

An inner model theoretic proof of Becker's theorem

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Abstract

We re-prove Becker's theorem from [1] by showing that $AD^{L(\mathbb{R})}$ implies that $L(\mathbb{R}) \models$ “ ω_2 is δ_1^2 -supercompact”. Our proof uses inner model theoretic tools instead of Baire category. We also show that ω_2 is $< \Theta$ -strongly compact.

This article draws inspiration from the work of Neeman ([5]) who, using inner model theoretic tools, showed that under $AD^{L(\mathbb{R})}$, ω_1 is $< \Theta$ -supercompact. We have also been influenced by the work of Becker ([1]), Becker-Jackson ([2]) and Jackson ([4]). In [1], Becker showed that assuming $AD+V = L(\mathbb{R})$, ω_2 is δ_1^2 -supercompact. In [2], Becker and Jackson showed that, under $AD+V = L(\mathbb{R})$, all projective cardinals are δ_1^2 -supercompact. Finally, in [4], Jackson showed that under $AD+V = L(\mathbb{R})$ all Suslin cardinals and their successors are δ_1^2 -supercompact.

In this short note, we re-prove Becker's theorem using inner model theoretic tools. The paper assumes familiarity with what is commonly called HOD analysis. The reader can find this background exposted in [5] and in [8]. The point of re-proving such results is to find more applications of inner model theory in descriptive set theory.

1 Measures on $\mathcal{O}_{\omega_2}(\lambda)$

We assume $AD+V = L(\mathbb{R})$. Fix $\lambda < \Theta$. Let A be an OD set of reals such that $\gamma_{A,\infty} \geq \lambda$. Suppose \mathcal{R} is a suitable premouse that is A -iterable. It is customary to let $\delta^{\mathcal{R}}$ be the Woodin cardinal of \mathcal{R} . Assume that $\lambda \in \text{rng}(\pi_{(\mathcal{R},A),\infty})$. We then let $\lambda^{\mathcal{R}}$ be such that $\pi_{(\mathcal{R},A),\infty}(\lambda^{\mathcal{R}}) = \lambda$.

We let $\text{Code}(A, \lambda) \subseteq \mathbb{R}$ be the set of reals x such that x codes a pair $(\mathcal{R}_x, \alpha_x)$ such that \mathcal{R}_x is an A -iterable suitable pre-mouse such that $\lambda^{\mathcal{R}}$ is defined and $\alpha_x < \lambda^{\mathcal{R}}$. Let $\leq_{A,\lambda}$ be the natural pre-wellordering of $\text{Code}(\mathcal{Q}, A, \lambda)$ given by: $x \leq_{\mathcal{Q},A,\lambda} y$ if and

only if whenever S is an A -iterate of \mathcal{R}_x and an A -iterate of \mathcal{R}_y , $\pi_{(\mathcal{R}_x, A), (S, A)}(\alpha_x) \leq \pi_{(\mathcal{R}_y, A), (S, A)}(\alpha_y)$. We have that $\leq_{A, \lambda}$ has length λ . Given $x \in \text{Code}(A, \lambda)$ let

$$c(x) = \pi_{(\mathcal{R}_x, A), \infty}(\alpha_x) = |x|_{\leq_{A, \lambda}}.$$

Let S be a tree of a Σ_1^2 -scale on a universal Σ_1^2 -set. Given x and y we write $x \sim_S y$ if and only if $x \in L[S, y]$ and $y \in L[S, x]$. We then say that d is an S -degree if d is an \sim_S -class. We write $d \leq_S e$ if $d \in L[S, e]$. Let now $C(A, \lambda) = \{d : \text{Code}(A, \lambda) \cap HC^{L[S, d]} \neq \emptyset\}$.

The following is a corollary to the Harrington-Kechris theorem (see [3], and see [6] and the references there for some uses of it).

Corollary 1.1 *There is a formula ϕ such that whenever $d \in C(A, \lambda)$, g is $< \omega_1$ -generic over $L[S, d]$ and $\mathcal{R} \in L_{\omega_1}[T, d][g]$,*

$$\mathcal{R} \text{ is a suitable pre-mouse if and only if } L[S, d][g] \models \phi[\mathcal{R}].$$

Moreover, there is a formula ψ such that for any A -iterable suitable $\mathcal{Q}, \mathcal{R} \in L_{\omega_1}[T][g]$ and for any π ,

$$\mathcal{R} \text{ is an } A\text{-iterate of } \mathcal{Q} \text{ and } \pi : H_A^{\mathcal{Q}} \rightarrow H_A^{\mathcal{R}} \text{ is the } A\text{-iteration embedding if and only if } L[S, d][g] \models \psi[\mathcal{Q}, \mathcal{R}, \pi, \tau_A],$$

where τ_A is the term relation for A in $L[S, d]^{Coll(\omega, < \omega_1)}$.

The formulas ϕ and ψ essentially repeat the definitions of suitability and A -iterability. Another important lemma that we need is a consequence of what is usually called *generic comparisons* (see [8]). The proof is a standard generic comparison argument which we leave to the reader.

Lemma 1.2 *Suppose $d \in C(A, \lambda)$ and g is $< \omega_1$ -generic over $L[S, d]$. Suppose ϕ is as in Corollary 1.1, and for some $\mathcal{R} \in L[S, d][g]$, $L[S, d][g] \models \phi[\mathcal{Q}, \mathcal{R}]$. Then there is an \emptyset -iterate \mathcal{S} of \mathcal{R} such that $\mathcal{S} \in L_{\omega_1}[T, d]$.*

Given $d \in C(A, \lambda)$ we let B_d be the set of β such that there is $x \in \text{Code}(A, \lambda)$ with the property that $(\mathcal{R}_x, \alpha_x) \in L[S, d]$ and $c(\mathcal{R}_x, \alpha_x) = \beta$. As $|L_{\omega_1}[T, d]| = \omega_1$, we have that $B_d \in \wp_{\omega_2}(\gamma)$. Lemma 1.2 has the following easy corollary.

Corollary 1.3 *Suppose $d_0 \in C(A, \lambda)$ and d is a S -degree such that $L[S, d]$ is a $< \omega_1$ -generic extension of $L[S, d_0]$. Then $B_{d_0} = B_d$.*

We now define $\mu(A, \lambda)$ on $\wp_{\omega_2}(\lambda)$ by setting $B \in \mu(A, \lambda)$ if and only if for an S -cone of d , $B_d \in B$.

Lemma 1.4 $\mu(A, \lambda)$ is an ω_2 -complete ultrafilter on $\wp_{\omega_2}(\lambda)$.

Proof. Clearly $\mu(A, \lambda)$ is an ultrafilter. Let $(B_\xi : \xi < \omega_1)$ be such that $B_\xi \in \mu(A, \lambda)$ for all $\xi < \omega_1$. Let WO be the set of reals coding a countable ordinal. Using the coding lemma we can find $y \in \mathbb{R}$ and a $\Sigma_2^1(y)$ -set $D \subseteq WO \times \mathbb{R}$ such that

1. $[y]_S \in C(A, \lambda)$,
2. for every $x \in WO$, $D_x \neq \emptyset$ (here $D_x = \{z : (x, z) \in D\}$),
3. for every $x \in WO$, $D_x \subseteq \{z : [z]_S \text{ is a base of a cone witnessing that } B_{|x|} \in \mu(A, \lambda)\}^1$.

Let $d \in C$ be such that $y \in L[S, d]$. We claim that $B_d \in B_\xi$ for every $\xi < \omega_1$. To see this, fix $\xi < \omega_1$. Let $g \subseteq Coll(\omega, \xi)$ be $L[S, d]$ -generic and u be a real such that $L[S, d][g] = L[S, u]$. Let $x \in \mathbb{R}^{L[S, u]}$ be such that $|x| = \xi$. Because D is $\Sigma_2^1(y)$ we have that there is $z \in D_x \cap L[S, u]$. Because $[z]_S \leq [u]_S$, we must have that $B_{[u]_S} \in B_\xi$. However, it follows from Corollary 1.3 that $B_d = B_{[u]_S}$. Hence, $B_d \in B_\xi$.

As d was arbitrary, we have shown that for any d that is S -above $[y]_S$, $B_d \in \bigcap_{\xi < \omega_1} B_\xi$. It follows that $\bigcap_{\xi < \omega_1} B_\xi \in \mu(A, \lambda)$. \square

2 ω_2 is δ_1^2 -supercompact and $< \Theta$ -strongly compact

Proposition 2.1 For every $\lambda < \Theta$ and an ordinal definable $A \subseteq \mathbb{R}$ such that $\gamma_{A, \infty} \geq \lambda$, $\mu(A, \lambda)$ is superfine, i.e., for every $B \in \wp_{\omega_2}(\lambda)$,

$$\{D \in \wp_{\omega_2}(\lambda) : B \subseteq D\} \in \mu(A, \lambda).$$

Proof. Fix B and let $f : \omega_1 \rightarrow B$ be a bijection. Let $B_\xi = \{x \in Code(A, \lambda) : c(x) = f(\xi)\}$. Using the coding lemma find $y \in \mathbb{R}$ and $D \subseteq WO \times \mathbb{R}$ such that

1. $[y]_S \in C(A, \lambda)$,
2. $D \in \Sigma_2^1(y)$,
3. for every $x \in WO$, $D_x \neq \emptyset$,

¹where $[z]_S$ is the S -degree given by z and $|x|$ is the ordinal coded by x .

4. for every $x \in WO$, $D_x \subseteq \{z \in Code(A, \lambda) : c(z) = f(|x|)\}$.

We claim that for every d such that $[y]_S \leq_S d$, $B \subseteq B_d$. To see this, fix d such that $[y]_S \leq_S d$. Fix $\zeta \in B$. We want to see that $\zeta \in B_d$. Let $\xi = f^{-1}(\zeta)$, and fix $u \in \mathbb{R}$ such that $L[S, u]$ is a generic extension of $L[S, d]$ and ξ is countable in $L[S, u]$. Fix $x \in WO \cap L[S, u]$ such that $|x| = \xi$. Because $D \in \Sigma_2^1(y)$, we have that $D_x \cap L[S, u] \neq \emptyset$. Fix then $z \in Code(A, \lambda) \cap D_x \in L[S, u]$. It follows that $c(z) = f(\xi)$. Since $c(z) \in B_{[u]_T} = B_d$, we have that $\zeta \in B_d$. \square

Putting Proposition 1.4 and Proposition 2.1 we get the following corollary.

Corollary 2.2 *Assume $AD+V = L(\mathbb{R})$. Then ω_2 is a $< \Theta$ -strongly compact. More precisely, for every $\lambda < \Theta$ there an ω_2 -complete superfine ordinal definable ultrafilter on $\wp_{\omega_2}(\lambda)$.*

Theorem 2.3 (Becker, [1]) *Assume $AD+V = L(\mathbb{R})$. Then ω_2 is δ_1^2 -supercompact.*

Proof. Set $\lambda = \delta_1^2$. Suppose \mathcal{R} is an \emptyset -iterable suitable pre-mouse. Recall that if ν is the least cardinal that is $< \delta^{\mathcal{R}}$ -strong in \mathcal{R} then $\pi_{(\mathcal{R}, \emptyset), \infty}(\nu) = \lambda$ (see [7, Chapter 8]). We now want to show that $\mu =_{def} \mu(\emptyset, \lambda)$ is an ω_2 -supercompactness measure. Proposition 1.4 shows that μ is ω_2 -complete and Proposition 2.1 shows that μ is fine. It remains to show that μ is normal. The following lemma is the first step towards normality. Set $Code =_{def} Code(\emptyset, \lambda)$ and $\leq^* = \leq_{\emptyset, \lambda}$.

Lemma 2.4 *Suppose $F : \wp_{\omega_2}(\lambda) \rightarrow \lambda$ is such that for an S -cone of d , $F(B_d) \in B_d$. Then for an S -cone of d there is $x \in (\mathbb{R}^{L[S, d]} \cap Code)$ such that $c(x) = F(B_d)$.*

Proof. Assume not. Fix an S -degree d_0 such that whenever d is S -above d_0 , for every $x \in (\mathbb{R}^{L[S, d]} \cap Code)$, $c(x) \neq F(B_d)$. Fix $(\mathcal{R}, \alpha) \in L[S, d_0]$ such that $\pi_{(\mathcal{R}, \emptyset), \infty}(\alpha) = F(B_{d_0})$.

Let $\nu < \omega_1$ be any cardinal of $L[S, d_0]$ such that $(\mathcal{R}, \alpha) \in L_\nu[S, d_0]$ and let $g \subseteq Coll(\omega, (\nu^+)^{L[S, d_0]})$ be $L[S, d_0]$ -generic. Let $x \in \mathbb{R}$ be such that $L[S, d_0][g] = L[S, x]$. We then have that $B_{d_0} = B_{[x]_S}$ (see Corollary 1.1). This is a contradiction as we can find $y \in L[S, x] \cap \mathbb{R}$ coding (\mathcal{R}, α) . \square

Lemma 2.5 *μ is normal.*

Proof. Suppose μ is not normal. Let $F : \wp_{\omega_2}(\lambda) \rightarrow \lambda$ be such that for an S -cone of d , $F(B_d) \in B_d$ but F is not constant on a μ -measure one set. Let $e_0 \in C$ be a base for the cone of the previous sentence.

Let $e \in C$ be S -above e_0 and such that for every d such that $e \in L[S, d]$, there is $x \in (\mathbb{R}^{L[S, d]} \cap \text{Code})$ with the property that $c(x) = F(B_d)$. We now follow an idea of Becker from [1].

Given an ordinal $\xi < \lambda$ let $D_\xi = \{d : F(B_d) \neq \xi\}$. We have that for each ξ , D_ξ contains an S -cone. Let then $C_\xi = \{x \in \mathbb{R} : [x]_S \text{ is a base of a cone contained in } D_\xi\}$. It follows from the coding lemma that there is a real y and a set D such that

1. $e \leq_S [y]_S$,
2. $D \subseteq \text{Code} \times \mathbb{R}$ is $\Sigma_1^2(y)$,
3. if $(x, z) \in D$ then $z \in C_{c(x)}$,
4. for every $x \in \text{Code}$ there is z such that $(x, z) \in D$.

Set $d = [y]_S$. Let $x \in \text{Code} \cap L[S, d]$ be such that $c(x) = F(B_d)$. Because D is $\Sigma_1^2(y)$ we have that $D_x \cap L[S, d] \neq \emptyset$. Let then $z \in D_x \cap L[S, d]$. It follows that $[z]_S \leq_S d$. Hence, $d \in D_{c(x)}$. It follows that $F(B_d) \neq c(x) = F(B_d)$, contradiction.

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3 A covering conjecture

Again we assume $AD + V = L(\mathbb{R})$. Suppose $\kappa < \lambda < \Theta$ and A is an ordinal definable set of reals such that $\gamma_{A, \infty} \geq \lambda$. Given $X \in \wp_\kappa(\lambda)$ we say that X is an A -iteration set if for every $\alpha \in X$ there is an A -iterable \mathcal{Q} such that $\alpha \in \text{rng}(\pi_{(\mathcal{Q}, A), \infty})$ and $\pi_{(\mathcal{Q}, A), \infty}[\gamma_A^{\mathcal{Q}}] \cap \lambda \subseteq X$.

Conjecture 3.1 *Assume $AD + V = L(\mathbb{R})$. Suppose κ is either a Suslin cardinal or a successor of a Suslin cardinal. Then for every $\lambda < \Theta$, an OD set $A \subseteq \mathbb{R}$ such that $\lambda \leq \gamma_{A, \infty}$, and $B \in \wp_\kappa(\lambda)$ there is an A -iteration set $X \in \wp_\kappa(\lambda)$ such that $B \subseteq X$.*

The conjecture is clearly true for $\kappa = \omega_1$. Proposition 2.1 shows that the conjecture is true for $\kappa = \omega_2$. We expect that the validity of the full conjecture will follow once a link is made between Kechris-Woodin generic codes and iteration sets.

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