$ be either the last model of $\mathcal{U}$ or $\mathcal{M}^{\mathcal{U}}_c$.

We then must have one of the following cases: either

1. $\mathcal{T}$ has a last model $\mathcal{Y}_2$, $\pi^{\mathcal{T}}$ exists, $\mathcal{Y}$ is generic over the extender algebra of $\mathcal{Y}_2$ at $\pi^{\mathcal{T}}(\delta_{a,0})$ and $\mathcal{Y}_2 \lhd \mathcal{R}_1$, or

2. $\mathcal{T}$ is of limit length, $\mathcal{Q}(\mathcal{T})$ does not exist, and letting $\mathcal{Y}_2 = \mathcal{W}(\mathcal{M}(\mathcal{T}))$, $\mathcal{Y}_2 \trianglelefteq \mathcal{R}_1$.

Set $\mathcal{X} = \mathcal{Y}_2$. Notice that because $\rho(\mathcal{R}^*) = \omega$, we must have that $rud(\mathcal{R}_1) \vDash$ "ζ is not a Woodin cardinal" where ζ is the Woodin of $\mathcal{X}$. Let $\mathcal{R}_2 \trianglelefteq \mathcal{R}_1$ be the longest such that $\mathcal{R}_2 \vDash$ "ζ is a Woodin cardinal". Let $\mathcal{R} \trianglelefteq \mathcal{R}_2$ be the longest such that $\mathcal{R} \vDash$ "$(\zeta^+)^{\mathcal{X}}$ is a cardinal". We claim $(\mathcal{X}, \mathcal{R})$ is as desired.

Next we need to show that $\mathcal{R}$ is not δ-iterable in $\mathcal{M}[g]$ above $Ord \cap \mathcal{X}$. Assume not. Let $\eta \in (\delta_{a,0}, \delta)$ be an $\mathcal{M}$-cardinal such that $\mathbb{P} \in \mathcal{M}|\eta$. Let $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over $\mathcal{X}$ using extenders with critical point $> \eta$. Then $\mathcal{R} \ntrianglelefteq \mathcal{P}$ which means that $\mathcal{R}$ must outiterate $\mathcal{P}$ (this can be shown by considering the comparison of $\mathcal{R}$ with the construction producing $\mathcal{P}$). We then have a tree $\mathcal{K}$ on $\mathcal{R}$ such that $\mathcal{M}(\mathcal{K}) = \mathcal{P}$ implying that $\mathcal{P}$ cannot compute unboundedly many successors correctly contradicting Theorem 2.2.

Finally we need to show that $\mathcal{X}$ and $\mathcal{R}$ are countable in $\mathcal{N}$. Assume not. We then must have that the construction of $\mathcal{T}$ and $\mathcal{U}$ lasts $\omega_1^{\mathcal{N}}$ steps. We now claim that we must also have a branch for $\mathcal{T}$ in $\mathcal{N}$. Indeed, let $\pi : H \to \mathcal{N}|(\omega_2^{\mathcal{N}})$ be countable in $\mathcal{N}$ such that $\mathcal{T}, \mathcal{U}, c \in rng(\pi)$. Then $\pi^{-1}(c) \in H$ is the branch of $\pi^{-1}(\mathcal{U})$. Let $\xi = \omega_1^H$. Notice that $\mathcal{Q}(\mathcal{T} \restriction \xi)$ exists and the branch of $\mathcal{T} \restriction \xi$ chosen in $\mathcal{T}$ for $\mathcal{T} \restriction \xi$ is the unique branch b such that $\mathcal{Q}(\mathcal{T} \restriction \xi, b)$ exists and is equal to $\mathcal{Q}(\mathcal{T} \restriction \xi)$. But we have that $\pi^{-1}(\mathcal{R}_2) = \mathcal{Q}(\mathcal{T} \restriction \xi)$, and because $\pi^{-1}(\mathcal{R}_2) \in H$, the branch of $\mathcal{T} \restriction \xi$ is in H. Let b be this branch. It then follows that $\pi(b)$ is a branch of $\mathcal{T}$. The usual comparison argument now implies that the iteration must have lasted $< \omega_1^{\mathcal{N}}$ steps.
□

Let Λ be the strategy of $\mathcal{R}$ in $\mathcal{N}$. We would like to find a Λ-iterate $\mathcal{S}$ of $\mathcal{R}$ such that $\mathcal{S}$ is a *minimal counterexample* to δ-iterability. Below we define what this notion means.

Given a finite stack of normal trees $\vec{\mathcal{T}} \in \mathcal{M}|\delta[g]$ on $\mathcal{R}$, we say $\vec{\mathcal{T}}$ is Λ-correct if in $\mathcal{M}[g]$, there is a club of countable $X \prec \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$ such that letting $\pi_X : N_X \to \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$ be the transitive collapse, $\pi_X^{-1}(\vec{\mathcal{T}})$ is according to Λ. We now look for an iterate of $\mathcal{R}$ that is a *minimal counterexample* to $< \delta$-iterability among Λ-correct iterates of $\mathcal{R}$. Below we make the notion more precise.

Suppose $\vec{\mathcal{T}} \in \mathcal{M}|\delta[g]$ is a finite Λ-correct stack on $\mathcal{R}$ with last model $\mathcal{K}$. Let $\mathcal{S} \trianglelefteq \mathcal{K}$. We say $(\vec{\mathcal{T}}, \mathcal{S})$ is a minimal counterexample to δ-iterability if there is an $\mathcal{S}$-cardinal η such that

1. $\mathcal{S}$ is a mouse over $\mathcal{S}|\eta$ (i.e., η is not overlapped in $\mathcal{S}$),

2. $\rho(\mathcal{S}) \leq \eta$ and $\mathcal{S}$ is η-sound,

3. whenever $\mathcal{U} \in \mathcal{M}|\delta[g]$ is a normal tree on $\mathcal{S}$ above η with last model $\mathcal{W}^*$ such that $\vec{\mathcal{T}}^\frown\mathcal{U}$ is Λ-correct, for any $\mathcal{W}$ such that $\mathcal{S}|\eta \triangleleft \mathcal{W} \triangleleft \mathcal{W}^*$ and for any $\mathcal{W}$-cardinal ν such that ν is a cutpoint of $\mathcal{W}$ and $\rho(\mathcal{W}) \leq \nu$, $\mathcal{M}[g] \vDash$ "$\mathcal{W}$ is δ-iterable above ν".

It is not difficult to see that there is a minimal counterexample to δ-iterability. Towards a contradiction, assume there is no minimal counterexample to δ-iterability. We know that $\mathcal{R}$ is not δ-iterable. Therefore, it is not a minimal counterexample to δ-iterability. We can then construct a sequence $(\mathcal{R}_i^+, \mathcal{R}_i, \mathcal{T}_i, \nu_i : i \in [1, \omega))$ such that

1. $\mathcal{R}_i$ is a ν_i-sound mouse over $\mathcal{R}_i|\nu_i$ such that $\rho(\mathcal{R}_i) \leq \nu_i$,

2. $\mathcal{T}_i$ is a tree on $\mathcal{R}_i$ above ν_i such that $\oplus_{k \leq i}\mathcal{T}_k$ is Λ-correct,

3. $\mathcal{R}_{i+1}^+$ is the last model of $\mathcal{T}_i$,

4. $\mathcal{R}_{i+1} \triangleleft \mathcal{R}_{i+1}^+$ is such that for some ν_{i+1}, $\mathcal{R}_{i+1}$ is a ν_{i+1}-sound mouse over $\mathcal{R}_{i+1}|\nu_{i+1}$ such that $\rho(\mathcal{R}_{i+1}) \leq \nu_{i+1}$ and $\mathcal{R}_{i+1}$ is not δ-iterable above ν_{i+1} in $\mathcal{M}[g]$.

Suppose then $X \prec \mathcal{M}|(\delta^+)^{\mathcal{M}}[g]$ is such that it witnesses that for each i, $\oplus_{k \leq i}\mathcal{T}_k$ is Λ-correct. It follows that $\pi_X^{-1}(\oplus_{i \in \omega}\mathcal{T}_i)$ witnesses that Λ is not an iteration strategy for $\mathcal{R}$.

Let now $\mathcal{S}$ be a minimal counterexample to δ-iterability and let $\vec{\mathcal{T}}$ be the finite Λ-correct stack on $\mathcal{R}$ producing $\mathcal{S}$. Thus, $\mathcal{S}$ is an initial segment of the last model of $\vec{\mathcal{T}}$. Let η be an $\mathcal{S}$-cardinal witnessing that $\mathcal{S}$ is a minimal counterexample to δ-iterability. We then have that $\mathcal{S}$ is a mouse over $\mathcal{S}|\eta$, $\mathcal{S}$ is η-sound and $\rho(\mathcal{S}) \leq \eta$.

Assume that $\mathcal{S}$ has a Woodin cardinal. Let ν be its least Woodin cardinal. Let $\mathcal{P}$ be the output of the fully backgrounded construction of $\mathcal{M}|\delta[g]$ done over $\mathcal{S}|\eta$ using extenders with critical point $> |\eta|^{\mathcal{M}[g]}$. We now compare $\mathcal{S}|\nu$ with the construction producing $\mathcal{P}$. The $\mathcal{P}$-side of such a comparison doesn't move. However, since $\mathcal{S}$ is not fully iterable, we need to describe a strategy for picking branches on the $\mathcal{S}$-side. Let $(\mathcal{P}_\xi^*, \mathcal{P}_\xi, E_\xi : \xi < \delta)$ be the models of the aforementioned construction.

Suppose then $\mathcal{U} \in \mathcal{M}|\delta[g]$ is a tree of limit length that has been built on $\mathcal{S}$ via the aforementioned comparison process. We would like to describe a branch for it. As an inductive hypothesis, we maintain that $\vec{\mathcal{T}}^\frown\mathcal{U}$ is Λ-correct. Thus, the branch b we pick for $\mathcal{U}$ has to have the property that $\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{M}_b^{\mathcal{U}}\}$ is Λ-correct. There can be at most one such branch. It is then enough to show that there is such a branch. The description of b splits into two cases.

First recall the definition of a fatal drop [5, Definition 1.27]. Given a tree $\mathcal{W}$ on a premouse $\mathcal{Q}$ we say $\mathcal{W}$ has a fatal drop if there is $\alpha < lh(\mathcal{W})$, ξ and $\mathcal{K} \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ such that $\mathcal{K}$ is a mouse over $\mathcal{K}|\xi$, $\rho(\mathcal{K}) \leq \xi$ and the rest of the tree is a tree on $\mathcal{K}$ above ξ.

$\mathcal{U}$ doesn't have a fatal drop.

We have that there is some $\xi < \delta$ such that $\mathcal{M}(\mathcal{U}) \triangleleft \mathcal{P}_\xi$. Because $\delta(\mathcal{U}) < \delta$, we have that $\mathcal{M} \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal". It follows that there is a mouse $\mathcal{Q}$ over $\mathcal{M}(\mathcal{U})$ that is obtained via the S-construction that translates $\mathcal{M}$ into a mouse over $\mathcal{P}_\xi|\delta(\mathcal{U})$ such that $\mathcal{Q}$ is $\delta(\mathcal{U})$-sound, $\rho(\mathcal{Q}) = \delta(\mathcal{U})$ and $rud(\mathcal{Q}) \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal". We claim that

Claim 2. there is a branch b of $\mathcal{U}$ such that $\mathcal{Q}(b, \mathcal{U})$ exists and $\mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}$.

Proof. Indeed, let $X \prec \mathcal{M}|(\delta^+)^\mathcal{M}[g]$ be countable such that $\vec{\mathcal{T}}, \mathcal{S}, \mathcal{U}, \mathcal{P}_\xi, \mathcal{Q} \in X$, and letting $\pi_X : \mathcal{N}_X \to \mathcal{M}|(\delta^+)^\mathcal{M}[g]$ be the transitive collapse of X, $\pi_X^{-1}(\vec{\mathcal{T}}^\frown\mathcal{U})$ is Λ-correct. Set $\bar{\mathcal{Q}} = \pi^{-1}(\mathcal{Q})$. Notice that it follows from Claim 1 and Proposition 1.7 that $\bar{\mathcal{Q}}$ is δ-iterable in $\mathcal{N}$. Let then $c = \Lambda(\pi_X^{-1}(\vec{\mathcal{T}}^\frown\mathcal{U}))$. We must have that $\mathcal{Q}(c, \mathcal{U})$ exists and $\mathcal{Q}(c, \mathcal{U}) = \bar{\mathcal{Q}}$. By absoluteness $c \in \mathcal{N}_X$. It is now not hard to check that $b =_{def} \pi_X(c)$ is as desired. b is the unique branch of $\mathcal{U}$ such that $\mathcal{Q}(b, \mathcal{U})$ exists and $\mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}$.

□

$\mathcal{U}$ has a fatal drop.

Let $\xi < lh(\mathcal{U})$ be such that the fatal drop happens at $\mathcal{M}_\xi^\mathcal{U}$. Let ζ and $\mathcal{W} \triangleleft \mathcal{M}_\xi^\mathcal{U}$ be such that $\mathcal{M}_\xi^\mathcal{U}|\zeta \triangleleft \mathcal{W}$, $\rho(\mathcal{W}) = \zeta$ and the rest of $\mathcal{U}$ is a tree on $\mathcal{W}$ above ζ. Because $\mathcal{S}$ is a minimal counterexample to δ-iterability, we have that $\mathcal{W}$ is δ-iterable in $\mathcal{M}|\delta[g]$. Let then b be the branch of $\mathcal{U}$ according to the unique strategy of $\mathcal{W}$. Again a Skolem hull argument and Claim 1 show that $\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{M}_b^\mathcal{U}\}$ is Λ-correct.

This finishes our description of branches that payer II plays in the comparison game between $\mathcal{S}|\nu$ and the construction producing $\mathcal{P}$. Let then $\mathcal{U}$ be the tree on $\mathcal{S}|\nu$ of maximal length constructed in the manner described above.

Notice that for unboundedly many $\theta < \delta$, $\mathcal{P}$ computes θ^+ correctly. This is because $\mathcal{X} \in \mathcal{P}$ and if $\mathcal{P}^*$ is the output of the fully backgrounded construction of $\mathcal{P}$ done over $\mathcal{X}$ with large enough critical points then $\mathcal{P}^*$ computes unboundedly many successors correctly.

It now follows that that $\mathcal{U}$ cannot length δ as then $\mathcal{M}(\mathcal{U}) = \mathcal{P}$. Also, $\mathcal{U}$ must have a last model $\mathcal{S}^*$. Indeed, if $\mathcal{U}$ doesn't have a last model then it is of limit length. Because $\delta(\mathcal{U}) < \delta$, we have that $\mathcal{M} \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal", implying that our method of picking branches of $\mathcal{U}$ does produce a branch for $\mathcal{U}$. Because $\mathcal{S}$-side lost the comparison, $\pi^{\mathcal{U}}$ must exist. Let $\zeta = \pi^{\mathcal{U}}(\nu)$ (because $\vec{\mathcal{T}}^\frown\mathcal{U}$ is Λ-correct, $\mathcal{U}$ can be applied to $\mathcal{S}$).

Because $\mathcal{M} \vDash$ "ζ is not a Woodin cardinal", we can find sound $\mathcal{S}^*|\zeta$-mouse $\mathcal{W} \in \mathcal{M}|\delta[g]$ such that $\rho(\mathcal{W}) \leq \zeta$ and $rud(\mathcal{W}) \vDash$ "ζ is not a Woodin cardinal". We claim that

Claim 2. $\mathcal{W} = \mathcal{S}^*$.

Proof. Indeed, let $X \prec \mathcal{M}|(\delta^+)^{\mathcal{M}[g]}$ be countable such that $\vec{\mathcal{T}}, \mathcal{S}, \mathcal{U}, \mathcal{W}, \mathcal{S}^* \in X$, and letting $\pi_X : \mathcal{N}_X \to \mathcal{M}|(\delta^+)^{\mathcal{M}[g]}$ be the transitive collapse, $\pi_X^{-1}(\vec{\mathcal{T}}^\frown\mathcal{U})$ is according to Λ. Let $\mathcal{K} = \pi_X^{-1}(\vec{\mathcal{T}}^\frown\mathcal{U}\}$, $\bar{\mathcal{W}} = \pi_X^{-1}(\mathcal{W})$, $\bar{\mathcal{S}} = \pi_X^{-1}(\mathcal{S}^*)$ and $\phi = \pi_X^{-1}(\zeta)$. Because $\vec{\mathcal{T}}^\frown\mathcal{U}$ is Λ-correct, we have that in $\mathcal{N}$, $\bar{\mathcal{S}}$ is a δ-iterable mouse over $\bar{\mathcal{S}}|\phi$, $\bar{\mathcal{S}} \vDash$ "ϕ is a Woodin cardinal" and $rud(\bar{\mathcal{S}}) \vDash$ "ϕ is not a Woodin cardinal".

Notice now that $\bar{\mathcal{W}}$ has exactly the same properties as $\bar{\mathcal{S}}$ in $\mathcal{N}$. The only non-trivial part is δ-iterability, which follows from Claim 1. It follows that $\bar{\mathcal{W}} = \bar{\mathcal{S}}$. Hence, $\mathcal{W} = \mathcal{S}^*$. □

Since $\mathcal{W}$ is δ-iterable in $\mathcal{M}[g]$ above η and $\pi^{\mathcal{U}}$ exists (see Proposition 1.5), we have that $\mathcal{S}$ is also δ-iterable in $\mathcal{M}[g]$ above η. This is a contradiction as $\mathcal{S}$ is not δ-iterable in $\mathcal{M}[g]$ above η.

The case when $\mathcal{S}$ has no Woodin cardinals is very similar. Now we compare $\mathcal{S}$ with the fully backgrounded constructions producing a tree $\mathcal{U}$ on $\mathcal{S}$ such that $\vec{\mathcal{T}}^\frown\mathcal{U}$ is Λ-correct. Because $\mathcal{S}$ has no Woodin cardinals, handling limit stages of the construction of $\mathcal{U}$ is very similar. Assuming $\mathcal{U}$ has been built up to stage γ, we consider, as above, two cases. If $\mathcal{U} \restriction \gamma$ has no fatal drops then we proceed as in the "no fatal drop case" of the above argument. Otherwise, we proceed in the "fatal drop" case of the above argument. We leave the details to the reader. □

We believe that the project of characterizing in mice the exact cardinals κ that permit stationary tower like embeddings with critical point κ is a very nice project.

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