HOD IN INNER MODELS WITH WOODIN CARDINALS

SANDRA MÜLLER AND GRIGOR SARGSYAN

Abstract. We analyze the hereditarily ordinal definable sets HOD in the canonical inner model with n Woodin cardinals $M_n(x, g)$ for a Turing cone of reals x, where g is generic over $M_n(x)$ for the Lévy collapse up to its bottom inaccessible cardinal. We prove that assuming $\mathbf{\Pi}^1_{n+2}$-determinacy, for a Turing cone of reals x, $\mathrm{HOD}^{M_n(x,g)} = M_n(M_\infty, \Lambda)$, where M_∞ is a direct limit of iterates of an initial segment of M_{n+1} and Λ is a partial iteration strategy for M_∞. This implies that under the same hypothesis $\mathrm{HOD}^{M_n(x,g)}$ is a fine structural model and therefore satisfies GCH. These results generalize to HOD^M for self-iterable canonical inner models M, for example M_ω, the least mouse with ω Woodin cardinals, or initial segments of the least non-tame mouse M_{nt}.

1. Introduction

An essential question regarding the theory of inner models is the analysis of the class of all hereditarily ordinal definable sets HOD inside various inner models M of the set theoretic universe V under appropriate determinacy hypotheses. Examples for such inner models M are $L(\mathbb{R})$, $L[x]$, and the canonical proper class x-mouse with n Woodin cardinals $M_n(x)$, but nowadays also larger models of determinacy M are considered.

One motivation for analyzing the internal structure of these models HOD^M is given by Woodin's results in [KW10] that under determinacy hypotheses these models contain large cardinals. He showed in [KW10] for example that assuming Δ^1_2 determinacy there is a Turing cone of reals x such that $\omega_2^{L[x]}$ is a Woodin cardinal in the model $\mathrm{HOD}^{L[x]}$. This result generalizes to higher levels in the projective hierarchy. That means for $n \geq 1$ assuming $\mathbf{\Pi}^1_{n+1}$ determinacy and Π^1_{n+2} determinacy there is a cone of reals x such that $\omega_2^{M_n(x)}$ is a Woodin cardinal in the model $\mathrm{HOD}^{M_n(x)|\delta_x}$, where $M_n(x)$ denotes the canonical proper class x-mouse with n Woodin cardinals and δ_x is the least Woodin cardinal in $M_n(x)$. Moreover, Woodin showed a similar result for $\mathrm{HOD}^{L(\mathbb{R})}$. If we let Θ denote the supremum of all ordinals α such that there exists a surjection $\pi : \mathbb{R} \to \alpha$, then assuming $\mathrm{ZF} + \mathrm{AD}$, he

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showed that $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$ (see [KW10]). The fact that these models of the form HOD^M can have large cardinals as for example Woodin cardinals motivates the question if they are in some sense fine structural as for example the models $L[x]$, $M_n(x)$, and $L(\mathbb{R})$ are. A good test question for this is whether these models HOD^M satisfy the generalized continuum hypothesis GCH. If it turns out that HOD^M is in fact a fine structural model, it would follow that it satisfies the GCH and even stronger combinatorial principles as for example the $\diamondsuit$ principle.

The first model which was analyzed in this sense was $\mathrm{HOD}^{L(\mathbb{R})}$ under the assumption that every set of reals in $L(\mathbb{R})$ is determined (short: $\mathrm{AD}^{L(\mathbb{R})}$). Using purely descriptive set theoretic methods Becker showed in [Be80] under this hypothesis that GCH_α, i.e. $2^\alpha = \alpha^+$, holds in $\mathrm{HOD}^{L(\mathbb{R})}$ for all $\alpha < \omega_1^{L(\mathbb{R})}$. Later J. R. Steel and W. H. Woodin independently were able to push the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ forward using more recent advances in inner model theory. In 1993 they first showed that the reals in $\mathrm{HOD}^{L(\mathbb{R})}$ are the same as the reals in M_ω, the least proper class iterable premouse with ω Woodin cardinals. Then they showed in §4 of [St93] that $\mathrm{HOD}^{L(\mathbb{R})}$ in fact agrees with the inner model N up to $\mathcal{P}(\omega_1^{L(\mathbb{R})})$, where N denotes the $\omega_1^{L(\mathbb{R})}$-th linear iterate of M_ω by its least measure and its images. Building on this, John R. Steel was able to show in [St95] that $\mathrm{HOD}^{L(\mathbb{R})}$ agrees with the inner model $\mathcal{M}_\infty$ up to $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$, where $\mathcal{M}_\infty$ is a direct limit of iterates of M_ω and $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ is the supremum of all ordinals α such that there exists a surjection $\pi : \mathbb{R} \to \alpha$ which is $\Delta_1^{L(\mathbb{R})}$ definable. Finally, in 1996 W. Hugh Woodin extended this (see [StW16]) and showed that in fact $\mathrm{HOD}^{L(\mathbb{R})} = L[\mathcal{M}_\infty, \Lambda]$, where Λ is a partial iteration strategy for $\mathcal{M}_\infty$. For even larger models of determinacy M the corresponding model HOD^M was first analyzed in [Sa09], where the second author showed that it is fine structural using a layered hierarchy. Models of this form are nowadays called *hod mice*. A different approach for the fine structure of hod mice called the least branch hierarchy is studied in [St16].

The question if $\mathrm{HOD}^{L[x]}$ is a fine structural model or even a model of GCH for a Turing cone of reals x under a suitable determinacy hypothesis remains open until today. What has been done is the analysis of the model $\mathrm{HOD}^{L[x][G]}$, where G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $\mathrm{HOD}^{L[x]}$ for the least inaccessible cardinal κ_x in $L[x]$. Woodin showed in the 1990's (see [StW16]) that assuming Δ^1_2 determinacy there is a Turing cone of reals x such that $\mathrm{HOD}^{L[x][G]} = L[\mathcal{M}_\infty, \Lambda]$, where $\mathcal{M}_\infty$ is a direct limit of mice (which are iterates of M_1) and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

In this article, we analyze HOD in the canonical premouse $M_n(x, g)$ for any real x of sufficiently high Turing degree under the assumption that every $\boldsymbol{\Pi}^1_{n+2}$ set of reals is determined. Here g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$, where κ denotes the least inaccessible of $M_n(x)$. We first show that the countable direct limit model $\mathcal{M}_{\infty,\kappa}$, obtained from iterates of suitable premice together with their iteration embeddings, agrees up to its bottom Woodin cardinal $\delta_{\infty,\kappa}$ with $\mathrm{HOD}^{M_n(x,g)}$. In a second step, we show that the full model $\mathrm{HOD}^{M_n(x,g)}$ is in fact of the form $M_n(\mathcal{M}_{\infty,\kappa}, \Lambda)$, where Λ is a partial iteration strategy for $\mathcal{M}_{\infty,\kappa}$. This yields that $\mathrm{HOD}^{M_n(x,g)}$ is a model of GCH, $\Diamond$, and other combinatorial principles which are consequences of fine structure.

In the statement of the following main theorem and in fact everywhere in this article whenever we write HOD^M for some premouse M we mean $\mathrm{HOD}^{\lfloor M \rfloor}$, where $\lfloor M \rfloor$ denotes the universe of the model M. In particular, we do not allow the extender sequence of M as a parameter. It will be clear from the context if we consider the model M or the universe $\lfloor M \rfloor$ of M, therefore we decided for the sake of readability to not distinguish the notation for these two objects.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $n < \omega$ and assume $\boldsymbol{\Pi}^1_{n+2}$-determinacy. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_{\infty,\kappa}, \Lambda),$$

where g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$, κ denotes the least inaccessible cardinal of $M_n(x)$, $\mathcal{M}_{\infty,\kappa}$ is a direct limit of iterates of an initial segment of M_{n+1}, and Λ is a partial iteration strategy for $\mathcal{M}_{\infty,\kappa}$.

Corollary 1.2. *Assume $\boldsymbol{\Pi}^1_{n+2}$-determinacy. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_n(x,g)} \vDash \mathrm{GCH},$$

where g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$ and κ denotes the least inaccessible cardinal of $M_n(x)$

Remark. In fact the full strength of $\boldsymbol{\Pi}^1_{n+2}$-determinacy is not needed for these results. It suffices to assume that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x (or equivalently $\boldsymbol{\Pi}^1_{n+1}$-determinacy, see [MSW] and [Ne02]) and that $M_{n+1}^\#$ exists and is ω_1-iterable. This is all we will use in the proof.

The above mentioned results generalize to self-iterable canonical inner models as follows. For definitions and futher explanations see Section 6 below.

Theorem 6.4. *Let φ be a formula expressing the existence of a fixed number of Woodin and strong cardinals (in a fixed order) and assume that $M_\varphi(x)$ exists and is self-iterable for all $x \in {}^\omega\omega$. Moreover, assume that $M_{\delta,\varphi}^\#$ exists. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_\varphi(x,g)} = M_\varphi(\mathcal{M}_\infty, \Lambda),$$

where κ is the least inaccessible cardinal in $M_\varphi(x)$, g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_\varphi(x)$, $\mathcal{M}_\infty$ is a direct limit of iterates of an initial segment of $M_{\delta,\varphi}$, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

Since we assume self-iterability, our method is at the moment restricted to tame[1] mice. Using [Schl14, Theorem 4.5] it extends to inital segments of the least non-tame mouse M_{nt}. See Theorem 6.7 for the precise statement.

Finally, we summarize some open questions related to these results.

Question 1. *Assume $\mathbf{\Delta}^1_2$ determinacy. Is $HOD^{L[x]}$ for a cone of reals x a fine structural model?*

Question 2. *Assume $\mathbf{\Pi}^1_{n+2}$ determinacy. Is $HOD^{M_n(x)}$ for a cone of reals x a fine structural model?*

Question 3. *Is it possible to extend this analysis of $HOD^{M_\varphi(x,g)}$ to non-tame mice?*

This article is structured as follows. In Section 2 we recall some preliminaries and fix the basic notation. In Section 3 we recall the relevant notions from [Sa13] and define the direct limit system converging to $\mathcal{M}_{\infty,\kappa}$, before we compute $\mathrm{HOD}^{M_n(x,g)}$ up to its Woodin cardinal in Section 4. In Section 5 we then show how this can be used to compute the full model $\mathrm{HOD}^{M_n(x,g)}$, i.e. we finish the proof of Theorem 1.1. Finally, we will argue in Section 6 that our analysis can be extended to canonical inner models with more Woodin cardinals or for example a strong cardinal above the Woodin cardinals, as well as to initial segments of the least non-tame mouse. The authors thank Farmer Schlutzenberg for the helpful discussions related to the generalizations in Section 6 of this article during the 4th Münster conference on inner model theory in the summer of 2017.

2. Preliminaries and notation

Whenever we say *reals* we mean elements of the Baire space ${}^\omega\omega$. We also write $\mathbb{R}$ for ${}^\omega\omega$. HOD denotes the class of all hereditarily ordinal definable sets. Moreover HOD_x for any $x \in {}^\omega\omega$ denotes the class of all sets which are hereditarily ordinal definable over $\{x\}$.[2] That means we let $A \in \mathrm{OD}_x$ iff there is a formula φ such that $A = \{v \mid \varphi(v, \alpha_1, \dots, \alpha_n, x)\}$ for some ordinals $\alpha_1, \dots, \alpha_n$. Then $A \in \mathrm{HOD}_x$ iff $\mathrm{TC}(\{A\}) \subset \mathrm{OD}_x$, where $\mathrm{TC}(\{A\})$ denotes the transitive closure of the set $\{A\}$.

We use the notions of premice and iterability from [St10, §1 − 4] and assume that the reader is familiar with the basic concepts defined there. When we say that a premouse is ω_1-iterable we in fact mean that it is $(\omega, \omega_1, \omega_1)$-iterable in the sense of Definition 4.4 in [St10] and if not stated otherwise we will always assume this amount of iterability for our mice. We say a *cutpoint* of a premouse $\mathcal{M}$ is an infinite $\mathcal{M}$-cardinal γ such that there is no extender E on the $\mathcal{M}$-sequence with $\mathrm{crit}(E) \leq \gamma \leq \mathrm{lh}(E)$.[3]

[1]A premouse M is called *tame* iff for every extender E on the M-sequence and every ordinal δ with $crit(E) \leq \delta < lh(E)$, $M|lh(E) \vDash$ "δ is not Woodin".

[2]In the literature this is sometimes also called $\mathrm{HOD}_{\{x\}}$.

[3]Such a cutpoint γ is often also called a strong cutpoint cardinal.

For some premouse $\mathcal{M}$ with $\mathcal{M} \vDash \mathrm{ZFC}$ and some real $x \in \mathcal{M}$ we write $L[E](x)^{\mathcal{M}}$ for the result of a fully backgrounded extender construction above x inside $\mathcal{M}$ in the sense of [MS94], generalized to ω-small premice. In Section 6 we will consider fully backgrounded extender constructions generalized to tame mice which do not have arbitrarily large Woodin cardinals, for details see [St93]. Moreover, we let for a cutpoint η of $\mathcal{M}$ and a premouse $\mathcal{N} \in \mathcal{M}|(\eta+1)$ such that $\eta \subseteq \mathcal{N} \subseteq H_\eta^{\mathcal{M}}$, $\mathcal{P}^{\mathcal{M}}(\mathcal{N})$ denote the result of a $\mathcal{P}$-construction over $\mathcal{N}$ inside the model $\mathcal{M}$ in the sense of [SchSt09] or [Sa13, Proposition 2.3 and Definition 2.4].

For $x \in {}^\omega\omega$ and $n \leq \omega$ we let $M_n^\#(x)$, if it exists, denote a countable, sound, ω_1-iterable x-premouse which is not n-small but all of whose proper inital segments are n-small. If we assume that some $M_n^\#(x)$ exists for all $x \in {}^\omega\omega$ (or equivalently $\mathbf{\Pi}^1_{n+1}$-determinacy), then every $M_n^\#(x)$ is in fact unique (see for example Lemma 2.2.8 in [Uh16]). If $M_n^\#(x)$ exists and is unique, we let $M_n(x)$ be the proper class premouse obtained by iterating the top extender of $M_n^\#(x)$ out of the universe. In the rest of this article, whenever we say that $M_n^\#(x)$ exists for some $x \in {}^\omega\omega$ we mean that it exists, is ω_1-iterable, and unique.

We fix some standard coding of hereditarily countable sets by reals. So whenever A is a hereditarily countable set in V, we might identify it with its natural coding $x_A \in {}^\omega\omega$. In this case we say that $x_A \in {}^\omega\omega$ *canonically codes* A. If $g \in V$ is $\mathrm{Col}(\omega, \kappa)$-generic over $M_n(x)$ for some cutpoint κ of $M_n(x)$ which is countable in V, we can construe $M_n(x)[g]$ as a y-premouse, for a real y canonically coding x and g. We denote this premouse by $M_n(x, g)$ and identify it with $M_n(x)[g]$. See for example [SchSt09] for the fine structural details. The same holds for a $\mathrm{Col}(\omega, {<}\kappa)$-generic $g \in V$ and an inaccessible cutpoint κ of $M_n(x)$.

Let $x \in {}^\omega\omega$ and let $\mathcal{N}$ be some countable x-premouse. Then, if it exists and is unique, we write $M_n^\#(\mathcal{N})$ for the smallest x-premouse $\mathcal{M} \trianglerighteq \mathcal{N}$ with $\rho_\omega(\mathcal{M}) \leq \mathcal{N} \cap \mathrm{Ord}$ which is ω_1-iterable above $\mathcal{N} \cap \mathrm{Ord}$, sound above $\mathcal{N} \cap \mathrm{Ord}$, and such that either $\mathcal{M}$ is not fully sound or $\mathcal{M}$ is not n-small above $\mathcal{N} \cap \mathrm{Ord}$. In case it exists and is unique, we moreover write $M_n(\mathcal{N})$ for the proper class premouse obtained from $M_n^\#(\mathcal{N})$ by iterating the top extender out of the universe. It will be clear from the context whether we mean the x-premice $M_n^\#(\mathcal{N})$ and $M_n(\mathcal{N})$ in this sense or the $x_{\mathcal{N}}$-premice $M_n^\#(x_{\mathcal{N}})$ and $M_n(x_{\mathcal{N}})$ for a real $x_{\mathcal{N}}$ canonically coding the countable premouse $\mathcal{N}$.

3. The direct limit system

To show that $\mathrm{HOD}^{M_n(x,g)}$ is a fine structural inner model, we will use an extension of the direct limit system introduced in [Sa13]. For the readers convenience we will first recall the relevant definitions and results from [Sa13], obtaining a direct limit system which is definable in $M_n(x)$. Then we discuss the changes we need to make to obtain a direct limit system definable in $M_n(x,g)$. Another application of a similar but slightly different direct limit system as in [Sa13] can be found in [SaSch].

Fix an arbitrary natural number n. Throughout this and the next two sections we will assume $\mathbf{\Pi}^1_{n+1}$ determinacy or equivalently that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x (see [Ne95] and [MSW] for a proof of this equivalence due to Itay Neeman and W. Hugh Woodin).

The first direct limit system. Following the notation of [Sa13], we let $S_n(x) = M_n(x) \cap \mathbb{R}$ for all $x \in \mathbb{R}$. Moreover, we define for countable sets a and b that $b \in S_n(a)$ iff for comeager many g which are $\mathrm{Col}(\omega, a)$-generic over V, whenever $a_g, b_g \in \mathbb{R}$ are coding a and b relative to g, then $b_g \in S_n(a_g)$.

Definition 3.1. *A countable premouse $\mathcal{N}$ is n-*suitable *iff there is an ordinal δ such that*

(1) *$\mathcal{N} \vDash \mathrm{ZFC}^-$ and $\mathcal{N} \cap \mathrm{Ord} = \sup_{i<\omega}(\delta^{+i})^{\mathcal{N}}$,*
(2) *$\mathcal{N} \vDash$ "δ is the only Woodin cardinal",*
(3) *for every cutpoint $\gamma < \delta$ of $\mathcal{N}$, $S_{n+1}(\mathcal{N}|\gamma) = \mathcal{N}|(\gamma^+)^{\mathcal{N}}$, and*
(4) *$S_n(\mathcal{N}|(\delta^{+i})^{\mathcal{N}}) = \mathcal{N}|(\delta^{+(i+1)})^{\mathcal{N}}$ for all $i < \omega$.*

If $\mathcal{N}$ is an n-suitable premouse we denote the ordinal $\delta < \omega_1^V$ from Definition 3.1 by $\delta^{\mathcal{N}}$. Then $\mathcal{N} = M_n(\mathcal{N}|\delta^{\mathcal{N}})|((\delta^{\mathcal{N}})^{+\omega})^{M_n(\mathcal{N}|\delta^{\mathcal{N}})}$ for every n-suitable premouse $\mathcal{N}$, in particular the proper class premouse $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ exists.

Remark. If it exists, $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ is n-suitable, where δ_0 is the least Woodin cardinal in M_{n+1}. We denote this premouse by M_{n+1}^- and write $\Sigma_{M_{n+1}^-}$ for its canonical iteration strategy induced by the canonical iteration strategy $\Sigma_{M_{n+1}}$ for M_{n+1}.

We start with some definitions indicating how n-suitable premice can be iterated. The goal is to approximate the iteration strategy $\Sigma_{M_{n+1}^-}$ inside $M_n(x)$, where x is some real coding M_{n+1}^-.

Definition 3.2. *Let $\mathcal{N}$ be an n-suitable premouse and let $\mathcal{T}$ be a normal iteration tree on $\mathcal{N}$ of length $< \omega_1^V$.*

(1) *$\mathcal{T}$ is* correctly guided *iff for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, if b is the branch choosen for $\mathcal{T} \upharpoonright \lambda$ in $\mathcal{T}$ and $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda)$ exists*[4]*, then $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) \trianglelefteq M_n(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$.*
(2) *$\mathcal{T}$ is* short *iff $\mathcal{T}$ is correctly guided and in case $\mathcal{T}$ has limit length there is a cofinal well-founded branch b through $\mathcal{T}$ such that $\mathcal{T}^\frown b$ is correctly guided.*
(3) *$\mathcal{T}$ is* maximal *iff $\mathcal{T}$ is correctly guided and not short.*

Definition 3.3. *Let $\mathcal{N}$ be an n-suitable premouse. We say $\mathcal{N}$ is* short tree iterable *iff whenever $\mathcal{T}$ is a short tree on $\mathcal{N}$,*

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[4]By saying that the $\mathcal{Q}$-structure $\mathcal{Q}(b, \mathcal{T})$ exists for some tree $\mathcal{T}$ of limit length and cofinal well-founded branch b through $\mathcal{T}$, we mean that it exists and is ω_1-iterable.

(i) *if $\mathcal{T}$ has a last model, then every putative*[5] *iteration tree $\mathcal{U}$ extending $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ has a well-founded last model, and*

(ii) *if $\mathcal{T}$ has limit length, then there exists a unique cofinal well-founded branch b through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.*

This can be generalized to stacks of correctly guided normal trees.

Definition 3.4. *Let $\mathcal{N}$ be an n-suitable premouse and $m < \omega$. Then we say $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ is a* correctly guided finite stack *on $\mathcal{N}$ iff*

(i) *$\mathcal{N}_0 = \mathcal{N}$,*

(ii) *$\mathcal{N}_i$ is n-suitable and $\mathcal{T}_i$ is a correctly guided normal iteration tree on $\mathcal{N}_i$ which acts below $\delta^{\mathcal{N}_i}$ for all $i \leq m$,*

(iii) *for every $i < m$ either $\mathcal{T}_i$ has a last model which is equal to $\mathcal{N}_{i+1}$ and the iteration embedding $i^{\mathcal{T}_i} : \mathcal{N}_i \rightarrow \mathcal{N}_{i+1}$ exists or $\mathcal{T}_i$ is maximal and $\mathcal{N}_{i+1} = M_n(\mathcal{M}(\mathcal{T}_i))|(\delta(\mathcal{T}_i)^{+\omega})^{M_n(\mathcal{M}(\mathcal{T}_i))}$.*

Moreover, we say that $\mathcal{M}$ is the last model of $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ iff either

(i) *$\mathcal{T}_m$ has a last model which is equal to $\mathcal{M}$,*

(ii) *$\mathcal{T}_m$ is of limit length and short and there is a cofinal well-founded branch b through $\mathcal{T}_m$ such that $\mathcal{Q}(b, \mathcal{T})$ exists, $\mathcal{T}_m {}^\frown b$ is correctly guided, and $\mathcal{M} = \mathcal{M}_b^{\mathcal{T}}$, or*

(iii) *$\mathcal{T}_m$ is maximal and $\mathcal{M} = M_n(\mathcal{M}(\mathcal{T}_m))|(\delta(\mathcal{T}_m)^{+\omega})^{M_n(\mathcal{M}(\mathcal{T}_m))}$.*

Finally, we say that $\mathcal{M}$ is a correct iterate *of $\mathcal{N}$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$. In case there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ of length 1, i.e. such that $m = 0$, we say that $\mathcal{M}$ is a* pseudo-normal iterate *(or just* pseudo-iterate*) of $\mathcal{N}$.*

Analogous to Theorem 3.14 in [StW16] we also have a version of the comparison lemma for short tree iterable premice and pseudo-normal iterates.

Lemma 3.5 (Pseudo-comparison lemma)**.** *Let $\mathcal{N}$ and $\mathcal{M}$ be n-suitable premice which are short tree iterable. Then there is a common pseudo-normal iterate $\mathcal{R} \in M_n(y)$ such that $\delta^{\mathcal{R}} \leq (\max(\delta^{\mathcal{N}}, \delta^{\mathcal{M}})^+)^{M_n(y)}$, where y is a real canonically coding $\mathcal{N}$ and $\mathcal{M}$.*

The proof of Lemma 3.5 is similar to the proof of Theorem 3.14 in [StW16], so we omit it. Similarly, we have an analogue to the pseudo-genericity iteration (see Theorem 3.16 in [StW16]).

Lemma 3.6 (Pseudo-genericity iterations)**.** *Let $\mathcal{N}$ be an n-suitable premouse which is short tree iterable and let z be a real. Then there is a pseudo-normal iterate $\mathcal{R}$ of $\mathcal{N}$ in $M_n(y, z)$ such that z is $\mathbb{B}^{\mathcal{R}}$-generic over $\mathcal{R}$ and $\delta^{\mathcal{R}} \leq ((\delta^{\mathcal{N}})^+)^{M_n(y,z)}$, where y is a real canonically coding $\mathcal{N}$ and $\mathbb{B}^{\mathcal{R}}$ denotes Woodin's extender algebra inside $\mathcal{R}$.*

[5] An iteration tree $\mathcal{U}$ is a *putative iteration tree* if $\mathcal{U}$ satisfies all properties of an iteration tree, but in case $\mathcal{U}$ has a last model we allow this last model to be ill-founded.

For the definition of the direct limit system converging to HOD we need the notion of s-iterability. To define this, we first introduce some notation.

For an n-suitable premouse $\mathcal{N}$, a finite sequence of ordinals s, and some $k < \omega$ let

$$T_{s,k}^{\mathcal{N}} = \{(t, \ulcorner\phi\urcorner) \in [((\delta^{\mathcal{N}})^{+k})^{\mathcal{N}}]^{<\omega} \times \omega \mid \phi \text{ is a } \Sigma_1\text{-formula and } M_n(\mathcal{N}|\delta^{\mathcal{N}}) \vDash \phi[t,s]\},$$

where $\ulcorner\phi\urcorner$ denotes the Gödel number of ϕ. Let $Hull_1^{\mathcal{N}}$ denote an uncollapsed Σ_1 hull in $\mathcal{N}$. Then we let

$$\gamma_s^{\mathcal{N}} = \sup(Hull_1^{\mathcal{N}}(\{T_{s,k}^{\mathcal{N}} \mid k < \omega\}) \cap \delta^{\mathcal{N}})$$

and

$$H_s^{\mathcal{N}} = Hull_1^{\mathcal{N}}(\gamma_s^{\mathcal{N}} \cup \{T_{s,k}^{\mathcal{N}} \mid k < \omega\}).$$

Then $\gamma_s^{\mathcal{N}} = H_s^{\mathcal{N}} \cap \delta^{\mathcal{N}}$. For $s_m = (u_1, \dots, u_m)$ the sequence of the first m uniform indiscernibles, we write $\gamma_m^{\mathcal{N}} = \gamma_{s_m}^{\mathcal{N}}$ and $H_m^{\mathcal{N}} = H_{s_m}^{\mathcal{N}}$. Then we have that $\sup_{m\in\omega} \gamma_m^{\mathcal{N}} = \delta^{\mathcal{N}}$ (see Lemma 5.3 in [Sa13]).

Definition 3.7. *Let $\mathcal{N}$ be an n-suitable premouse and s a finite sequence of ordinals. Then $\mathcal{N}$ is* s-iterable *iff for every correctly guided finite stack $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ on $\mathcal{N}$ with last model $\mathcal{M}$ there is a sequence of branches $(b_i \mid i \leq m)$ and a sequence of embeddings $(\pi_i \mid i \leq m)$ such that*

- *(i) if $\mathcal{T}_i$ has successor length $\alpha + 1$, then $b_i = [0, \alpha]_{T_i}$ and $\pi_i = i_{0,\alpha}^{\mathcal{T}_i}$ is the corresponding iteration embedding for $i \leq m$,*
- *(ii) if $\mathcal{T}_m$ is short, then b_m is the unique cofinal well-founded branch through $\mathcal{T}_m$ such that $\mathcal{Q}(b_m, \mathcal{T}_m)$ exists and $\mathcal{T}_m{}^\frown b_m$ is correctly guided and $\pi_m = i_{b_m}^{\mathcal{T}_m}$ is the corresponding iteration embedding,*
- *(iii) if $\mathcal{T}_i$ is maximal, then b_i is a cofinal well-founded branch through $\mathcal{T}_i$ such that $\mathcal{M}_{b_i}^{\mathcal{T}_i} = \mathcal{N}_{i+1}$ if $i < m$ or $\mathcal{M}_{b_i}^{\mathcal{T}_i} = \mathcal{M}$ if $i = m$, and $\pi_i = i_{b_i}^{\mathcal{T}_i}$ is the corresponding iteration embedding for $i \leq m$, and*
- *(iv) if we let $\pi = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_0$ then for every $k < \omega$,*

$$\pi(T_{s,k}^{\mathcal{N}}) = T_{s,k}^{\mathcal{M}}.$$

In this case we say that the sequence $\vec{b} = (b_i \mid i \leq m)$ witnesses s-iterability *for $\vec{\mathcal{T}} = (\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ or that $\vec{b}$ is an* s-iterability branch *for $\vec{\mathcal{T}}$ and we write $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi$.*

Now for every two s-iterability branches for $\vec{\mathcal{T}}$ on $\mathcal{N}$ their corresponding iteration embeddings agree on $H_s^{\mathcal{N}}$.

Lemma 3.8 (Uniqueness of s-iterability embeddings, Lemma 5.5 in [Sa13])**.** *Let $\mathcal{N}$ be an n-suitable premouse, s a finite sequence of ordinals, and $\vec{\mathcal{T}}$ a correctly guided finite stack on $\mathcal{N}$. Moreover let $\vec{b}$ and $\vec{c}$ be s-iterability branches for $\vec{\mathcal{T}}$. Then*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_s^{\mathcal{N}} = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction H_s^{\mathcal{N}}.$$

The following lemma shows that some form of s-iterability for an n-suitable premouse $\mathcal{N}$ already implies that $\mathcal{N}$ has a correct iteration strategy in the following sense. Recall that s_m denotes the sequence of the first m uniform indiscernibles.

Lemma 3.9 (Lemma 5.6 in [Sa13])**.** *Let $\mathcal{N}$ be an n-suitable premouse which is s_m-iterable for every $m < \omega$. Then $\mathcal{N}$ has a correct iteration strategy, i.e. $\mathcal{N}$ has an ω_1-iteration strategy Σ such that whenever $\mathcal{T}$ is a correctly guided normal iteration tree of limit length on $\mathcal{N}$ and $b = \Sigma(\mathcal{T})$, then $\mathcal{M}_b^{\mathcal{T}}$ is n-suitable and $\mathcal{T}^\frown b$ is correctly guided.*

The uniqueness of s-iterability embeddings (Lemma 3.8) yields that for every n-suitable, s-iterable $\mathcal{N}$, every correctly guided finite stack $\vec{\mathcal{T}}$ on $\mathcal{N}$ and every s-iterability branch $\vec{b}$ for $\vec{\mathcal{T}}$, the embedding $\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_s^{\mathcal{N}}$ is independent of the choice of $\vec{b}$, but it might still depend on $\vec{\mathcal{T}}$. This motivates the following definition.

Definition 3.10. *Let $\mathcal{N}$ be an n-suitable premouse and s a finite sequence of ordinals. Then $\mathcal{N}$ is* strongly s-iterable *iff $\mathcal{N}$ is s-iterable and for every two correctly guided finite stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ on $\mathcal{N}$ with common last model $\mathcal{M}$ and s-iterability witnesses $\vec{b}$ and $\vec{c}$ for $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ respectively, we have that*

$$\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_s^{\mathcal{N}} = \pi_{\vec{\mathcal{U}},\vec{c}} \upharpoonright H_s^{\mathcal{N}}.$$

A so-called *bad sequence argument* shows the following lemma, which yields the existence of strongly s-iterable premice.

Lemma 3.11 (Lemma 5.9 in [Sa13])**.** *For every finite sequence of ordinals s and any short tree iterable n-suitable premouse $\mathcal{N}$ there is a pseudo-normal iterate $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}$ is strongly s-iterable.*

If $\mathcal{N}$ is strongly s-iterable and $\vec{\mathcal{T}}$ is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$, let $\pi_{\mathcal{N},\mathcal{M},s} : H_s^{\mathcal{N}} \rightarrow H_s^{\mathcal{M}}$ denote the embedding given by any s-iterability branch $\vec{b}$ for $\vec{\mathcal{T}}$. As $\mathcal{N}$ is strongly s-iterable, the embedding $\pi_{\mathcal{N},\mathcal{M},s}$ does not depend on the choice of $\vec{\mathcal{T}}$ and $\vec{b}$.

From now on, we will assume that M_{n+1}, the least proper class premouse with $n+1$ Woodin cardinals which are countable in V, exists and is ω_1-iterable and unique. Moreover we assume as above that $M_n^{\#}(y)$ exists and is ω_1-iterable for all reals y. By a theorem of Woodin these assumptions follow for example from $\mathbf{\Pi}_{n+2}^1$ determinacy (see [MSW]).

As before, let $\Sigma_{M_{n+1}}$ denote the canonical iteration strategy for M_{n+1}. Recall that we write $M_{n+1}^- = M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$, where δ_0 is the least Woodin cardinal in M_{n+1}, and $\Sigma_{M_{n+1}^-}$ for the canonical iteration strategy for M_{n+1}^- induced by $\Sigma_{M_{n+1}}$. Then M_{n+1}^- is n-suitable and strongly s_m-iterable for every m. Moreover, if $\vec{\mathcal{T}}$ is a correctly guided finite stack on M_{n+1}^- with last model $\mathcal{M}$, then $\pi_{M_{n+1}^-,\mathcal{M},s_m}$ agrees with the iteration embedding according to the canonical iteration strategy $\Sigma_{M_{n+1}^-}$ on $H_{s_m}^{M_{n+1}^-}$. The first direct limit system we define will consist of iterates of M_{n+1}^-.

Definition 3.12. *Let*

$$\mathcal{F}^+ = \{\mathcal{N} \mid \mathcal{N} \text{ is an iterate of } M_{n+1}^- \text{ via } \Sigma_{M_{n+1}^-}\}$$

and for $\mathcal{N}, \mathcal{M} \in \mathcal{F}^+$ let $\mathcal{N} \leq^+ \mathcal{M}$ iff $\mathcal{M}$ is an iterate of $\mathcal{N}$ via the tail strategy $\Sigma_{\mathcal{N}}$ as witnessed by some finite stack of iteration trees. Then we let $\mathcal{M}_\infty^+$ be the direct limit of $(\mathcal{F}^+, \leq^+)$ under the iteration maps.

Remark. The prewellordering $\leq^+$ on $\mathcal{F}^+$ is directed and the direct limit $\mathcal{M}_\infty^+$ is well-founded as the limit system $(\mathcal{F}^+, \leq^+)$ only consists of iterates of M_{n+1}^- via the canonical iteration strategy $\Sigma_{M_{n+1}^-}$.

Since $\mathcal{F}^+$ is not definable enough for our purposes, we now introduce another direct limit system which has the same direct limit $\mathcal{M}_\infty^+$.

Definition 3.13. *Let*

$$\mathcal{I} = \{(\mathcal{N}, s) \mid \mathcal{N} \text{ is } n\text{-suitable}, s \in [\mathrm{Ord}]^{<\omega}, \text{ and } \mathcal{N} \text{ is strongly } s\text{-iterable}\}$$

and

$$\mathcal{F} = \{H_s^{\mathcal{N}} \mid (\mathcal{N}, s) \in \mathcal{I}\}.$$

For $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}$ we let $(\mathcal{N}, s) \leq_{\mathcal{I}} (\mathcal{M}, t)$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let $\pi_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\mathcal{N}} \to H_t^{\mathcal{M}}$ denote the canonical corresponding embedding.

Remark. The prewellordering $\leq_{\mathcal{I}}$ on $\mathcal{I}$ is directed: Let $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}$. By Lemma 3.11 there exists an n-suitable premouse $\mathcal{R}$ which is strongly $(s \cup t)$-iterable. Let $\mathcal{S}$ be the result of simultaneously comparing $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{R}$ in the sense of Lemma 3.5. Then $(\mathcal{S}, s \cup t) \in \mathcal{I}$, $(\mathcal{N}, s) \leq_{\mathcal{I}} (\mathcal{S}, s \cup t)$, and $(\mathcal{M}, t) \leq_{\mathcal{I}} (\mathcal{S}, s \cup t)$, as desired.

Definition 3.14. *Let $\mathcal{M}_\infty$ be the direct limit of $(\mathcal{F}, \leq_{\mathcal{I}})$ under the embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$. For $(\mathcal{N}, s) \in \mathcal{I}$ let $\pi_{(\mathcal{N},s),\infty} : H_s^{\mathcal{N}} \to \mathcal{M}_\infty$ denote the corresponding direct limit embedding.*

The fact that $\mathcal{M}_\infty$ is well-founded follows from the next lemma.

Lemma 3.15 (Lemma 5.10 in [Sa13])**.** $\mathcal{M}_\infty = \mathcal{M}_\infty^+$.

A direct limit system in $\mathrm{HOD}^{M_n(x)}$**.** To obtain HOD of some inner model from the direct limit, we in particular need to show is that the direct limit is in fact contained in HOD of that inner model. In our setting we therefore need to internalize the direct limit system into the inner model $M_n(x, g)$ for some real x of sufficiently high Turing degree, where g is generic over $M_n(x)$ for the Lévy collapse up to its bottom inaccessible. Recall that we identify $M_n(x)[g]$ and $M_n(x, g)$ (see for example [SchSt09] on how this can be done in detail).

We first aim to define a direct limit system similar to $(\mathcal{F}, \leq_{\mathcal{I}})$ which is internal to $M_n(x)$ as in [Sa13]. In a second step, we then modify the system to obtain a direct limit system with the same direct limit which is definable in $M_n(x,g)$. For the rest of this article, we fix a real x that codes M_{n+1}^- and a generic g as above.

The notion of n-suitability from Definition 3.1 is already internal to $M_n(x)$ and $M_n(x,g)$, i.e. if $\mathcal{N} \in M_n(x)$ then $\mathcal{N}$ is n-suitable in V iff $\mathcal{N}$ is n-suitable in $M_n(x)$, because $M_n(x)$ is closed under the operation $y \mapsto S_n(y)$, similarly for $M_n(x,g)$. Moreover the definitions of short tree, maximal tree, and correctly guided finite stack we gave above are internal to $M_n(x)$ and $M_n(x,g)$ as well, as they can be defined only using the S_n-operation. The only notion we have to take care of is s-iterability since it is not even clear how the sets $T_{s,k}^{\mathcal{N}}$ can be identified inside $M_n(x)$.

This obstacle is solved by shrinking the direct limit system $(\mathcal{F}, \leq_{\mathcal{I}})$ to a dense subset as follows. Let κ be the least inaccessible cardinal in $M_n(x)$.

Definition 3.16. *Let*

$$\begin{aligned}\mathcal{G}_\kappa = \{\mathcal{N} \in M_n(x)|\kappa \mid \mathcal{N} \textit{ is } n\textit{-suitable and } M_n(x) \vDash \textit{“for some}\\ \textit{cutpoint } \eta, \delta^{\mathcal{N}} = \eta^+ \textit{ and } M_n(x)|\eta \textit{ is generic over } \mathcal{N} \textit{ for}\\ \textit{the } \delta^{\mathcal{N}}\textit{-generator version of the extender algebra at } \delta^{\mathcal{N}}\textit{”}\}.\end{aligned}$$

See for example Section 4.1 in [Fa] for an introduction to the δ-generator version of the extender algebra at some Woodin cardinal δ. For $\mathcal{N} \in \mathcal{G}_\kappa$ let $\eta^{\mathcal{N}}$ be the ordinal η witnessing that $\mathcal{N} \in \mathcal{G}_\kappa$. The following lemma shows how we can use $\mathcal{N} \in \mathcal{G}_\kappa$ to detect $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ inside $M_n(x)$. For some premouse $\mathcal{R} \in M_n(x)$ we denote the last model of a $\mathcal{P}$-construction above $\mathcal{R}$ performed inside $M_n(x)$ as introduced in [SchSt09] (see also Proposition 2.3 and Definition 2.4 in [Sa13]) by $\mathcal{P}^{M_n(x)}(\mathcal{R})$.

Lemma 3.17 (Lemma 5.11 in [Sa13])**.** *Let $\mathcal{N} \in M_n(x)|\kappa$ be an n-suitable premouse such that for some cutpoint $\eta < \delta^{\mathcal{N}}$ of $M_n(x)$, we have that $\mathcal{N}|\delta^{\mathcal{N}} \subseteq M_n(x)|(\eta^+)^{M_n(x)}$ and $M_n(x)|\eta$ is generic over $\mathcal{N}$ for the $\delta^{\mathcal{N}}$-generator version of the extender algebra at $\delta^{\mathcal{N}}$. Then $\mathcal{N} \in \mathcal{G}_\kappa$ and*

$$\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}) = M_n(\mathcal{N}|\delta^{\mathcal{N}}).$$

Using pseudo-genericity iterations (see Lemma 3.6) we can obtain the following corollary.

Corollary 3.18. *Let $\mathcal{N}$ be a short tree iterable n-suitable premouse such that $\mathcal{N} \in M_n(x)|\kappa$. Then there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ such that $\mathcal{M} \in \mathcal{G}_\kappa$ and $\mathcal{P}^{M_n(x)}(\mathcal{M}|\delta^{\mathcal{M}}) = M_n(\mathcal{M}|\delta^{\mathcal{M}})$.*

Now the following definition of s-iterability agrees with the previous one given in Definition 3.7 for n-suitable premice in $\mathcal{G}_\kappa$.

Definition 3.19. *For $\mathcal{N} \in \mathcal{G}_\kappa$, $s \in [\mathrm{Ord}]^{<\omega}$, and $k < \omega$ let*

$$T_{s,k}^{\mathcal{N},*} = \{(t, \ulcorner\phi\urcorner) \in [((\delta^{\mathcal{N}})^{+k})^{\mathcal{N}}]^{<\omega} \times \omega \mid \phi \text{ is a } \Sigma_1\text{-formula and } \mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}) \vDash \phi[t,s]\}.$$

Then we say for $\mathcal{N} \in \mathcal{G}_\kappa$ and $s \in [\mathrm{Ord}]^{<\omega}$ that $M_n(x) \vDash$ "$\mathcal{N}$ is s-iterable below κ" iff for every correctly guided finite stack $\vec{\mathcal{T}} = (\mathcal{T}_i, \mathcal{N}_i \mid i \leq m) \in M_n(x)|\kappa$ on $\mathcal{N}$ with last model $\mathcal{M} \in \mathcal{G}_\kappa$ and for every $\mathrm{Col}(\omega, |\mathcal{N} \cup \mathcal{M}|)$-generic G over $M_n(x)$, there is a sequence of branches $\vec{b} = (b_i \mid i \leq m) \in M_n(x)[G]$ and a sequence of embeddings $(\pi_i \mid i \leq m)$ satisfying $(1)-(3)$ in Definition 3.7 such that if we let $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_0$, then for every $k < \omega$,

$$\pi_{\vec{\mathcal{T}},\vec{b}}(T_{s,k}^{\mathcal{N},*}) = T_{s,k}^{\mathcal{M},*}.$$

In addition, we define $M_n(x) \vDash$ "$\mathcal{N}$ is strongly s-iterable below κ" analogous to Definition 3.10. For $\mathcal{N} \in \mathcal{G}_\kappa$, $s \in [\mathrm{Ord}]^{<\omega}$, and $k < \omega$, we have $T_{s,k}^{\mathcal{N},*} = T_{s,k}^{\mathcal{N}}$, so we will omit the $*$ for $\mathcal{N} \in \mathcal{G}_\kappa$. Using this, $\gamma_s^{\mathcal{N}}$ and $H_s^{\mathcal{N}}$ are defined as before. Then we can define the internal direct limit system as follows.

Definition 3.20. *Let*

$$\mathcal{I}_\kappa = \{(\mathcal{N}, s) \mid \mathcal{N} \in \mathcal{G}_\kappa, s \in [\mathrm{Ord}]^{<\omega}, \text{ and } M_n(x) \vDash \text{"}\mathcal{N} \text{ is strongly } s\text{-iterable below } \kappa\text{"}\}$$

and

$$\mathcal{F}_\kappa = \{H_s^{\mathcal{N}} \mid (\mathcal{N}, s) \in \mathcal{I}_\kappa\}.$$

Moreover, for $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}_\kappa$ we let $(\mathcal{N}, s) \leq_\kappa (\mathcal{M}, t)$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let as before $\pi_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\mathcal{N}} \to H_t^{\mathcal{M}}$ denote the canonical corresponding embedding.

Similar as before we have that for every $\mathcal{N} \in \mathcal{G}_\kappa$ and $s \in [\mathrm{Ord}]^{<\omega}$ there is a normal correct iterate $\mathcal{M}$ of $\mathcal{N}$ such that $(\mathcal{M}, s) \in \mathcal{I}_\kappa$. Using the fact that κ is inaccessible and a limit of cutpoints in $M_n(x)$ we can obtain the following lemma.

Lemma 3.21 (Lemma 5.14 in [Sa13])**.** *$\leq_\kappa$ is directed.*

Therefore we can again define the direct limit.

Definition 3.22. *Let $\mathcal{M}_{\infty,\kappa}$ be the direct limit of $(\mathcal{F}_\kappa, \leq_\kappa)$ under the embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$. Moreover, let $\delta_{\infty,\kappa} = \delta^{\mathcal{M}_{\infty,\kappa}}$ be the Woodin cardinal in $\mathcal{M}_{\infty,\kappa}$ and $\pi_{(\mathcal{N},s),\infty} : H_s^{\mathcal{N}} \to \mathcal{M}_{\infty,\kappa}$ be the direct limit embedding for all $(\mathcal{N}, s) \in \mathcal{I}_\kappa$.*

An argument similar to the one for Lemma 3.15 shows that this direct limit is well-founded as well. As we will use ideas from this proof in the next section, we will give some details here. We again first define another direct limit system which consists of iterates of M^-_{n+1} and then show that its direct limit $\mathcal{M}^+_{\infty,\kappa}$ is equal to $\mathcal{M}_{\infty,\kappa}$.

Definition 3.23. *Let*

$$\mathcal{F}^+_\kappa = \{\mathcal{Q} \in \mathcal{G}_\kappa \mid \mathcal{Q} \textit{ is the last model of a correctly guided finite stack on } M^-_{n+1} \textit{ via } \Sigma_{M^-_{n+1}}\}.$$

Moreover, let $\mathcal{P} \leq^+_\kappa \mathcal{Q}$ for $\mathcal{P}, \mathcal{Q} \in \mathcal{F}^+_\kappa$ iff there is a correctly guided finite stack on $\mathcal{P}$ according to the tail strategy $\Sigma_\mathcal{P}$ with last model $\mathcal{Q}$. In this case we let $i_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \rightarrow \mathcal{Q}$ denote the corresponding iteration embedding.

Then $\leq^+_\kappa$ on $\mathcal{F}^+_\kappa$ is directed, so we can define the direct limit.

Definition 3.24. *Let $\mathcal{M}^+_{\infty,\kappa}$ be the direct limit of $(\mathcal{F}^+_\kappa, \leq^+_\kappa)$ under the embeddings $i_{\mathcal{P},\mathcal{Q}}$. Moreover, let $i_{\mathcal{Q},\infty} : \mathcal{Q} \rightarrow \mathcal{M}^+_{\infty,\kappa}$ denote the direct limit embedding for all $\mathcal{Q} \in \mathcal{F}^+_\kappa$.*

Then it is easy to see that $\mathcal{M}^+_{\infty,\kappa}$ is well-founded as $\mathcal{F}^+_\kappa$ only consists of iterates of M^-_{n+1} according to the canonical iteration strategy $\Sigma_{M^-_{n+1}}$.

Lemma 3.25 (Lemma 5.15 in [Sa13])**.** *$\mathcal{M}^+_{\infty,\kappa} = \mathcal{M}_{\infty,\kappa}$ and hence $\mathcal{M}_{\infty,\kappa}$ is well-founded.*

Proof. We construct a sequence $(\mathcal{Q}_i \mid i < \omega)$ of iterates of M^-_{n+1} such that $\mathcal{Q}_i \in \mathcal{F}^+_\kappa$ for every $i < \omega$ and $(\mathcal{Q}_i \mid i < \omega)$ is cofinal in $\mathcal{G}_\kappa$, i.e. for every $\mathcal{N} \in \mathcal{G}_\kappa$ there is an $i < \omega$ such that $\mathcal{Q}_i$ is the last model of a correctly guided finite stack on $\mathcal{N}$.

We define $(\mathcal{Q}_i \mid i < \omega)$ together with a strictly increasing sequence $(\eta_i \mid i < \omega)$ of cutpoints of $M_n(x)|\kappa$ by induction on $i < \omega$. So let $\mathcal{Q}_0 = M^-_{n+1}$ and let $\eta_0 < \kappa$ be a cutpoint of $M_n(x)$. Moreover assume that we already constructed $(\mathcal{Q}_i \mid i \leq j)$ and $(\eta_i \mid i \leq j)$ with the above mentioned properties such that in addition $(\mathcal{Q}_i \mid i \leq j) \in M_n(x)|\eta_j$. Let $\mathcal{Q}^*_{j+1}$ be the result of simultaneously pseudo-comparing (in the sense of Lemma 3.5) all n-suitable premice $\mathcal{M}$ such that $\mathcal{M} \in \mathcal{G}_\kappa \cap M_n(x)|\eta_j$. Then in particular $\mathcal{Q}^*_{j+1}$ is a normal iterate of $\mathcal{Q}_j$ according to the canonical tail iteration strategy $\Sigma_{\mathcal{Q}_j}$, but $\mathcal{Q}^*_{j+1}$ might not be in $\mathcal{G}_\kappa$. Let ν be a cutpoint of $M_n(x)$ such that $\eta_j < \nu < \kappa$ and $\mathcal{Q}^*_{j+1} \in M_n(x)|\nu$. Note that such a ν exists as κ is inaccessible and a limit of cutpoints in $M_n(x)$. Let $\mathcal{Q}_{j+1}$ be the normal iterate of $\mathcal{Q}^*_{j+1}$ according to the canonical tail strategy $\Sigma_{\mathcal{Q}^*_{j+1}}$ of $\Sigma_{\mathcal{Q}_j}$ obtained by Woodin's genericity iteration such that $M_n(x)|\nu$ is generic over $\mathcal{Q}_{j+1}$ for the $\delta^{\mathcal{Q}_{j+1}}$-generator version of the extender algebra (see for example Section 4.1 in [Fa]). Then $\mathcal{Q}_{j+1} \in \mathcal{G}_\kappa$ is as desired. Finally choose $\eta_{j+1} < \kappa$ such that $\eta_{j+1} > \eta_j$ is a cutpoint in $M_n(x)$ and $(\mathcal{Q}_i \mid i \leq j+1) \in M_n(x)|\eta_{j+1}$.

Now we define an embedding $\sigma : \mathcal{M}_{\infty,\kappa} \rightarrow \mathcal{M}^+_{\infty,\kappa}$ as follows. Let $x \in \mathcal{M}_{\infty,\kappa}$. Since $(\mathcal{Q}_i \mid i < \omega)$ is cofinal in $\mathcal{G}_\kappa$, there are $i, m < \omega$ such that $(\mathcal{Q}_i, s_m) \in \mathcal{I}_\kappa$ and $x = \pi_{(\mathcal{Q}_i,s_m),\infty}(\bar{x})$ for some $\bar{x} \in H^{\mathcal{Q}_i}_{s_m} \subseteq \mathcal{Q}_i$. Then we let $\sigma(x) = i_{\mathcal{Q}_i,\infty}(\bar{x})$.

It follows as in the proof of Lemma 5.10 in [Sa13] that the definition of σ does not depend on the choice of $i, m < \omega$ and in fact $\sigma = \mathrm{id}$. □

Moreover, it is possible to compute $\delta_{\infty,\kappa}$.

Lemma 3.26 (Lemma 5.16 in [Sa13]). $\delta_{\infty,\kappa} = (\kappa^+)^{M_n(x)}$.

Direct limit systems in $\mathrm{HOD}^{M_n(x,g)}$**.** Finally, we will argue that $\mathcal{M}_{\infty,\kappa} \in \mathrm{HOD}^{M_n(x,g)}$ by first defining direct limit systems in various premice $M(y)$ satisfying certain properties definable in $M_n(x,g)$ and then showing that the direct limits $\mathcal{M}_{\infty,\kappa}^{M(y)}$ are equal to $\mathcal{M}_{\infty,\kappa}$. A similar approach but in a completely different setting can be found in [SaSch].
In what follows, we will let $(K(z))^N$ denote the core model constructed above a real z inside some n-small model N with n Woodin cardinals in the sense of [Sch06], i.e. the core model $K(z)$ is constructed between consecutive Woodin cardinals. Lemma 1.1 in [Sch06] (due to John Steel) implies that $(K(x))^{M_n(x)} = M_n(x)$. We will use this fact and consider more arbitrary premice with this property in the following definition. We state the definition in V, but we will later apply it inside $M_n(x)[g]$.

Definition 3.27. *Let $y \in {}^\omega\omega \cap M_n(x)[g]$. Then we say y is* pre-dlm-suitable *iff there is a proper class y-premouse $M(y)$ satisfying the following properties.*

(i) $M(y) \models \mathrm{ZFC}$,
(ii) *$M(y)$ is n-small and has n Woodin cardinals,*
(iii) *the least inaccessible cardinal in $M(y)$ is κ,*
(iv) *$M(y) = (K(y))^{M(y)}$, and*
(v) *there is a $\mathrm{Col}(\omega, {<}\kappa)$-generic h over $M(y)$ such that*

$$M(y)[h] = M_n(x)[g].$$

We also call such a y-premouse $M(y)$ pre-dlm-suitable *and say that $M(y)$* witnesses that y is pre-dlm-suitable.

Using this, we can define a version of the direct limit system $\mathcal{F}_\kappa$ inside arbitrary pre-dlm-suitable y-premice $M(y)$.

Definition 3.28. *Let $y \in {}^\omega\omega$ be pre-dlm-suitable as witnessed by $M(y)$. Then we let*

$$\mathcal{G}_\kappa^{M(y)} = \{\mathcal{N} \in M(y)|\kappa \mid \mathcal{N} \textit{ is } n\textit{-suitable and } M(y) \models \textit{“for some cutpoint } \eta, \delta^{\mathcal{N}} = \eta^+ \textit{ and } M(y)|\eta \textit{ is generic over } \mathcal{N} \textit{ for the } \delta^{\mathcal{N}}\textit{-generator version of the extender algebra at } \delta^{\mathcal{N}}\textit{”}\}.$$

Analogous as before, we can now define when an n-suitable premouse $\mathcal{N}$ is strongly s-iterable (below κ) inside some $M(y)$ by referring to $\mathcal{P}^{M(y)}(\mathcal{N}|\delta^{\mathcal{N}})$ in the definition of $(T_{s,k}^{\mathcal{N},*})^{M(y)}$. Let $H_s^{\mathcal{N},M(y)}$ be defined analogous to $H_s^{\mathcal{N}}$

inside $M(y)$ using $(T^{\mathcal{N},*}_{s,k})^{M(y)}$. For $M(y) = M_n(x)$ and $\mathcal{N} \in \mathcal{G}_\kappa$ this agrees with our previous definition of s-iterability.

Definition 3.29. *Let $y \in {}^\omega\omega$ be pre-dlm-suitable as witnessed by $M(y)$. Then we let*

$$\mathcal{I}^{M(y)}_\kappa = \{(\mathcal{N}, s) \mid \mathcal{N} \in \mathcal{G}^{M(y)}_\kappa, s \in [\mathrm{Ord}]^{<\omega}, \text{ and } M(y) \vDash \text{“}\mathcal{N} \text{ is strongly } s\text{-iterable below } \kappa\text{”}\}$$

and

$$\mathcal{F}^{M(y)}_\kappa = \{H^{\mathcal{N},M(y)}_s \mid (\mathcal{N}, s) \in \mathcal{I}^{M(y)}_\kappa\}.$$

Moreover, for $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}^{M(y)}_\kappa$ we let $(\mathcal{N}, s) \leq_{\mathcal{F}^{M(y)}_\kappa} (\mathcal{M}, t)$ iff in $M(y)$ there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let $\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{M},t)} : H^{\mathcal{N},M(y)}_s \to H^{\mathcal{M},M(y)}_t$ denote the canonical corresponding embedding in $M(y)$. Finally, let $\mathcal{M}^{M(y)}_{\infty,\kappa}$ denote the direct limit of $(\mathcal{F}^{M(y)}_\kappa, \leq_{\mathcal{F}^{M(y)}_\kappa})$ under these embeddings.

We will now strengthen this and define when a real $y \in {}^\omega\omega$ (or a y-premouse $M(y)$) is dlm-suitable. Again, we will apply this definition inside $M_n(x)[g]$ later.

Definition 3.30. *Let $y \in {}^\omega\omega \cap M_n(x)[g]$ be pre-dlm-suitable as witnessed by some y-premouse $M(y)$. We say say that y is* dlm-suitable *(witnessed by $M(y)$) iff*

(*i*) *for every $s \in [\mathrm{Ord}]^{<\omega}$ there is a premouse $\mathcal{N}$ such that $(\mathcal{N}, s) \in \mathcal{I}^{M(y)}_\kappa$ and*
(*ii*) *for every $\mathcal{N} \in \mathcal{G}^{M(y)}_\kappa$,*
$$\mathcal{P}^{M(y)}(\mathcal{N}|\delta^\mathcal{N}) = K^{M_n(x)[g]}(\mathcal{N}|\delta^\mathcal{N}).$$

Lemma 3.31. *$M_n(x)$ witnesses that x is dlm-suitable.*

Proof. The fact that $M_n(x)$ satisfies (i) follows from Lemma 3.11 and Corollary 3.18, so we only have to show (ii). Let $\mathcal{N} \in \mathcal{G}_\kappa$. Then $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^\mathcal{N}) = M_n(\mathcal{N}|\delta^\mathcal{N})$ by Lemma 3.17. Moreover, there is some G generic over the result of the $\mathcal{P}$-construction $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^\mathcal{N})$ for the $\delta^\mathcal{N}$-generator version of the extender algebra at $\delta^\mathcal{N}$ with $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^\mathcal{N})[G] = M_n(x)$. That means

$$M_n(\mathcal{N}|\delta^\mathcal{N})[G] = M_n(x).$$

Now,

$$K^{M_n(x)[g]}(\mathcal{N}|\delta^\mathcal{N}) = K^{M_n(\mathcal{N}|\delta^\mathcal{N})[G][g]}(\mathcal{N}|\delta^\mathcal{N}) = K^{M_n(\mathcal{N}|\delta^\mathcal{N})}(\mathcal{N}|\delta^\mathcal{N}) = M_n(\mathcal{N}|\delta^\mathcal{N}) = \mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^\mathcal{N}),$$

by generic absoluteness of the core model and Lemma 1.1 in [Sch06] (due to Steel). □

Condition (*ii*) in Definition 3.30 will ensure that for any dlm-suitable y-premouse $M(y)$ and $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}_\kappa \cap \mathcal{I}_\kappa^{M(y)}$ with $(\mathcal{N}, s) \leq_{\mathcal{F}_\kappa} (\mathcal{M}, t)$ and $(\mathcal{N}, s) \leq_{\mathcal{F}_\kappa^{M(y)}} (\mathcal{M}, t)$, the induced embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$ and $\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{M},t)}$ agree. Hence we can show in the following lemma that the direct limit $\mathcal{M}^{M(y)}_{\infty,\kappa}$ defined inside some dlm-suitable $M(y)$ will in fact be the same as the direct limit $\mathcal{M}_{\infty,\kappa}$ defined inside $M_n(x)$.

Lemma 3.32. *Let $y \in {}^\omega\omega$ be dlm-suitable as witnessed by $M(y)$. Then $\mathcal{F}_\kappa$ and $\mathcal{F}_\kappa^{M(y)}$ have cofinally many points in common and hence $\mathcal{M}_{\infty,\kappa} = \mathcal{M}^{M(y)}_{\infty,\kappa}$.*

Proof. Let h be $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M(y)$ such that $M(y)[h] = M_n(x)[g]$. Let $(\mathcal{N}, s) \in \mathcal{I}_\kappa$ and $(\mathcal{N}', s') \in \mathcal{I}_\kappa^{M(y)}$. We aim to show that there is some $(\mathcal{M}, t) \in \mathcal{I}_\kappa \cap \mathcal{I}_\kappa^{M(y)}$ such that $(\mathcal{N}, s) \leq_{\mathcal{F}_\kappa} (\mathcal{M}, t)$, i.e. $(\mathcal{N}, s) \leq_\kappa (\mathcal{M}, t)$, and $(\mathcal{N}', s') \leq_{\mathcal{F}_\kappa^{M(y)}} (\mathcal{M}, t)$. As condition (*ii*) in Definition 3.30 yields that the embeddings associated to $\mathcal{F}_\kappa$ and $\mathcal{F}_\kappa^{M(y)}$ agree, this suffices to show that $\mathcal{M}_{\infty,\kappa} = \mathcal{M}^{M(y)}_{\infty,\kappa}$.

Let $t = s \cup s'$. By assumption, there is a t-iterable premouse $\mathcal{R}$ in $M_n(x)$ and a t-iterable premouse $\mathcal{R}'$ in $M(y)$. Therefore we can assume that $\mathcal{N}$ and $\mathcal{N}'$ are both t-iterable in $M_n(x)$ and $M(y)$ respectively as we can replace them by the result of their coiteration with $\mathcal{R}$ and $\mathcal{R}'$ respectively.

By the choice of $M(y)$ and generic absoluteness of the core model we have

$$\begin{aligned}(1)\qquad M(y) &= (K(y))^{M(y)} = (K(y))^{M(y)[h]} \\ &= (K(y))^{M_n(x)[g]} = (K(y))^{M_n(x)[g\restriction\xi]},\end{aligned}$$

where $\xi < \kappa$ is such that $y \in M_n(x)[g \restriction \xi]$. Analogously, using Lemma 1.1 in [Sch06] due to Steel and generic absoluteness of the core model again,

$$\begin{aligned}(2)\qquad M_n(x) &= (K(x))^{M_n(x)} = (K(x))^{M_n(x)[g]} \\ &= (K(x))^{M(y)[h]} = (K(x))^{M(y)[h\restriction\xi']},\end{aligned}$$

where $\xi' < \kappa$ is such that $x \in M(y)[h \restriction \xi']$. In particular, every cutpoint in $M_n(x)[g \restriction \xi]$ is a cutpoint in $M(y)$ and hence we have the following claim.

Claim 1. *$M(y)$ and $M_n(x)$ have cofinally many common cutpoints below κ.*

Moreover, the equations yield

$$M(y) \subset M_n(x)[g \restriction \xi] \subset M(y)[h \restriction \zeta],$$

where $\xi' < \zeta < \kappa$ is such that $g \restriction \xi \in M(y)[h \restriction \zeta]$. By the intermediate model theorem (see for example Lemma 15.43 in [Je03]) this implies that $M_n(x)[g \restriction \xi]$ is a generic extension of $M(y)$ for a forcing of size less than κ.[6] Since $M_n(x)[g \restriction \xi]$ is a generic extension of $M_n(x)$ for a forcing of size

[6] I.e. $M(y)$ is a ground of $M_n(x)[g \restriction \xi]$. See for example [FHR15] or [Us17] for an introduction to the theory of grounds.

less than κ as well, this implies by Theorem 1.3 in [Us17] that there is some common inner model $W \subseteq M_n(x) \cap M(y)$ such that $M_n(x)[g \upharpoonright \xi]$ is a generic extension of W for a forcing of size less than κ.

As every generic extension via a forcing of size less than κ can be absorbed by the collapse of some ordinal $\beta < \kappa$, this yields that we can fix some ordinal $\beta < \kappa$ and some $\mathrm{Col}(\omega, \beta)$-generic $b \in M_n(x)[g]$ over W such that $x, y, \mathcal{N}, \mathcal{N}' \in W[b]$. Then $M_n(x)$ and $M(y)$ exist in $W[b]$ as definable subclasses because

$$(K(x))^{W[b]} = (K(x))^{M_n(x)[g \upharpoonright \xi]} = (K(x))^{M_n(x)} = M_n(x)$$

and similarly

$$(K(y))^{W[b]} = (K(y))^{M_n(x)[g \upharpoonright \xi]} = (K(y))^{M_n(x)[g]} = (K(y))^{M(y)[h]} = M(y)$$

by generic absoluteness of the core model again. Let $\dot{x}, \dot{y}, \dot{\mathcal{N}}$ and $\dot{\mathcal{N}'}$ be $\mathrm{Col}(\omega, \beta)$-names for $x, y, \mathcal{N}$ and $\mathcal{N}'$ in W. Moreover, let $p \in \mathrm{Col}(\omega, \beta)$ force all properties we need about $\dot{x}, \dot{y}, \dot{\mathcal{N}}$ and $\dot{\mathcal{N}'}$. For $q \leq_{\mathrm{Col}(\omega,\beta)} p$ let b_q be the $\mathrm{Col}(\omega, \beta)$-generic filter over W such that $\bigcup b_q$ agrees with q on $\mathrm{dom}(q)$ and with $\bigcup b$ everywhere else.

Now we construct $(\mathcal{M}, t) \in \mathcal{I}_\kappa \cap \mathcal{I}_\kappa^{M(y)}$. Let $\eta < \kappa$ be a cutpoint of both $M(y)$ and $M_n(x)$ such that $\xi, \xi' < \eta$, which exists by Claim 1. Then in fact $(\eta^+)^{M_n(x)} = (\eta^+)^{M(y)}$ as by Equations (1) and (2) at the beginning of the proof

$$(\eta^+)^{M(y)} \leq (\eta^+)^{M_n(x)[g \upharpoonright \xi]} = (\eta^+)^{M_n(x)} \leq (\eta^+)^{M(y)[h \upharpoonright \xi']} = (\eta^+)^{M(y)}.$$

By the same argument, $(\eta^+)^{K(\dot{x}^{b_q})} = (\eta^+)^{K(\dot{y}^{b_q})}$ for all $q \leq_{\mathrm{Col}(\omega,\beta)} p$.

Work in $W[b]$. Using Lemmas 3.5 and 3.6, we obtain an inner model $\mathcal{M}$ by pseudo-comparing all $(\dot{\mathcal{N}})^{b_q}$ and $(\dot{\mathcal{N}'})^{b_q}$ for $q \leq_{\mathrm{Col}(\omega,\beta)} p$ and simultaneously pseudo-genericity iterating such that $K(\dot{x}^{b_q})|\eta$ and $K(\dot{y}^{b_q})|\eta$ are generic over $\mathcal{M}$ and $\delta^{\mathcal{M}} = (\eta^+)^{K(\dot{x}^{b_q})} = (\eta^+)^{K(\dot{y}^{b_q})}$. Since $\mathcal{M}$ is definable in $W[b]$ from $\{b_q \mid q \leq_{\mathrm{Col}(\omega,\beta)} p\}$ and parameters from W, we have that in fact $\mathcal{M} \in W \subseteq M_n(x) \cap M(y)$, as $\mathcal{M}$ does not depend on the choice of the generic b. Moreover, $\mathcal{M}$ is a correct iterate of $\mathcal{N}$ in $M_n(x)$ and a correct iterate of $\mathcal{N}'$ in $M(y)$.

As argued above, we can assume that $\mathcal{N}$ and $\mathcal{N}'$ are t-iterable in $M_n(x)$ and $M(y)$ respectively for $t = s \cup s'$. Therefore $\mathcal{M}$ is t-iterable in both, $M_n(x)$ and $M(y)$. Hence, $(\mathcal{M}, t) \in \mathcal{I}_\kappa \cap \mathcal{I}_\kappa^{M(y)}$, $(\mathcal{N}, s) \leq_{\mathcal{F}_\kappa} (\mathcal{M}, t)$, and $(\mathcal{N}', s') \leq_{\mathcal{F}_\kappa^{M(y)}} (\mathcal{M}, t)$, as desired. □

This yields that $\mathcal{M}_{\infty,\kappa} \in \mathrm{HOD}^{M_n(x,g)}$.

4. HOD below $\delta_{\infty,\kappa}$

In this section we will show that $\mathrm{HOD}^{M_n(x,g)}$ and $\mathcal{M}_{\infty,\kappa}$ agree up to $\delta_{\infty,\kappa}$ by generalizing the arguments in Section 3.4 in [StW16].

Definition 4.1. *Let $\mathcal{N}$ be an n-suitable premouse and let $\kappa^{\mathcal{N}} > \delta^{\mathcal{N}}$ be the least inaccessible cardinal in $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ above $\delta^{\mathcal{N}}$. Let H be* $\mathrm{Col}(\omega, {<}\kappa^{\mathcal{N}})$*-generic over $M_n(\mathcal{N}|\delta^{\mathcal{N}})$. Then we say that $M_n(\mathcal{N}|\delta^{\mathcal{N}})[H]$ is a* derived model *of $\mathcal{N}$.*

We have the following useful lemma for $\mathcal{G}_\kappa = \mathcal{G}_\kappa^{M_n(x)}$.

Lemma 4.2. *Let $\mathcal{N}$ be an n-suitable premouse such that $\mathcal{N} \in \mathcal{G}_\kappa$. Then $M_n(x, g)$ is a derived model of $\mathcal{N}$.*

Proof. As $\mathcal{N} \in \mathcal{G}_\kappa$ we have that $M_n(x)|\eta$ is generic over $\mathcal{N}$ for the $\delta^{\mathcal{N}}$-generator version of the extender algebra at $\delta^{\mathcal{N}}$ for some cutpoint η of $M_n(x)$ such that $\delta^{\mathcal{N}} = \eta^+ < \kappa$. Moreover, using Lemma 3.17 we have that $M_n(\mathcal{N}|\delta^{\mathcal{N}})[G] = M_n(x)$ for some G generic over $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ for the $\delta^{\mathcal{N}}$-generator version of the extender algebra at $\delta^{\mathcal{N}}$. Hence κ is the least inaccessible cardinal of $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ above $\delta^{\mathcal{N}}$, i.e. $\kappa = \kappa^{\mathcal{N}}$.
Since g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$ it follows by the factor lemma that there is an H which is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ such that $M_n(x, g) = M_n(\mathcal{N}|\delta^{\mathcal{N}})[H]$. Thus $M_n(x, g)$ is a derived model of $\mathcal{N}$. $\square$

Using this lemma we can prove a generalization of Woodin's derived model resemblance (see Lemma 3.39 in [StW16]). To do this we need to expand our direct limit $\mathcal{M}_{\infty,\kappa}$ to the proper class premouse $\hat{\mathcal{M}}_{\infty,\kappa} = M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa})$ and define a direct limit system $\hat{\mathcal{F}}_\kappa$ of expansions of the elements of $\mathcal{F}_\kappa$ which converges to $\hat{\mathcal{M}}_{\infty,\kappa}$.
For an n-suitable premouse $\mathcal{N}$ and $s \in [\mathrm{Ord}]^{<\omega}$ with $max(s)$ a uniform indiscernible above the Woodin cardinals in $M_n(\mathcal{N}|\delta^{\mathcal{N}})$, we let $\hat{\mathcal{N}} = M_n(\mathcal{N}|\delta^{\mathcal{N}})$ be the proper class expansion of $\mathcal{N}$, $s^- = s \setminus max(s)$,

$$\gamma_s^{\hat{\mathcal{N}}} = \sup(Hull_1^{\hat{\mathcal{N}}|max(s)}(s^-) \cap \delta^{\mathcal{N}}),$$

and

$$H_s^{\hat{\mathcal{N}}} = Hull_1^{\hat{\mathcal{N}}|max(s)}(\gamma_s^{\hat{\mathcal{N}}} \cup s^-).$$

Now we let

$$\hat{\mathcal{F}}_\kappa = \{H_s^{\hat{\mathcal{N}}} \mid (\mathcal{N}, s) \in \mathcal{I}_\kappa\}$$

and for $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}_\kappa$ with $(\mathcal{N}, s) \leq_\kappa (\mathcal{M}, t)$ we let $\hat{\pi}_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\hat{\mathcal{N}}} \to H_t^{\hat{\mathcal{M}}}$ denote the canonical corresponding embedding obtained by lifting the correctly guided finite stack on $\mathcal{N}$ witnessing that $(\mathcal{N}, s) \leq_\kappa (\mathcal{M}, t)$ to a finite stack on $\hat{\mathcal{N}}$.
Finally, let $\hat{\mathcal{M}}_{\infty,\kappa}$ be the direct limit of $(\hat{\mathcal{F}}_\kappa, \leq_\kappa)$ under the embeddings $\hat{\pi}_{(\mathcal{N},s),(\mathcal{M},t)}$. Then it is easy to see that $M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}) = \hat{\mathcal{M}}_{\infty,\kappa}$ and moreover the embeddings $\hat{\pi}_{(\mathcal{N},s),(\mathcal{M},t)}$ extend the corresponding embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$ from $\mathcal{F}_\kappa$. Let $\hat{\pi}_{(\mathcal{N},s),\infty} : H_s^{\hat{\mathcal{N}}} \to \hat{\mathcal{M}}_{\infty,\kappa}$ for $(\mathcal{N}, s) \in \mathcal{I}_\kappa$ denote the direct limit embedding.

Analogously, we also define a direct limit system $\hat{\mathcal{F}}_\kappa^+$ of premice expanding the n-suitable premice in $\mathcal{F}_\kappa^+$. We let $\delta^{\hat{\mathcal{Q}}}$ be the least Woodin cardinal in the premouse $\hat{\mathcal{Q}}$, if it exists, and

$$\hat{\mathcal{F}}_\kappa^+ = \{\hat{\mathcal{Q}} \mid \hat{\mathcal{Q}} \text{ is the last model of a correctly guided finite}$$
$$\text{stack on } M_{n+1} \text{ via } \Sigma_{M_{n+1}} \text{ and } \hat{\mathcal{Q}}|((\delta^{\hat{\mathcal{Q}}})^{+\omega})^{\hat{\mathcal{Q}}} \in \mathcal{G}_\kappa\},$$

where correctly guided finite stacks on the proper class model M_{n+1} are defined by lifting the iteration trees in a correctly guided finite stack on the n-suitable premouse M_{n+1}^- to trees on the proper class expansions. Analogous to $\mathcal{F}_\kappa^+$ we can define a directed prewellordering $\hat{\mathcal{P}} \leq_\kappa^+ \hat{\mathcal{Q}}$ and maps $\hat{i}_{\hat{\mathcal{P}},\hat{\mathcal{Q}}} : \hat{\mathcal{P}} \rightarrow \hat{\mathcal{Q}}$ induced by $\Sigma_{M_{n+1}}$ for $\hat{\mathcal{P}}, \hat{\mathcal{Q}} \in \hat{\mathcal{F}}_\kappa^+$. Let $\hat{\mathcal{M}}_{\infty,\kappa}^+ = M_n(\mathcal{M}_{\infty,\kappa}^+|\delta^{\mathcal{M}_{\infty,\kappa}^+})$ be the direct limit of $(\hat{\mathcal{F}}_\kappa^+, \leq_\kappa^+)$ via these embeddings and let $\hat{i}_{\hat{\mathcal{P}},\infty} : \hat{\mathcal{P}} \rightarrow \hat{\mathcal{M}}_{\infty,\kappa}^+$ for $\hat{\mathcal{P}} \in \hat{\mathcal{F}}_\kappa^+$ be the corresponding direct limit embedding.

Fix an H which is $\operatorname{Col}(\omega, {<}\kappa_\infty)$-generic over $M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa})$, where κ_∞ is the least inaccessible cardinal of $M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa})$ above $\delta_{\infty,\kappa}$. Moreover, we choose for any ordinal α an arbitrary $(\mathcal{N}, s) \in \mathcal{I}_\kappa$ such that $\alpha \in s^-$ and let $\alpha^* = \hat{\pi}_{(\mathcal{N},s),\infty}(\alpha)$. Note that the value of α^* does not depend on the choice of $(\mathcal{N}, s)$. We also let $t^* = \{\alpha^* \mid \alpha \in t\}$ for $t \in [\mathrm{Ord}]^{<\omega}$.

Now we can prove the generalization of Woodin's derived model resemblance to our setting.

Lemma 4.3 (Derived model resemblance for $M_n(x, g)$)**.** *Let $\mathcal{N}$ be an n-suitable premouse and $s \in [\mathrm{Ord}]^{<\omega}$ such that $(\mathcal{N}, s) \in \mathcal{I}_\kappa$. Let $\bar{\xi} < \gamma_s^{\mathcal{N}}$, $\xi = \pi_{(\mathcal{N},s),\infty}(\bar{\xi})$ and $t \in [\mathrm{Ord}]^{<\omega}$. Moreover let $\varphi(v_0, v_1, v_2)$ be a formula in the language of set theory. Then the following are equivalent.*

(*a*) $M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa})[H] \vDash \varphi(\mathcal{M}_{\infty,\kappa}, \xi, t^*)$,

(*b*) $M_n(x, g) \vDash$ *"there is some dlm-suitable $y \in {}^\omega\omega$ witnessed by $M(y)$ with $\mathcal{N} \in M(y)$ and a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M} \in M(y)$ such that whenever $\mathcal{R} \in \mathcal{G}_\kappa^{M(y)}$ is the last model of a correctly guided finite stack on $\mathcal{M}$, then $\varphi(\mathcal{R}, \pi_{(\mathcal{N},s),(\mathcal{R},s)}^{M(y)}(\bar{\xi}), t)$".*

Proof. To prove that (*a*) implies (*b*) we assume toward a contradiction that (*b*) is false. So for all dlm-suitable $y \in {}^\omega\omega$ and $M(y)$ witnessing this with $\mathcal{N} \in M(y)$ and all correctly guided finite stacks on $\mathcal{N}$ with last model $\mathcal{M} \in M(y)$, there is a correctly guided finite stack on $\mathcal{M}$ with last model $\mathcal{R} \in \mathcal{G}_\kappa^{M(y)}$ such that $M_n(x, g) \vDash \neg\varphi(\mathcal{R}, \pi_{(\mathcal{N},s),(\mathcal{R},s)}^{M(y)}(\bar{\xi}), t)$.

We can assume without loss of generality that $\mathcal{N} \in M_n(x)$ is the last model of a correctly guided finite stack on M_{n+1}^- via the canonical iteration strategy $\Sigma_{M_{n+1}^-}$ and strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M_{n+1}^-}$. Moreover, we can assume that $\mathcal{N} \in \mathcal{F}_\kappa^+$, i.e. that in addition $\mathcal{N} \in \mathcal{G}_\kappa$. If this is not already the case, we replace $\mathcal{N}$ by a pseudo-iterate of the result of the pseudo-comparison of $\mathcal{N}$ with M_{n+1}^- using Lemma 3.5 and Corollary 3.18.

Claim 1. *There are n-suitable premice $\mathcal{N}_k \in \mathcal{F}_\kappa^+$ for $k < \omega$ which are cofinal in $\mathcal{F}_\kappa^+$ such that $\mathcal{N}_0 = \mathcal{N}$ and for all $k < \omega$,*

$$M_n(x, g) \vDash \neg\varphi(\mathcal{N}_k, \bar{\xi}_k, t),$$

where $\bar{\xi}_k = i_{\mathcal{N}_0,\mathcal{N}_k}(\bar{\xi})$ is the image of $\bar{\xi}$ under the iteration map induced by $\Sigma_{M_{n+1}^-}$.

Proof. Let $(\mathcal{Q}_i \mid i < \omega)$ be an enumeration of $\mathcal{F}_\kappa^+$ and $\mathcal{N}_0 = \mathcal{N}$. Then we construct $\mathcal{N}_{k+1}$ inductively. So assume that we already constructed $\mathcal{N}_k$ and pseudo-coiterate $\mathcal{N}_k$ with $\mathcal{Q}_k$ to some model $\mathcal{N}_k^*$ (see Lemma 3.5). By assumption (b) is false, so let $\mathcal{R}$ be a counterexample witnessing this for $\mathcal{N}_k^*$ and the dlm-suitable premouse $M_n(x)$. That means $\mathcal{R} \in \mathcal{G}_\kappa$ is the last model of a correctly guided finite stack on $\mathcal{N}_k^*$ such that $M_n(x, g) \vDash \neg\varphi(\mathcal{R}, i_{\mathcal{N},\mathcal{R}}(\bar{\xi}), t)$ as $i_{\mathcal{N},\mathcal{R}} \restriction H_s^{\mathcal{N}} = \pi_{(\mathcal{N},s),(\mathcal{R},s)}$. But $\mathcal{R} \in \mathcal{F}_\kappa^+$ since $\mathcal{R} \in \mathcal{G}_\kappa$ and it is a correct iterate of $\mathcal{Q}_k$. Thus we can let $\mathcal{N}_{k+1} = \mathcal{R}$. □

Since $\mathcal{N}_k \in \mathcal{F}_\kappa^+$, in particular $\mathcal{N}_k \in \mathcal{G}_\kappa$ and Lemma 4.2 implies that $M_n(x, g)$ is a derived model of $\mathcal{N}_k$. Hence it follows that

$$M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k}) \vDash \text{“}1 \Vdash^{\mathbb{P}_k} \neg\varphi(\mathcal{N}_k, \bar{\xi}_k, t)\text{”},$$

where the forcing $\mathbb{P}_k$ is the Lévy collapse up to the first inaccessible cardinal of $M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k})$ above $\delta^{\mathcal{N}_k}$. Since $(\mathcal{N}_k \mid k < \omega)$ is cofinal in $\mathcal{F}_\kappa^+$, it follows that the direct limit of $(\mathcal{N}_k, i_{\mathcal{N}_k,\mathcal{N}_l} \mid k < l < \omega)$ is equal to $\mathcal{M}_{\infty,\kappa}^+$. Let $\hat{\mathcal{N}_k} = M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k})$ and let $\hat{i}_{\hat{\mathcal{N}}_k,\infty} : \hat{\mathcal{N}_k} \to M_n(\mathcal{M}_{\infty,\kappa}^+|\delta^{\mathcal{M}_{\infty,\kappa}^+}) = \hat{\mathcal{M}}_{\infty,\kappa}^+$ be the corresponding extension of the direct limit map $i_{\mathcal{N}_k,\infty}$.
Then we have for all sufficently large k that

$$M_n(\mathcal{M}_{\infty,\kappa}^+|\delta^{\mathcal{M}_{\infty,\kappa}^+}) \vDash \text{“}1 \Vdash^{\mathbb{P}} \neg\varphi(\mathcal{M}_{\infty,\kappa}^+, i_{\mathcal{N}_k,\infty}(\bar{\xi}_k), \hat{i}_{\hat{\mathcal{N}}_k,\infty}[t])\text{”},$$

where $\mathbb{P}$ denotes the Lévy collapse up to the first inaccessible cardinal of $M_n(\mathcal{M}_{\infty,\kappa}^+|\delta^{\mathcal{M}_{\infty,\kappa}^+})$ above $\delta^{\mathcal{M}_{\infty,\kappa}^+}$. Since we assumed that $\mathcal{N}$ is strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M_{n+1}^-}$ and $\bar{\xi} < \gamma_s^{\mathcal{N}}$, it follows that $i_{\mathcal{N}_k,\infty}(\bar{\xi}_k) = i_{\mathcal{N},\infty}(\bar{\xi}) = \pi_{(\mathcal{N},s),\infty}(\bar{\xi}) = \xi$ as $\bar{\xi}_k = i_{\mathcal{N},\mathcal{N}_k}(\bar{\xi})$.
Let $k < \omega$ be large enough such that $(\mathcal{N}_k, s \cup t) \in \mathcal{I}_\kappa$ and $\hat{i}_{\hat{\mathcal{N}_l},\hat{\mathcal{N}}_{l+1}}(s) = s$ for all $l \geq k$. Such a k exists by a so-called bad sequence argument similar to the one in the proof of Lemma 5.8 in [Sa13]. Then consider the map

$$\hat{\sigma} : M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}) \to M_n(\mathcal{M}_{\infty,\kappa}^+|\delta^{\mathcal{M}_{\infty,\kappa}^+})$$

which is the canonical extension of the map $\sigma : \mathcal{M}_{\infty,\kappa} \to \mathcal{M}_{\infty,\kappa}^+$ defined in the proof of Lemma 3.25, i.e. for $x \in \hat{\mathcal{M}}_{\infty,\kappa}$, say $x = \hat{\pi}_{(\mathcal{N}_k,s),\infty}(\bar{x})$ for some $\bar{x} \in H_s^{\hat{\mathcal{N}}_k}$ and $k < \omega$, let $\hat{\sigma}(x) = \hat{i}_{\hat{\mathcal{N}}_k,\infty}(\bar{x})$. Then it follows as in the proof of Lemma 3.25 that $\hat{\sigma} \restriction (\delta_{\infty,\kappa} + 1) = \sigma \restriction (\delta_{\infty,\kappa} + 1) = \mathrm{id}$ and moreover we have that $\hat{\sigma}[t^*] = \hat{\sigma}(\hat{\pi}_{(\mathcal{N}_k, s \cup t),\infty}[t]) = \hat{i}_{\hat{\mathcal{N}}_k,\infty}[t]$. Therefore pulling back under $\hat{\sigma}$ yields that

$$M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}) \vDash \text{“}1 \Vdash^{\mathbb{P}} \neg\varphi(\mathcal{M}_{\infty,\kappa}, \xi, t^*)\text{”},$$

for $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$. This is the desired contradiction to (a).

To show that (b) implies (a) we now assume that (b) is true. Let $M(y)$ be the dlm-suitable premouse with $\mathcal{N} \in M(y)$ given by (b). As before we can assume without loss of generality that $\mathcal{N}$ is the last model of a correctly guided finite stack on M_{n+1}^- via the canonical iteration strategy $\Sigma_{M_{n+1}^-}$, strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M_{n+1}^-}$, and that $\mathcal{N} \in \mathcal{G}_\kappa \cap \mathcal{G}_\kappa^{M(y)}$ using Lemma 3.32.

Claim 2. *There are n-suitable premice $\mathcal{N}_k \in \mathcal{F}_\kappa^+$ for $k < \omega$ which are cofinal in $\mathcal{F}_\kappa^+$ such that $\mathcal{N}_0 = \mathcal{N}$ and for all $k < \omega$,*

$$M_n(x, g) \vDash \varphi(\mathcal{N}_k, \bar{\xi}_k, t),$$

where $\bar{\xi}_k = i_{\mathcal{N}_0,\mathcal{N}_k}(\bar{\xi})$ is the image of $\bar{\xi}$ under the iteration map induced by $\Sigma_{M_{n+1}^-}$.

Proof. As before let $(\mathcal{Q}_i \mid i < \omega)$ be an enumeration of $\mathcal{F}_\kappa^+$, let $\mathcal{N}_0 = \mathcal{N}$ and construct $\mathcal{N}_{k+1}$ inductively. Assume that we already constructed $\mathcal{N}_k$ and let $\mathcal{M}$ be the last model of a correctly guided finite stack on $\mathcal{N}$ in $M_n(x, g)$ witnessing that (b) is true. Simultaneously pseudo-coiterate $\mathcal{M}$ with $\mathcal{N}_k$ and $\mathcal{Q}_k$ to some premouse $\mathcal{N}_k^*$. Using genericity iterations and Lemma 3.32, there is a pseudo-iterate $\mathcal{R}$ of $\mathcal{N}_k^*$ such that $\mathcal{R} \in \mathcal{G}_\kappa \cap \mathcal{G}_\kappa^{M(y)}$ (see also Corollary 3.18). In particular, we have that $M_n(x, g) \vDash \varphi(\mathcal{R}, i_{\mathcal{N},\mathcal{R}}(\bar{\xi}), t)$ using (b) as $i_{\mathcal{N},\mathcal{R}}(\bar{\xi}) = \pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\xi}) = \pi^{M(y)}_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\xi})$ by dlm-suitability of $M(y)$. Moreover, $\mathcal{R}$ is the last model of a correctly guided finite stack on $\mathcal{Q}_k$ and thus $\mathcal{R} \in \mathcal{F}_\kappa^+$, so we can let $\mathcal{N}_{k+1} = \mathcal{R}$. □

As before we can use this claim to obtain that

$$M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}) \vDash \text{“}1 \Vdash^{\mathbb{P}} \varphi(\mathcal{M}_{\infty,\kappa}, \xi, t^*)\text{”},$$

for $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$, which proves (a). □

By the derived model resemblance we have that $\hat{\mathcal{M}}_{\infty,\kappa}[H]$ is elementarily equivalent to $M_n(x, g)$. Therefore we can consider the direct limit system $\hat{\mathcal{F}}_\kappa$ of the model $\hat{\mathcal{M}}_{\infty,\kappa}[H]$ and call it $\hat{\mathcal{F}}^* = (\hat{\mathcal{F}}_\kappa)^{\hat{\mathcal{M}}_{\infty,\kappa}[H]}$. Moreover, the derived model resemblance implies that $\mathcal{M}_{\infty,\kappa}$ is strongly s^*-iterable in $\hat{\mathcal{M}}_{\infty,\kappa}[H]$ for all $s \in [\mathrm{Ord}]^{<\omega}$. So we can consider its direct limit embedding

$$\pi_\infty = \bigcup\{(\hat{\pi}_{(\mathcal{M}_{\infty,\kappa}, s^*),\infty})^{\hat{\mathcal{F}}^*} \mid s \in [\mathrm{Ord}]^{<\omega}\}$$

in the system $\hat{\mathcal{F}}^*$.

Lemma 4.4. *For all $\eta < \delta_{\infty,\kappa}$ we have that $\pi_\infty(\eta) = \eta^*$.*

Proof. This is again a consequence of the derived model resemblance (Lemma 4.3). Consider the dlm-suitable premouse $M_n(x)$. Let $\eta = \pi_{(\mathcal{N},s),\infty}(\bar{\eta})$ for some $(\mathcal{N}, s) \in \mathcal{I}_\kappa$ and $\bar{\eta} < \gamma_s^{\mathcal{N}}$ and consider the formula

$$\varphi(v_0, v_1, v_2, v_3) = \text{``}(v_0, v_1) \in \mathcal{I}_\kappa, v_2 < \gamma_{v_1}^{v_0}, \text{ and } \pi_{(v_0,v_1),\infty}(v_2) = v_3\text{''}.$$

Then we have for every $\mathcal{R} \in \mathcal{G}_\kappa$ which is the last model of a correctly guided finite stack on $\mathcal{N}$ that $M_n(x, g) \vDash \varphi(\mathcal{R}, s, \pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\eta}), \eta)$. So Lemma 4.3 yields that $\hat{\mathcal{M}}_{\infty,\kappa}[H] \vDash \varphi(\mathcal{M}_{\infty,\kappa}, s^*, \eta, \eta^*)$ and thus $(\pi_{(\mathcal{M}_{\infty,\kappa},s^*),\infty})^{\hat{\mathcal{F}}^*}(\eta) = \eta^*$, as desired. □

Theorem 4.5. $V_{\delta_{\infty,\kappa}}^{\mathrm{HOD}^{M_n(x,g)}} = V_{\delta_{\infty,\kappa}}^{\mathcal{M}_{\infty,\kappa}}$.

Proof. By the internal definition of $\mathcal{M}_{\infty,\kappa}$ from Lemma 3.32 we have that $V_{\delta_{\infty,\kappa}}^{\mathcal{M}_{\infty,\kappa}} \subseteq V_{\delta_{\infty,\kappa}}^{\mathrm{HOD}^{M_n(x,g)}}$. For the other inclusion we first show the following claim.

Claim 1. *$\pi_\infty \upharpoonright \alpha \in \hat{\mathcal{M}}_{\infty,\kappa}$ for all $\alpha < \delta_{\infty,\kappa}$.*

Proof. As $\alpha < \delta_{\infty,\kappa}$, there exists an $s \in [\mathrm{Ord}]^{<\omega}$ such that $\alpha < \gamma_{s^*}^{\hat{\mathcal{M}}_{\infty,\kappa}}$. For this s we have by definition that $\pi_\infty \upharpoonright \alpha = (\hat{\pi}_{(\mathcal{M}_{\infty,\kappa},s^*),\infty})^{\hat{\mathcal{F}}^*} \upharpoonright \alpha$. Since $\hat{\mathcal{F}}^* = (\hat{\mathcal{F}}_\kappa)^{\hat{\mathcal{M}}_{\infty,\kappa}[H]}$ it follows that $\pi_\infty \upharpoonright \alpha \in \hat{\mathcal{M}}_{\infty,\kappa}[H]$. Therefore $\pi_\infty \upharpoonright \alpha \in \hat{\mathcal{M}}_{\infty,\kappa}$ by homogeneity of the forcing $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$. □

Now let $A \in V_{\delta_{\infty,\kappa}}^{\mathrm{HOD}^{M_n(x,g)}}$ be arbitrary. Let $\alpha < \delta_{\infty,\kappa}$ be such that $A \subset \alpha$ is defined over $M_n(x, g)$ by a formula φ with ordinal parameters from $t \in [\mathrm{Ord}]^{<\omega}$ and let $\beta < \alpha$ be arbitrary. That means $\beta \in A$ iff $M_n(x, g) \vDash \varphi(\beta, t)$. The derived model resemblance (Lemma 4.3) yields that this is the case iff $\hat{\mathcal{M}}_{\infty,\kappa}[H] \vDash \varphi(\beta^*, t^*)$. Since $\beta < \alpha < \delta_{\infty,\kappa}$, we have that $\beta^* = \pi_\infty(\beta)$ by Lemma 4.4. Moreover, we have by Claim 1 that $\pi_\infty \upharpoonright \alpha \in \hat{\mathcal{M}}_{\infty,\kappa}$. Therefore, it follows by homogeneity of the forcing $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$ that $A \in \hat{\mathcal{M}}_{\infty,\kappa}$ since t^* is a fixed parameter in $\hat{\mathcal{M}}_{\infty,\kappa}$. Thus $A \in V_{\delta_{\infty,\kappa}}^{\mathcal{M}_{\infty,\kappa}}$, as desired. □

5. The full HOD in $M_n(x, g)$

To compute the full model $\mathrm{HOD}^{M_n(x,g)}$, i.e. prove Theorem 1.1, we first show the following lemma.

Lemma 5.1. *$\mathrm{HOD}^{M_n(x,g)} = M_n(A)$ for some $M_n(x, g)$-definable set $A \subseteq \omega_2^{M_n(x,g)}$.*

Proof. Let $\mathbb{V}$ denote the Vopěnka algebra in $M_n(x, g)$ for making a real generic over $\mathrm{HOD}^{M_n(x,g)}$. By Vopěnka's theorem (see for example Theorem 15.46 in [Je03] or Theorem 9.0.1 in [La17]) there is a $\mathbb{V}$-generic G_x over $\mathrm{HOD}^{M_n(x,g)}$ such that $x \in \mathrm{HOD}^{M_n(x,g)}[G_x]$ and in fact $\mathrm{HOD}^{M_n(x,g)}[G_x] =$

$\mathrm{HOD}_x^{M_n(x,g)}$. We aim to show that $M_n(\mathbb{P}) = \mathrm{HOD}^{M_n(x,g)}$, where $\mathbb{P} = \mathbb{V} \times \mathrm{Col}(\omega, {<}\kappa)$ and κ is as before the least inaccessible cardinal in $M_n(x)$.

First of all notice that it suffices to show that $M_n(\mathbb{P}) = \mathrm{HOD}^{M_n(x,g)}$ as $\mathbb{P}$ is definable over $M_n(x,g)$ and $\mathbb{P} \subseteq \omega_2^{M_n(x,g)}$, in particular $\mathbb{P} \in \mathrm{HOD}^{M_n(x,g)}$. Let λ be the least inaccessible cutpoint of $M_n(x)$ above κ which is a limit of cutpoints in $M_n(x)$.

Claim 1. *$M_n(\mathbb{P})|\lambda \subseteq \mathrm{HOD}^{M_n(x,g)}$.*

Proof. Let $G \in V$ be $\mathrm{Col}(\omega, \omega_2^{M_n(x,g)})$-generic over $M_n(x,g)$, then $\mathbb{P}$ is countable in $M_n(x,g)[G]$. Let $L[E](\mathbb{P})^{M_n(x,g)[G]}$ be the result of an $L[E]$-construction in the sense of [MS94] above $\mathbb{P}$ performed inside $M_n(x,g)[G]$ and generalized to n-small premice. Then $L[E](\mathbb{P})^{M_n(x,g)[G]}$ inherits the iterability from $M_n(x,g)[G]$. We aim to compare it with $M_n(\mathbb{P})$, but as our premice are only ω_1-iterable we need to give a short argument how we can perform successful comparisons.

Let z be a real in V coding $M_n^\#(x,g)[G]$ and $M_n^\#(\mathbb{P})$. Using Lemma 2.2.8 in [Uh16] we can then successfully coiterate the proper class $\mathbb{P}$-premice $L[E](\mathbb{P})^{M_n(x,g)[G]}$ and $M_n(\mathbb{P})$ inside $M_n(z)$ using the fact that the premice $M_n^\#(x,g)[G]$ and $M_n^\#(\mathbb{P})$ (from which they inherit the iterability) are countable there and $\omega_2^{M_n(z)} < \omega_1^V$. As both proper class premice are n-small they successfully coiterate to the same proper class model. Therefore, $M_n(\mathbb{P})$ and $L[E](\mathbb{P})^{M_n(x,g)[G]}$ agree up to the minimum of their least measurable cardinals. In particular, λ is an inaccessible cutpoint of both $L[E](\mathbb{P})^{M_n(x,g)[G]}$ and $M_n(\mathbb{P})$ and

$$M_n(\mathbb{P})|\lambda = L[E](\mathbb{P})^{M_n(x,g)[G]}|\lambda.$$

By the definability of the $L[E]$-construction and of the extender sequence in $M_n(x,g)[G]$ (see Lemma 1.1 in [Sch06] due to J. Steel) we have that $L[E](\mathbb{P})^{M_n(x,g)[G]} \subseteq \mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)[G]}$. As $\mathrm{Col}(\omega, \omega_2^{M_n(x,g)})$ is homogeneous and ordinal-definable in $M_n(x,g)$, $\mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)[G]} \subseteq \mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)}$. Using that $\mathbb{P} \in \mathrm{HOD}^{M_n(x,g)}$ it follows that $\mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)} = \mathrm{HOD}^{M_n(x,g)}$ and thus $M_n(\mathbb{P})|\lambda \subseteq \mathrm{HOD}^{M_n(x,g)}$, as desired. $\square$

As G_x is $\mathbb{V}$-generic over $\mathrm{HOD}^{M_n(x,g)}$, Claim 1 yields that G_x is $\mathbb{V}$-generic over $M_n(\mathbb{P})$ as well since $|\mathbb{V}| < \lambda$. Moreover, we can show the following claim.

Claim 2. *$M_n(\mathbb{P})|\lambda[G_x] = M_n(x)|\lambda$.*

Proof. As G_x is $\mathbb{V}$-generic over $\mathrm{HOD}^{M_n(x,g)}$ for the Vopěnka algebra $\mathbb{V}$ in $M_n(x,g)$, Claim 1 implies that

$$M_n(\mathbb{P})|\lambda[G_x] \subseteq \mathrm{HOD}^{M_n(x)[g]}[G_x] = \mathrm{HOD}_x^{M_n(x)[g]}.$$

By the homogeneity and ordinal definability of the forcing $\mathrm{Col}(\omega, {<}\kappa)$ we in addition have that $\mathrm{HOD}_x^{M_n(x)[g]} \subseteq \mathrm{HOD}_x^{M_n(x)} \subseteq$[7] $M_n(x)$. Thus we have shown that $M_n(\mathbb{P})|\lambda[G_x] \subseteq M_n(x)|\lambda$.

For the other inclusion let ν be a cutpoint of $M_n(\mathbb{P})$ such that $(\kappa^+)^{M_n(x)} < \nu < \lambda$ and let H be $\mathrm{Col}(\omega, \nu)$-generic over $M_n(\mathbb{P})[G_x]$. Then $M_n(\mathbb{P})[G_x][H]$ can be considered as $M_n(y)$ for some real y (see for example [SchSt09] for the fine structural details) and we have $x \in M_n(y)$. Let $L[E](x)^{M_n(y)}$ denote the result of an $L[E]$-construction in the sense of [MS94] above x performed inside $M_n(y)$. Then $L[E](x)^{M_n(y)}$ inherits the iterability from $M_n(y)$ and we can successfully compare it with $M_n(x)$ using an argument as in Claim 1. As both are proper class n-small premice they coiterate to the same model and hence agree up to the minimum of their least measurable cardinals. In particular $M_n(x)|\lambda = L[E](x)^{M_n(y)}|\lambda$. But by definability of the extender sequence (see Lemma 1.1 in [Sch06] due to J. Steel) and of the $L[E]$-construction, $L[E](x)^{M_n(y)}|\lambda \subseteq \mathrm{HOD}_x^{M_n(y)} = \mathrm{HOD}_x^{M_n(\mathbb{P})[G_x][H]}$. By homogeneity and ordinal-definability of the forcing $\mathrm{Col}(\omega, \nu)$ it follows that $\mathrm{HOD}_x^{M_n(\mathbb{P})[G_x][H]} \subseteq \mathrm{HOD}_x^{M_n(\mathbb{P})[G_x]} \subseteq M_n(\mathbb{P})[G_x]$ and thus $M_n(x)|\lambda \subseteq M_n(\mathbb{P})|\lambda[G_x]$. $\square$

Now we can show that the lemma holds below λ.

Claim 3. *$M_n(\mathbb{P})|\lambda = \mathrm{HOD}^{M_n(x,g)}|\lambda$.*

Proof. We first show that $M_n(\mathbb{P})|\lambda[G_x] = \mathrm{HOD}^{M_n(x,g)}|\lambda[G_x]$. The inclusion $\subseteq$ follows from Claim 1. For the other inclusion we have that

$$\mathrm{HOD}^{M_n(x)[g]}[G_x] = \mathrm{HOD}_x^{M_n(x)[g]} \subseteq \mathrm{HOD}_x^{M_n(x)} \subseteq M_n(x),$$

using the homogeneity and ordinal definability of the forcing $\mathrm{Col}(\omega, {<}\kappa)$. Therefore by Claim 2

$$\mathrm{HOD}^{M_n(x,g)}|\lambda[G_x] \subseteq M_n(x)|\lambda = M_n(\mathbb{P})|\lambda[G_x].$$

Finally, we argue that the equality $M_n(\mathbb{P})|\lambda[G_x] = \mathrm{HOD}^{M_n(x,g)}|\lambda[G_x]$ also holds true without adding the generic G_x. As by Claim 1 we have $M_n(\mathbb{P})|\lambda \subseteq \mathrm{HOD}^{M_n(x,g)}|\lambda$, we are again left with proving the other inclusion. Note that (G_x, g) is $\mathbb{P}$-generic over both $M_n(\mathbb{P})|\lambda$ and $\mathrm{HOD}^{M_n(x,g)}|\lambda$ and that $M_n(\mathbb{P})|\lambda[G_x, g] = \mathrm{HOD}^{M_n(x,g)}|\lambda[G_x, g]$. Let $a \in \mathrm{HOD}^{M_n(x,g)}|\lambda$ be a set of ordinals. Then there is a $\mathbb{P}$-name $\sigma \in M_n(\mathbb{P})|\lambda$ such that $\sigma_{(G_x,g)} = a$. This is forced over $\mathrm{HOD}^{M_n(x,g)}|\lambda$, i.e. there is a $p \in \mathbb{P}$ such that $\mathrm{HOD}^{M_n(x,g)}|\lambda \vDash$ "$p \Vdash \sigma = \check{a}$". Thus $M_n(\mathbb{P})|\lambda$ can compute the elements of a using the forcing relation for $\mathbb{P}$ below p. Hence $a \in M_n(\mathbb{P})|\lambda$, as desired. $\square$

Now we are able to extend Claim 2 above λ.

Claim 4. *$M_n(\mathbb{P})[G_x] = M_n(x)$.*

[7]By Steel's result on the definability of the extender sequence (see Lemma 1.1 in [Sch06]) in fact $\mathrm{HOD}_x^{M_n(x)} = M_n(x)$.

Proof. We use $\mathcal{P}^{M_n(x,g)}(M_n(\mathbb{P})|\lambda)$ to denote the result of a $\mathcal{P}$-construction in the sense of [SchSt09] above $M_n(\mathbb{P})|\lambda$ inside the model $M_n(x,g)$. By Claim 2, $M_n(\mathbb{P})|\lambda[G_x] = M_n(x)|\lambda$, so $M_n(\mathbb{P})|\lambda[G_x][g] = M_n(x,g)|\lambda$ and this $\mathcal{P}$-construction is well-defined. Moreover, the following argument shows that the construction never projects across λ.

Assume toward a contradiction that there is a level $\mathcal{P}$ of the $\mathcal{P}$-construction above $M_n(\mathbb{P})|\lambda$ inside $M_n(x,g)$ such that $\rho_\omega(\mathcal{P}) = \rho < \lambda$. That means there is an $r\Sigma_{k+1}(\mathcal{P})$-definable set $a \subseteq \rho$ for some $k < \omega$ such that $a \notin \mathcal{P}$. As by the proof of Claim 1, $M_n(\mathbb{P})|\lambda \in \mathrm{HOD}^{M_n(x,g)}$ it follows by definability of the $\mathcal{P}$-construction and of the extender sequence (see Lemma 1.1 in [Sch06] due to J. Steel) that $\mathcal{P} \in \mathrm{HOD}^{M_n(x,g)}$. This means that in particular $a \in \mathrm{HOD}^{M_n(x,g)}$. But $a \subseteq \rho < \lambda$ and by Claim 3, $\mathrm{HOD}^{M_n(x,g)}|\lambda = M_n(\mathbb{P})|\lambda = \mathcal{P}|\lambda$, so $a \in \mathcal{P}$. Contradiction.

Now it follows by construction (see [SchSt09]) that

$$\mathcal{P}^{M_n(x,g)}(M_n(\mathbb{P})|\lambda)[G_x][g] = M_n(x,g).$$

But this yields that $\mathcal{P}^{M_n(x,g)}(M_n(\mathbb{P})|\lambda)[G_x] = M_n(x)$, without adding the generic g, by an argument as at the end of the previous claim. Moreover, $\mathcal{P}^{M_n(x,g)}(M_n(\mathbb{P})|\lambda) = M_n(\mathbb{P})$ and thus $M_n(\mathbb{P})[G_x] = M_n(x)$, as desired. □

Using this, we can extend Claim 1 beyond λ as well.

Claim 5. $M_n(\mathbb{P}) \subseteq \mathrm{HOD}^{M_n(x,g)}$.

Proof. Let $K(\mathbb{P})^{M_n(\mathbb{P})}$ be the core model constructed above $\mathbb{P}$ inside the model $M_n(\mathbb{P})$ in the sense of [Sch06], i.e. the core model is constructed between consecutive Woodin cardinals. Lemma 1.1 in [Sch06] (due to John Steel) implies that $K(\mathbb{P})^{M_n(\mathbb{P})} = M_n(\mathbb{P})$. Since $|\mathbb{V}| < \delta$, where δ is the least Woodin cardinal in $M_n(x)$, we can apply generic absoluteness of the core model and obtain $K(\mathbb{P})^{M_n(\mathbb{P})[G_x]} = K(\mathbb{P})^{M_n(\mathbb{P})}$. In fact, as by Claim 4 $M_n(\mathbb{P})[G_x] = M_n(x)$, g is generic over $M_n(\mathbb{P})[G_x]$ as well. Let in addition G be $\mathrm{Col}(\omega, \omega_2^{M_n(x,g)})$-generic over $M_n(x,g)$. Then generic absoluteness of the core model below δ implies that $K(\mathbb{P})^{M_n(\mathbb{P})} = K(\mathbb{P})^{M_n(\mathbb{P})[G_x][g][G]}$. By definability of the core model and of the extender sequence we have that

$$K(\mathbb{P})^{M_n(\mathbb{P})[G_x][g][G]} \subseteq \mathrm{HOD}_{\mathbb{P}}^{M_n(\mathbb{P})[G_x][g][G]} = \mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)[G]}.$$

Moreover, the homogeneity and ordinal-definability of $\mathrm{Col}(\omega, \omega_2^{M_n(x,g)})$ implies that

$$\mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)[G]} \subseteq \mathrm{HOD}_{\mathbb{P}}^{M_n(x,g)} = \mathrm{HOD}^{M_n(x,g)},$$

where the last equality follows from $\mathbb{P} \in \mathrm{HOD}^{M_n(x,g)}$. Thus $M_n(\mathbb{P}) \subseteq \mathrm{HOD}^{M_n(x,g)}$, as desired. □

Remark. We will give another proof of Claim 5 using $\mathcal{P}$-constructions instead of the core model in the proof of Theorem 6.4.

Claim 6. $M_n(\mathbb{P})[G_x] = \mathrm{HOD}^{M_n(x,g)}[G_x]$.

Proof. We have $M_n(\mathbb{P})[G_x] \subseteq \mathrm{HOD}^{M_n(x,g)}[G_x]$ by Claim 5. Moreover, as G_x is $\mathbb{V}$-generic over $\mathrm{HOD}^{M_n(x,g)}$ for the Vopěnka algebra $\mathbb{V}$ in $M_n(x,g)$ we have as in the proof of Claim 2,

$$\mathrm{HOD}^{M_n(x)[g]}[G_x] = \mathrm{HOD}_x^{M_n(x)[g]} \subseteq \mathrm{HOD}_x^{M_n(x)} \subseteq M_n(x),$$

using the homogeneity and ordinal definability of the forcing $\mathrm{Col}(\omega, {<}\kappa)$. This yields the claim as by Claim 4 we have that $M_n(\mathbb{P})[G_x] = M_n(x)$. □

Finally, the statement of Claim 6 also holds true without adding the generic G_x by the argument at the end of the proof of Claim 3. Hence $M_n(\mathbb{P}) = \mathrm{HOD}^{M_n(x,g)}$, as desired. □

Corollary 5.2. *Let $F(s) = s^*$ for $s \in [\mathrm{Ord}]^{<\omega}$. Then*

$$\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)}).$$

Proof. Note that $\mathcal{M}_{\infty,\kappa}$ and F are definable over $M_n(x,g)$ by construction. We first show that $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$ and $\mathrm{HOD}^{M_n(x,g)}$ agree up to their least inaccessible cardinal.

Let $L[E](\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})^{M_n(x,g)}$ be the result of a fully backgrounded extender construction in the sense of [MS94] inside $M_n(x,g)$ above $\mathcal{M}_{\infty,\kappa}$ and $F \restriction \omega_2^{M_n(x,g)}$. Compare the premice $L[E](\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})^{M_n(x,g)}$ and $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$ inside the model $M_n(z)$, where z is a real in V coding the premice $M_n^\#(x,g)$ and $M_n^\#(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$. As in the proof of Claim 1 in the proof of Lemma 5.1 the comparison is successful and the premice coiterate to a common proper class premouse. In particular they agree up to their least inaccessible γ which is below the least measurable cardinal in $L[E](\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})^{M_n(x,g)}$ and the least measurable cardinal in $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$. Using the definability of the extender sequence (see Lemma 1.1 in [Sch06] due to J. Steel), this yields that $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})|\gamma \subseteq \mathrm{HOD}^{M_n(x,g)}$ as $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})|\gamma$ can be obtained inside $M_n(x,g)$ as an initial segment of a fully backgrounded $L[E]$-construction.

For the other inclusion let $A \subseteq \omega_2^{M_n(x,g)}$ be as in the statement of Lemma 5.1, i.e. $M_n(x,g)$-definable with $\mathrm{HOD}^{M_n(x,g)} = M_n(A)$. Moreover, let φ be a formula defining A, i.e. $\xi \in A$ iff $M_n(x,g) \vDash \varphi(\xi)$. Then

$$\xi \in A \text{ iff } M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}) \vDash (1 \Vdash^{\mathbb{P}} \varphi(F(\xi)))$$

for $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$ and thus $A \in L[M_n(\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}), F \restriction \omega_2^{M_n(x,g)}]$. As $A \subseteq \delta_{\infty,\kappa} = \omega_2^{M_n(x,g)}$, it follows that $A \in M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$. Hence, we can consider the premouse $L[E](A)^{M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})}$ and as in the argument we gave above we can compare it with $M_n(A)$ inside the model $M_n(z)$, where z is a real in V coding $M_n^\#(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$ and $M_n^\#(A)$.

Again, they coiterate to the same model and if ν is the minimum of γ and the least inaccessible in $M_n(A)$, it follows that

$$M_n(A)|\nu = L[E](A)^{M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})}|\nu.$$

In particular,

$$M_n(A)|\nu \subset L[E](A)^{M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})} \subset M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)}).$$

Therefore, we obtain $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})|\nu = M_n(A)|\nu$ and we can rearrange $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})$ and $M_n(A)$ as $M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})|\nu$-premice. As such it follows that the following equalities for sets hold:

$$\begin{aligned} M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)}) &= M_n(M_n(\mathcal{M}_{\infty,\kappa}, F \restriction \omega_2^{M_n(x,g)})|\nu) \\ &= M_n(A) = \mathrm{HOD}^{M_n(x,g)}. \end{aligned}$$

□

The following corollary follows immediately from Lemma 4.4 and Corollary 5.2.

Corollary 5.3. $\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_{\infty,\kappa}, \pi_\infty \restriction \delta_{\infty,\kappa})$.

We now consider the iteration strategy for $\mathcal{M}_{\infty,\kappa}$. Let Λ be the restriction of $\Sigma_{M_{n+1}^-}$ to correctly guided finite stacks $\vec{\mathcal{T}}$ on $\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}$ such that $\vec{\mathcal{T}} \in \hat{\mathcal{M}}_{\infty,\kappa}|\kappa_\infty$, where κ_∞ is the least inaccessible cardinal of $\hat{\mathcal{M}}_{\infty,\kappa}$ above $\delta_{\infty,\kappa}$.

Lemma 5.4. $\Lambda \subseteq \mathrm{HOD}^{M_n(x,g)}$.

Proof. Let $\mathcal{T}$ be a maximal tree on $\mathcal{M}_{\infty,\kappa}|\delta_{\infty,\kappa}$ with $\mathcal{T} \in \hat{\mathcal{M}}_{\infty,\kappa}|\kappa_\infty$. Moreover, let $b = \Lambda(\mathcal{T})$. We aim to show that $b \in \mathrm{HOD}^{M_n(x,g)}$.
Let $\mathcal{R} = \mathcal{M}_b^{\mathcal{T}}$ be the last model of $\mathcal{T}^\frown b$. Then $\mathcal{R} \in \mathrm{HOD}^{M_n(x,g)}$. Moreover, let $\delta^*_{\infty,\kappa}$ be the least Woodin cardinal in $(\hat{\mathcal{M}}_{\infty,\kappa})^{\hat{\mathcal{M}}_{\infty,\kappa}}$, the direct limit of the system $\hat{\mathcal{F}}^* = (\hat{\mathcal{F}}_\kappa)^{\hat{\mathcal{M}}_{\infty,\kappa}[H]}$. Then $(\hat{\mathcal{M}}_{\infty,\kappa}|\delta^*_{\infty,\kappa})^{\hat{\mathcal{M}}_{\infty,\kappa}}$ is an iterate of $\mathcal{R}$. As $\pi_\infty \restriction \delta_{\infty,\kappa} \in \mathrm{HOD}^{M_n(x,g)}$, we can identify b inside $M_n(x,g)$ as the unique branch through $\mathcal{T}$ which is $(\pi_\infty \restriction \delta_{\infty,\kappa})$-realizable, i.e. such that there is an elementary embedding $\sigma : \mathcal{M}_b^{\mathcal{T}} \to (\hat{\mathcal{M}}_{\infty,\kappa}|\delta^*_{\infty,\kappa})^{\hat{\mathcal{M}}_{\infty,\kappa}}$ with $\pi_\infty \restriction \delta_{\infty,\kappa} = \sigma \circ i_b^{\mathcal{T}}$.
The same argument applies to pseudo-normal iterates $\mathcal{N}$ of $\mathcal{M}_{\infty,\kappa}$ with $\mathcal{N}|\delta^{\mathcal{N}} \in \hat{\mathcal{M}}_{\infty,\kappa}|\kappa_\infty$ and maximal iteration trees $\mathcal{T}$ on $\mathcal{N}|\delta^{\mathcal{N}}$ such that $\mathcal{T} \in \hat{\mathcal{M}}_{\infty,\kappa}|\kappa_\infty$, hence $\Lambda \subseteq \mathrm{HOD}^{M_n(x,g)}$. □

Similarly to Lemma 3.47 in [StW16] we finally need a method of Boolean-valued comparison. As the proof is analogous we omit it.

Lemma 5.5. *Let H be $\mathrm{Col}(\omega, {<}\kappa_\infty)$-generic over $\hat{\mathcal{M}}_{\infty,\kappa}$, and let $\mathcal{Q}$ be such that $\hat{\mathcal{M}}_{\infty,\kappa}[H] \vDash$ "$\mathcal{Q}$ is countable and n-suitable". Then there is an $\mathcal{R}$ such that*

(1) $\mathcal{R}$ *is a pseudo-normal iterate of* $\mathcal{Q}$*,*
(2) $\mathcal{R}$ *is a* $\Sigma_{M_{n+1}^-}$*-iterate of* $\mathcal{M}_{\infty,\kappa}$*, and*
(3) $\mathcal{R} \in \hat{\mathcal{M}}_{\infty,\kappa}$.

Finally, we can finish the proof of Theorem 1.1.

Theorem 5.6. $\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_{\infty,\kappa}, \pi_\infty \restriction \delta_{\infty,\kappa}) = M_n(\mathcal{M}_{\infty,\kappa}, \Lambda)$.

Proof. $\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_{\infty,\kappa}, \pi_\infty \restriction \delta_{\infty,\kappa})$ is Corollary 5.3. Moreover, the inclusion $M_n(\mathcal{M}_{\infty,\kappa}, \Lambda) \subseteq \mathrm{HOD}^{M_n(x,g)}$ follows from Lemma 5.4 analogous to the proof of Corollary 5.2. The last inclusion $M_n(\mathcal{M}_{\infty,\kappa}, \pi_\infty \restriction \delta_{\infty,\kappa}) \subseteq M_n(\mathcal{M}_{\infty,\kappa}, \Lambda)$ follows from Lemma 5.5. The direct limit of $\hat{\mathcal{F}}^*$ up to the ω-th successor of its bottom Woodin cardinal is the same as the direct limit of all Λ-iterates of $\mathcal{M}_{\infty,\kappa}$ which are an element of $\hat{\mathcal{M}}_{\infty,\kappa}|\kappa_\infty$ via the comparison maps. Moreover, we have that π_∞ is the canonical direct limit map of this system and therefore definable from $\hat{\mathcal{M}}_{\infty,\kappa}$ and Λ. Thus $\pi_\infty \restriction \delta_{\infty,\kappa} \in M_n(\mathcal{M}_{\infty,\kappa}, \Lambda)$, which analogous to the proof of Corollary 5.2 implies that $M_n(\mathcal{M}_{\infty,\kappa}, \pi_\infty \restriction \delta_{\infty,\kappa}) \subseteq M_n(\mathcal{M}_{\infty,\kappa}, \Lambda)$. □

6. HOD in mice with more Woodin and strong cardinals

Most of the arguments we gave in the previous sections generalize with only small changes to more arbitrary canonical self-iterable inner models, e.g. M_ω, $M_{\omega+42}$ or initial segments of the least non-tame mouse M_{nt}. In what follows we will make this precise and point out the changes that need to be made.

We first fix some notation.

Definition 6.1. *Let* $x \in {}^\omega\omega$ *and let* φ *be a formula in the language of* x*-premice. We say an* x*-premouse* M *is* φ-small *iff for every critical point* κ *of an extender on the* M*-sequence* $M|\kappa \nvDash \varphi$.

Typical examples for the formula φ are statements about the existence of a fixed number of Woodin and/or strong cardinals in a fixed order.

Definition 6.2. *Let* $x \in {}^\omega\omega$ *and let* φ *be a formula in the language of* x*-premice. If it exists and is unique,* $M_\varphi^\#(x)$ *denotes the countable, sound,* ω_1*-iterable* x*-premouse which is not* φ*-small, but all of whose proper initial segments are* φ*-small. Moreover, in this case* M_φ *denotes the unique* x*-premouse which is obtained from* $M_\varphi^\#$ *by iterating its top extender out of the universe.*

Example. Consider the formula

$$\varphi = \exists\delta_0\exists\delta_1(\delta_0 < \delta_1 \wedge \delta_0, \delta_1 \text{ are Woodin cardinals}).$$

Then $M_\varphi^\# = M_2^\#$.

In addition, if $x \in {}^\omega\omega$ and $\mathcal{N}$ is an arbitrary countable x-premouse, we write $M_\varphi^\#(\mathcal{N})$ for the smallest countable x-premouse $\mathcal{M} \trianglerighteq \mathcal{N}$ with $\rho_\omega(\mathcal{M}) \leq \mathcal{N} \cap \mathrm{Ord}$ which is ω_1-iterable above $\mathcal{N} \cap \mathrm{Ord}$, sound above $\mathcal{N} \cap \mathrm{Ord}$, and such that either $\mathcal{M}$ is not fully sound or $\mathcal{M}$ is not φ-small above $\mathcal{N} \cap \mathrm{Ord}$, if it exists and is unique. In this case, we moreover write $M_\varphi(\mathcal{N})$ for the proper class premouse obtained from $M_\varphi^\#(\mathcal{N})$ by iterating its top extender out of the universe.

Definition 6.3. *Let φ be a formula in the language of premice and let δ be an ordinal. If it exists and is unique, $M_{\delta,\varphi}^\#$ denotes the countable, sound, ω_1-iterable premouse such that*

(1) $M_{\delta,\varphi}^\# \vDash$ "δ is Woodin",
(2) for every $\eta < \delta$, $M_{\delta,\varphi}^\# \vDash$ "η is not Woodin",
(3) $M_{\delta,\varphi}^\#$ is not φ-small above δ, and
(4) every proper initial segment of $M_{\delta,\varphi}^\#$ is φ-small above δ.

Moreover, in this case $M_{\delta,\varphi}$ denotes the result of iterating the top extender of $M_{\delta,\varphi}^\#$ out of the universe.

Example. For $M_\varphi^\# = M_n^\#$, we have $M_{\delta,\varphi}^\# = M_{n+1}^\#$ for $\delta = \delta_0$ the least Woodin cardinal in $M_{n+1}^\#$. Moreover, for $M_\varphi^\# = M_\omega^\#$, we have $M_{\delta,\omega}^\# = M_\omega^\#$ for $\delta = \delta_0$ the least Woodin cardinal in $M_\omega^\#$.

Using this, we can now state the generalized version of Theorem 1.1. As we assume self-iterability, this result is essentially limited to tame mice.

Theorem 6.4. *Let φ be a formula expressing the existence of a fixed number of Woodin and strong cardinals (in a fixed order) and assume that $M_\varphi(x)$ exists and is self-iterable[8] for all $x \in {}^\omega\omega$. Moreover, assume that $M_{\delta,\varphi}^\#$ exists. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_\varphi(x,g)} = M_\varphi(\mathcal{M}_\infty, \Lambda),$$

where κ is the least inaccessible cardinal in $M_\varphi(x)$, g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_\varphi(x)$, $\mathcal{M}_\infty$ is a direct limit of iterates of an initial segment of $M_{\delta,\varphi}$, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

Remark. This is a generalization of Theorem 1.1 as under the assumptions in Theorem 1.1 the hypotheses of Theorem 6.4 hold for $M_\varphi = M_n$ (see [St95] for a proof of self-iterability).

For the special case $M_\varphi = M_\omega$ we have the following corollary (see the discussion after Corollary 7.21 in [St10, §7] for the self-iterability of M_ω).

[8] in the sense of Theorem 4.3 in [Schl14]

Corollary 6.5. *Assume that $M_\omega^\#$ exists. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_\omega(x,g)} = M_\omega(\mathcal{M}_\infty, \Lambda),$$

where κ is the least inaccessible cardinal in $M_\omega(x)$, g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_\omega(x)$, $\mathcal{M}_\infty$ is a direct limit of iterates of an initial segment of M_ω, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

Proof of Theorem 6.4. Most of the definitions and results in Sections 3 and 4 generalize straightforwardly for the corresponding definition of suitable premouse for $M_\varphi(x)$. The only point where we need to be careful is the usage of the core model in Definitions 3.27 and 3.30. But instead of e.g. $M(y) = (K(y))^{M(y)}$ in Definition 3.27 we in fact only need to use the core model below the bottom Woodin cardinal δ as we are only using generic absoluteness for the core model below δ. That means instead of $M(y) = (K(y))^{M(y)}$ it suffices to require that $M(y)|\delta = (K(y))^{M(y)|\delta}$ and then construct the rest of the model back on top. For the condition involving $K^{M_n(x)[g]}(\mathcal{N}|\delta^{\mathcal{N}})$ in Definition 3.30 we can argue similarly. Note that the proof of Lemma 1.1 in [Sch06] due to Steel generalizes to this setting and yields that $M_{\delta,\varphi}|\delta \vDash$ "$V = K$". Therefore, the proofs in Sections 3 and 4 still go through with these modifications.

For the generalization of Section 5 we replace every occurence of a fully backgrounded $L[E]$-construction by a local K^c-construction as in [St07] or a construction as in Section 5 of [Schl15], in case the formula φ asserts the existence of strong cardinals. In what follows, we will point out the other changes that need to be made in the argument in Section 5.

At several points in the proof we used the definability of the extender sequence to obtain that the background construction or $\mathcal{P}$-construction and their results, when performed in the inner model $M_\varphi(y)$, are in $\mathrm{HOD}^{M_\varphi(y)}$ for some $y \in {}^\omega\omega$. In this generalized setting this follows from the self-iterability of $M_\varphi(y)$ by Theorem 4.3 in [Schl14].

Finally, we need to give a different proof of Claim 5 in the proof of Lemma 5.1 as in this general setting there might be no core model available.

Claim. $M_\varphi(\mathbb{P}) \subseteq \mathrm{HOD}^{M_\varphi(x,g)}$.

Proof. By the generalization of Claim 2 in the proof of Lemma 5.1 we have $M_\varphi(\mathbb{P})|\lambda[G_x] = M_\varphi(x)|\lambda$, where G_x is $\mathbb{V}$-generic over $\mathrm{HOD}^{M_\varphi(x,g)}$ and λ is the least inaccessible cutpoint of $M_\varphi(x)$ above κ which is a limit of cutpoints in $M_\varphi(x)$. Hence $M_\varphi(\mathbb{P})|\lambda[G_x][g] = M_\varphi(x)[g]|\lambda$ and we can consider the $\mathcal{P}$-construction $\mathcal{P}^{M_\varphi(x,g)}(M_\varphi(\mathbb{P})|\lambda)$ above $M_\varphi(\mathbb{P})|\lambda$ inside $M_\varphi(x,g)$.

By the same argument as in Claim 4 in the proof of Lemma 5.1 it follows that this $\mathcal{P}$-construction never projects across λ. Therefore, $\mathcal{P}^{M_\varphi(x,g)}(M_\varphi(\mathbb{P})|\lambda) = M_\varphi(\mathbb{P})$. By the generalization of Claim 1 in the proof of Lemma 5.1 to this setting we have that $M_\varphi(\mathbb{P})|\lambda \in \mathrm{HOD}^{M_\varphi(x,g)}$. Hence by definablity of the $\mathcal{P}$-construction and the extender sequence it follows that $M_\varphi(\mathbb{P}) \subseteq \mathrm{HOD}^{M_\varphi(x,g)}$, as desired. $\square$

All other arguments in the proof of Lemma 5.1 and Theorem 1.1 straightforwardly generalize to this setting. $\square$

In [Schl14] Farmer Schlutzenberg showed self-iterability for initial segments of the least non-tame mouse M_{nt}. We can use this to extend our analysis to this case. By Lemma 4.4 in [Schl14], if a non-tame mouse exists, then there is a least one, in the sense of the following definition.

Definition 6.6. *Let M_{nt} denote the least non-tame mouse, if it exists. That means if M is a (ω_1+1)-iterable non-tame mouse such that no proper initial segment of M is non-tame, then $M_{nt} = cHull_1^M(\emptyset)$ [9]. Moreover, $\rho_1(M_{nt}) = \omega$ and $p_1(M_{nt}) = \emptyset$.*

This definition straightforwardly generalizes to $M_{nt}(x)$ for $x \in {}^\omega\omega$ and we have the following theorem.

Theorem 6.7. *Assume that $M_{nt}(x)$ exists and is ω_1-iterable for all $x \in {}^\omega\omega$. Moreover, assume that $M_{\delta,nt}$ (the least premouse with a Woodin cardinal δ which is non-tame above δ in the sense of Definition 6.6) exists and is (ω_1+1)-iterable. Then for a Turing cone of reals x, if we let ν be a sufficiently large limit cardinal of $M_{nt}(x)$ such that $M_{nt}(x,g)|\nu$ and $M_{nt}(\mathcal{M}_\infty,\Lambda)|\nu$ are both models of* ZFC, [10]

$$\mathrm{HOD}^{M_{nt}(x,g)|\nu} = M_{nt}(\mathcal{M}_\infty,\Lambda)|\nu,$$

where κ is the least inaccessible cardinal in $M_{nt}(x)$, g is $\mathrm{Col}(\omega,{<}\kappa)$-generic over $M_{nt}(x)$, $\mathcal{M}_\infty$ is a direct limit of iterates of an initial segment of $M_{\delta,nt}$, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

Proof of Theorem 6.7. By Theorem 4.5 in [Schl14] the extender sequence of $M_{nt}|\nu$ is definable over $M_{nt}|\nu$. Therefore, the argument from the proof of Theorem 6.4 works using a fully backgrounded $L[E]$-construction as in Section 5. $\square$

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[9] $cHull_1^M(\emptyset)$ denotes the transitive collapse of $Hull_1^M(\emptyset)$.

[10] This ensures that $\mathrm{HOD}^{M_{nt}(x,g)|\nu}$ is defined the usual way.

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Sandra Müller, Kurt Gödel Research Center, Universität Wien, Währinger Strasse 25, 1090 Wien, Austria

E-mail address: sandra.uhlenbrock@univie.ac.at

Grigor Sargsyan, Department of Mathematics, Rutgers University, Hill Center for the Mathematical Sciences, 110 Frelinghuysen Rd., Pisacataway, NJ 08854, USA

E-mail address: grigor@math.rutgers.edu