Covering with Chang models over derived models[*†‡]

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Abstract

We present a covering conjecture that we expect to be true below superstrong cardinals. We then show that the conjecture is true in hod mice. This work is a continuation of the work that started in [4], and the main conjecture of the current paper is a revision of the UB-Covering Conjecture of [4].

One of the main projects of inner model theory is to identify canonical structures like Gödel's L that are *close* to the universe of sets. While what exactly *closeness* should mean is open to interpretation, perhaps the simpliest way of interpreting it is by demanding that the successor of some singular cardinal is computed correctly. Perhaps the most well-known result of this kind is Jensen's covering lemma. A weaker version of it says that if $0^{\#}$ doesn't exist then L computes the successor of any singular cardinal correctly, i.e., if κ is singular then $(\kappa^+)^{\mathsf{L}} = \kappa^+$. In this paper, our goal is to introduce a covering principle that is based on the idea that the information coded into the universally Baire sets is enough to describe the successor cardinals. To state our conjecture we need to introduce *universally Baire sets* and *extenders*.

Recall from [1] that a set of reals is *universally Baire* if all of its preimages in topological spaces have the property of Baire. Following Woodin (see [13]), we let Γ^∞ be the set of universally Baire sets. Given a V-generic g we set $\Gamma^\infty_g = (\Gamma^\infty)^{V[g]}$ and $\mathbb{R}_g = \mathbb{R}^{V[g]}$.

An extender is a system of ultrafilters. More precisely, we say E is a (κ, λ)-extender if there is an embedding $j : M \to N$ such that

1. $\mathrm{crit}(j) = \kappa$,

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2. $j(\kappa) \geq \lambda$,

3. $E = \{(a, A) : a \in \lambda^{<\omega}, A \in \wp(\kappa) \cap M \text{ and } a \in j(A)\}$,

4. $N = \{j(f)(a) : f \in M, f : \kappa \to M \text{ and } a \in \lambda^{<\omega}\}$.

Notice that for each $a \in \lambda^{<\omega}$, $E_a = \{A : (a, A) \in E\}$ is an ultrafilter measuring $\wp(\kappa) \cap M$. The extenders we have defined are usually called *short*, where shortness refers to the fact that $j(\kappa) \geq \lambda$. The fact that E_a is an ultrafilter on κ is a consequence of shortness. Long extenders have measures concentrating on more than one cardinal. Just like ultrafilters, extenders can be defined without a reference to the embedding j. In our set up, N is the ultrapower of M by E, and it can be shown to be completely determined by the pair (M, E).

From now on, when we say we work in the short extender region we mean that we are in the region of large cardinals that can be defined using only short extenders. Large cardinal notions such as strong cardinals, Woodin cardinals, Shelah cardinals, superstrong cardinals and subcompact cardinals are all in the short extender region, while supercompactness is not. We will drop *short* from now on, and to state our results, will use Steel's NLE (see [12]), which stands for "no long extender", to state our results.

Conjecture 0.1 (Covering with Chang Models) *Assume* NLE *and suppose there are unboundedly many Woodin cardinals and strong cardinals. Let* κ *be a limit of Woodin cardinals and strong cardinals and such that either* κ *is a measurable cardinal or* $\mathrm{cf}(\kappa) = \omega$*. Then there is a transitive model* M *of* $\mathsf{ZFC} - \mathsf{Powerset}$ *such that*

1. $Ord \cap M = \kappa^+$,

2. *M has a largest cardinal ν,*

3. *for any $g \subseteq Coll(\omega, <\kappa)$, letting $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g \cap Coll(\omega,\alpha)]}$ and $\Gamma^* = \{A^g \cap \mathbb{R}^* : \exists \alpha < \kappa (A \in \Gamma^{\infty}_{g \cap Coll(\omega,\alpha)})\}$, in $V(\mathbb{R}^*)$,*

$$L(M, \bigcup_{\alpha<\nu} (M|\alpha)^{\omega}, \Gamma^*, \mathbb{R}^*) \vDash \mathsf{AD}.$$

4. *If in addition there is no inner model with a subcompact cardinal then $M \vDash \square_\nu$.*

We state the following two immediate corollaries of $\mathsf{Covering\ with\ Chang\ Models}$.

Corollary 0.2 *Assume* $\mathsf{Covering\ with\ Chang\ Models}$ *holds and there is no inner model with a subcompact cardinal. Assume further that there are unboundedly many Woodin cardinals and strong cardinals. Let* κ *be a limit of Woodin cardinals and strong cardinals such that either* κ *is a measurable cardinal or* $\mathrm{cf}(\kappa) = \omega$*. Then* $\square_\kappa$ *holds.*

Corollary 0.3 *Assume* PFA *and suppose* Covering with Chang Models *holds. Then there is an inner model with a subcompact cardinal.*

Much has been done towards establishing covering results using canonical inner models resembling L. For example, the reader can consult [2], [3], [8], [4], [9], [10] and [14]. All of these papers except the last one deal with the *short-extender-region*. The last one has conjectures in the region of supercompact cardinals. The introduction of [4] contains a lengthier discussion of Conjecture 0.1 and its role in inner model theory.

Conjecture 0.1 connects determinacy theories with other natural frameworks. Perhaps our approach to the program of finding canonical covering structures is somewhat unconventional as the covering structures that we propose are not constructed by conventional backgrounded constructions such as $L[\vec{E}]$ constructions or K^c constructions. The reader can learn more about the conventional approach from [2], which is based on the existence of K^c.

Our main motivation comes from the desire to isolate large models of determinacy in natural extensions of ZFC. We strongly believe that the information coded into canonical sets of reals describes the *core* of the universe. Thus, our approach is closer to the Ultimate L framework proposed by Woodin than the conventional approach. Our ideas are also heavily influences by Steel's inter-translatability ideas that appear in [11].

Besides stating Conjecture 0.1, our goal here is to verify that it holds in *hod mice*, which are canonical structures that appear in the analysis of HOD of models of determinacy. The reader can learn more about them by consulting [5], [6], [7] and [12]. The following is the main theorem of the paper. We heavily rely on the terminology of [12].

Theorem 0.4 *Suppose $\mathcal{V} \vDash$ ZFC is an lbr hod mouse that has a class of Woodin cardinals and suppose κ is a limit of Woodin cardinals of $\mathcal{V}$ such that either κ is regular or its cofinality is not a measurable cardinal. Then the Conjecture 0.1 is true in $\mathcal{V}$ at κ.*

More precisely, if $g \subseteq Coll(\omega, <\kappa)$ is generic then there is a transitive lbr hod premouse $\mathcal{M}$ such that

1. *$Ord \cap \mathcal{M} = \kappa^+$,*
2. *$\mathcal{M}$ has a largest cardinal ν,*
3. *for any $g \subseteq Coll(\omega, <\kappa)$, letting $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g\cap Coll(\omega,\alpha)]}$, in $V(\mathbb{R}^*)$,*

$$L(\mathcal{M}, \bigcup_{\alpha<\nu}(\mathcal{M}|\alpha)^\omega, \Gamma^\infty, \mathbb{R}) \vDash \mathsf{AD}.$$

4. *If in addition κ is not subcompact then $\mathcal{M} \vDash \square_\nu$.*

In fact we will give a precise definition of $\mathcal{M}$ as above and will prove a stronger claim than what is stated above (see Theorem 2.1).

1 Internal direct limit constructions

Suppose $\mathcal{V} \vDash ZFC$ is an lbr hod premouse. We will let $S^{\mathcal{V}}$ denote its strategy predicate, and we will always assume that $S^{\mathcal{V}}$ has *full normalization*. This means that if $\mathcal{T}$ is an iteration of $\mathcal{V}$ according to $S^{\mathcal{V}}$ with last model $\mathcal{Q}$ then there is a normal iteration $\mathcal{U}$ of $\mathcal{V}$ with last model $\mathcal{Q}$ and such that

1. $\pi^{\mathcal{T}}$ exists if and only if $\pi^{\mathcal{U}}$ exists, and
2. if $\pi^{\mathcal{T}}$ exists then $\pi^{\mathcal{T}} = \pi^{\mathcal{U}}$.

Suppose next κ is a limit of Woodin cardinals of $\mathcal{V}$ such that if $\mathrm{cf}^{\mathcal{V}}(\kappa) < \kappa$ then $\mathrm{cf}^{\mathcal{V}}(\kappa)$ is not a measurable cardinal. Suppose $g \subseteq Coll(\omega, <\kappa)$ is $\mathcal{V}$-generic. Set $\mathcal{P} = \mathcal{V}|(\kappa^+)^{\mathcal{V}}$. It is shown in [12] that $S^{\mathcal{V}}$ has a unique canonical extension in $\mathcal{V}[g]$. We set $\Sigma = S^{\mathcal{V}}$ and Σ^g be the extension of Σ in $\mathcal{V}[g]$.

A notational digression. As is customary, given an iteration strategy Λ for a hod premouse $\mathcal{M}$ and an iteration $\mathcal{T}$ according to Λ with last model $\mathcal{Q}$, we will let $\Lambda_{\mathcal{T},\mathcal{Q}}$ be the strategy of $\mathcal{Q}$ induced by Λ. More precisely, $\Lambda_{\mathcal{T},\mathcal{Q}}(\mathcal{U}) = \Lambda(\mathcal{T}^\frown\mathcal{U})$. Given $\xi \leq Ord \cap \mathcal{M}$, we let $\Lambda_{\mathcal{M}|\xi}$ be the strategy of $\mathcal{M}|\xi$ induced by Λ[1]. In our situation above, $\Sigma^g_{\mathcal{T},\mathcal{Q}}$ is independent of $\mathcal{T}$, and so we will drop $\mathcal{T}$ from our notation.

Continuing with $(\mathcal{M}, \Lambda)$ above, given an interval (α, β) with $\beta \leq Ord \cap \mathcal{M}$, we say that iteration $\mathcal{T}$ is based on $\mathcal{M}|(\alpha, \beta)$ or just (α, β) if all extenders used in $\mathcal{T}$ have critical points $> \alpha$ and for each $\gamma < lh(\mathcal{T})$, if $\pi^{\mathcal{T}}_{0,\gamma}$ exists then $lh(E_\gamma) < \pi^{\mathcal{T}}_{0,\gamma}(\beta)$. Similarly, we define the meaning of "$\mathcal{T}$ is based on $\mathcal{M}|\beta$".

Suppose $\mathcal{Q}$ is a Λ-iterate of $\mathcal{M}$ and $\mathcal{R}$ is a $\Lambda_{\mathcal{Q}}$-iterate of $\mathcal{Q}$. Suppose Λ has full normalization. Then we let $\mathcal{T}^{\Lambda}_{\mathcal{Q},\mathcal{R}}$ be the unique normal $\Lambda_{\mathcal{Q}}$-iteration of $\mathcal{Q}$ with last model $\mathcal{R}$. If $\pi^{\mathcal{T}^{\Lambda}_{\mathcal{Q},\mathcal{R}}}$ exists then we let $\pi^{\Lambda}_{\mathcal{Q},\mathcal{R}}$ be this embedding. When Λ is clear from context, we will drop it from our notation.

[1]One can think of $\Lambda_{\mathcal{M}|\xi}$ as id-pullback of Λ.

We continue with $\mathcal{V}, \mathcal{P}, g$ as above. For $\alpha < \kappa$, let $g_\alpha = g \cap Coll(\omega, \alpha)$. Let $\mathcal{I}^{g_\alpha}(\mathcal{P})$ be the set of Σ^{g_α}-iterates of $\mathcal{P}$ that are obtained via iterations $\mathcal{T} \in V[g_\alpha]$ such that $lh(\mathcal{T}) \leq \kappa$, $\mathcal{T}$ is based on $\mathcal{P}|\kappa$, $\pi^{\mathcal{T}}$ is defined and $\pi^{\mathcal{T}}(\kappa) = \kappa$. Set $\mathcal{I}^g(\mathcal{P}) = \cup_{\alpha<\kappa}\mathcal{I}^{g_\alpha}(\mathcal{P})$.

Given $\mathcal{Q} \in \mathcal{I}^{g_\alpha}(\mathcal{P})$ and $\beta \in [\alpha, \kappa)$, we let $\mathcal{F}^{g_\beta}_{\mathcal{Q}}$ be the set of $\Sigma^{g_\beta}_{\mathcal{Q}}$-iterates $\mathcal{R}$ of $\mathcal{Q}$ such that the iteration $lh(\mathcal{T}_{\mathcal{Q},\mathcal{R}}) < \kappa$ and $\pi_{\mathcal{Q},\mathcal{R}}$ is defined. Set $\mathcal{F}^g_{\mathcal{Q}} = \cup_{\beta\in(\alpha,\kappa)}\mathcal{F}^{g_\beta}_{\mathcal{Q}}$.

Because Σ^g has full normalization, given $\mathcal{R}, \mathcal{S} \in \mathcal{F}^g_{\mathcal{Q}}$, we have $\mathcal{W} \in \mathcal{F}^g_{\mathcal{Q}}$ such that $\mathcal{W}$ is a common iterate of $\mathcal{R}$ and $\mathcal{S}$ via respectively $\Sigma^g_{\mathcal{R}}$ and $\Sigma^g_{\mathcal{S}}$. Moreover, $\pi_{\mathcal{R},\mathcal{W}} \circ \pi_{\mathcal{Q},\mathcal{R}} = \pi_{\mathcal{S},\mathcal{W}} \circ \pi_{\mathcal{Q},\mathcal{S}}$. It follows that $(\mathcal{F}^g_{\mathcal{Q}}, \pi_{\mathcal{R},\mathcal{S}} : \mathcal{R}, \mathcal{S} \in \mathcal{F}^g_{\mathcal{Q}})$ is a directed system. We let $\mathcal{M}_\infty(\mathcal{Q})$ be the direct limit of $(\mathcal{F}^g_{\mathcal{Q}}, \pi_{\mathcal{R},\mathcal{S}} : \mathcal{R}, \mathcal{S} \in \mathcal{F}^g_{\mathcal{Q}})$. Let $\pi^{\mathcal{Q}}_{\mathcal{R},\infty} : \mathcal{R} \to \mathcal{M}_\infty(\mathcal{Q})$ be the direct limit embedding. Set $\kappa^{\mathcal{Q}}_\infty = \pi^{\mathcal{Q}}_{\mathcal{Q},\infty}(\kappa)$.

Terminological digression. Suppose $\mathcal{M}$ is a transitive model of set theory. We say $w = (\eta^w, \delta^w)$ is a *window* of $\mathcal{M}$ if $\mathcal{M} \vDash$ "there are no Woodin cardinals in the interval (η^w, δ^w) and δ^w is a Woodin cardinal". We say w is a *maximal window* if in addition η_w is the least $\mathcal{M}$-inaccessible $> \sup\{\xi : \mathcal{M}| \vDash$ "ξ is a Woodin cardinal"$\}$. Given two windows w and v, we write $w <_W v$ if $\delta^w < \delta^v$. We say that an iteration $\mathcal{T}$ of $\mathcal{M}$ is based on w if $\mathcal{T}$ is based on $\mathcal{M}|\delta^w$ and is above η^w. We let $\mathsf{EA}^{\mathcal{M}}_w$ be the extender algebra of $\mathcal{M}$ associated with δ^w that uses extenders whose critical points are $> \eta^w$.

Suppose $\mathcal{R} \in \mathcal{I}^g(\mathcal{P})$. We say $\mathcal{Q}$ is a *window-based iterate* of $\mathcal{R}$ if there is $\iota < \kappa$ such that $\mathcal{R} \in V[g_\iota]$, an $<_W$-increasing sequence of windows $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ of $\mathcal{R}$ and a sequence $(\mathcal{Q}_\alpha : \alpha < \mathrm{cf}(\kappa)) \subseteq \mathcal{F}^{g_\iota}_{\mathcal{R}}$ (in $V[g_\iota]$) such that

1. $\mathcal{Q}_0 \in \mathcal{F}^{g_\iota}(\mathcal{R})$ and $\mathcal{T}_{\mathcal{R},\mathcal{Q}_0}$ is based on $\mathcal{R}|\eta_0$,
2. $\mathcal{Q}_{\alpha+1} \in \mathcal{F}^{g_\iota}_{\mathcal{Q}_\alpha}$,
3. $\mathcal{Q}_{\alpha+1}$ is obtained from $\mathcal{Q}_\alpha$ via an iteration according to $\Sigma^{g_\iota}_{\mathcal{Q}_\alpha}$ that is based on $\pi_{\mathcal{R},\mathcal{Q}_\alpha}(w_\alpha)$,
4. for limit ordinals λ, $\mathcal{Q}_\lambda$ is the direct limit of $(\mathcal{Q}_\alpha, \pi_{\mathcal{Q}_\alpha,\mathcal{Q}_\beta} : \alpha < \beta < \lambda)$,
5. $\mathcal{Q}$ is the direct limit of $(\mathcal{Q}_\alpha, \pi_{\mathcal{Q}_\alpha,\mathcal{Q}_\beta} : \alpha < \beta < \mathrm{cf}(\kappa))$.

We say that $\mathcal{Q}$ is a *genericity iterate* of $\mathcal{R}$ if it is a window-based iterate of $\mathcal{R}$ as witnessed by $(w_\alpha : \alpha < \mathrm{cf}(\kappa))$ such that

1. if $x \in \mathbb{R}_g$ then for some $\alpha < \kappa$, x is generic for the extender algebra $\mathsf{EA}^{\mathcal{Q}}_{\pi_{\mathcal{P},\mathcal{Q}(w_\alpha)}}$,

2. for each $\alpha < \mathrm{cf}(\kappa)$, $w_\alpha \in rng(\pi_{\mathcal{P},\mathcal{R}})$.

Suppose $\mathcal{Q}$ is a genericity iterate of $\mathcal{R}$. There is then $h \subseteq Coll(\omega, <\kappa)$ with $h \in \mathcal{V}[g]$ such that h is $\mathcal{Q}$-generic and $\mathbb{R}^{\mathcal{Q}[h]} = \mathbb{R}_g$. We call such h *maximal generics.*

Lemma 1.1 *Suppose $\mathcal{Q}$ is a genericity iterate of $\mathcal{P}$. Let $h \subseteq Coll(\omega, <\kappa)$ be a maximal $\mathcal{Q}$-generic. Then $\mathcal{M}_\infty(\mathcal{Q}) = (\mathcal{M}_\infty(\mathcal{Q}))^{\mathcal{Q}[h]}$.*

Proof. This is simply because $\mathcal{F}^g_{\mathcal{Q}} = (\mathcal{F}^h_{\mathcal{Q}})^{\mathcal{Q}[h]}$. □

Suppose $\iota < \kappa$ and $\mathcal{Q} \in \mathcal{I}^{g_\iota}(\mathcal{P})$ is a window-based iterate of $\mathcal{P}$ as witnessed by $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \mathrm{cf}(\kappa))$. Set $\eta_{\alpha,\beta} = \pi_{\mathcal{P},\mathcal{Q}_\alpha}(\eta_\beta)$, $\eta^\alpha = \eta_{\alpha,\alpha}$, $\delta_{\alpha,\beta} = \pi_{\mathcal{P},\mathcal{Q}_\alpha}(\delta_\beta)$ and $\delta^\alpha = \delta_{\alpha,\alpha}$. Suppose $\mathcal{R} \in \mathcal{F}^g_{\mathcal{Q}}$ and $y \in \mathcal{R}|\kappa$. Let $\alpha_{\mathcal{R},y}$ be the least α such that $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$ is based on $\mathcal{Q}|\eta^\alpha$ and letting $\mathcal{R}'$ be the last model of $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$ when we apply it to $\mathcal{Q}_\alpha$[2] $y \in \mathcal{R}'|\pi_{\mathcal{Q},\mathcal{R}'}(\eta^\alpha)$. Given $\beta \geq \alpha_{\mathcal{R},y}$, let $\mathcal{W}(\mathcal{R}, y, \beta)$ be the last model of $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$ when we regard it as an iteration of $\mathcal{Q}_\beta$. The following lemma is an easy consequence of full normalization.

Lemma 1.2 *Let $\xi \in [\iota, \kappa]$ be such that $\mathcal{R} \in \mathcal{F}^{g_\xi}_{\mathcal{Q}}$. For $\alpha_{\mathcal{R},y} \leq \beta \leq \gamma$, $\mathcal{W}(\mathcal{R}, y, \gamma)$ is a normal $\Sigma^{g_\xi}_{\mathcal{W}(\mathcal{R},y,\beta)}$-iterate of $\mathcal{W}(\mathcal{R}, y, \beta)$ via a normal iteration $\mathcal{U}$ such that $\mathcal{U}$ is above $\pi_{\mathcal{Q}_\beta,\mathcal{W}(\mathcal{R},y,\beta)}(\eta^\beta)$, $\mathcal{U}$ is based on $\mathcal{W}(\mathcal{R}, y, \beta)|\pi_{\mathcal{Q}_\beta,\mathcal{W}(\mathcal{R},y,\beta)}(\delta_{\beta,\gamma})$ and $\pi^{\mathcal{U}}$ exists.*

The following is the main theorem of this section.

Theorem 1.3 *Suppose $\mathcal{Q}$ is a window-based iterate of $\mathcal{P}$. Then*

$$\mathcal{M}_\infty(\mathcal{Q}) = \mathcal{M}_\infty(\mathcal{P}).$$

Proof. We define $j : \mathcal{M}_\infty(\mathcal{Q})|\kappa^{\mathcal{Q}}_\infty \to \mathcal{M}_\infty(\mathcal{P})|\kappa^{\mathcal{P}}_\infty$ as follows. Given $x \in \mathcal{M}_\infty(\mathcal{Q})|\kappa^{\mathcal{Q}}_\infty$ let $\mathcal{R} \in \mathcal{F}_{\mathcal{Q}}$ be such that $\pi^{\mathcal{Q}}_{\mathcal{R},\infty}(y) = x$ for some $y \in \mathcal{R}$. Set

$$j(x) = \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\alpha_{\mathcal{R},y}),\infty}(y).$$

Claim 1. For $\beta \geq \alpha_{\mathcal{R},y}$, $j(x) = \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\beta),\infty}(y)$.

[2] There is an abuse of notation here. What we really mean is that we confuse $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$ with an iteration $\mathcal{U}$ of $\mathcal{Q}_\alpha$ such that $\pi_{\mathcal{Q}_\alpha,\mathcal{Q}}$-copy of $\mathcal{U}$ is $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$. Because $\pi_{\mathcal{Q}_\alpha,\mathcal{Q}} \restriction \eta^\alpha = id$, $\mathcal{U}$ and $\mathcal{T}_{\mathcal{Q},\mathcal{R}}$ have the same extenders and branches, just the models are different.

Proof. Set $\alpha = \alpha_{\mathcal{R},y}$. Because $\text{crit}(\pi_{\mathcal{W}(\mathcal{R},y,\alpha),\mathcal{W}(\mathcal{R},y,\beta)}) > \pi_{\mathcal{Q}_\alpha,\mathcal{W}(\mathcal{R},y,\alpha)}(\eta^\alpha)$, we have the following equalities.

$$\begin{aligned}\pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\alpha),\infty}(y) &= \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\beta),\infty}(\pi_{\mathcal{W}(\mathcal{R},y,\alpha),\mathcal{W}(\mathcal{R},y,\beta)}(y)) \\ &= \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\beta),\infty}(y)\end{aligned}$$

□

Claim 2. j is well-defined.

Proof. Fix $\xi \in [\iota, \kappa)$, $\mathcal{R}, \mathcal{R}' \in \mathcal{F}^{g_\iota}_{\mathcal{Q}}$, $y \in \mathcal{R}$ and $y' \in \mathcal{R}'$ such that $\pi^{\mathcal{Q}}_{\mathcal{R},\infty}(y) = \pi^{\mathcal{Q}}_{\mathcal{R}',\infty}(y') = x$. Let $\beta > \max(\alpha_{\mathcal{R},y}, \alpha_{\mathcal{R}',y'})$. Because of Claim 1 above, it is enough to show that

$$\pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},y,\beta),\infty}(y) = \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R}',y',\beta),\infty}(y')$$

Set $\mathcal{W} = \mathcal{W}(\mathcal{R}, y, \beta)$ and $\mathcal{W}' = \mathcal{W}(\mathcal{R}', y', \beta)$. Let $\eta = \pi_{\mathcal{Q}_\beta,\mathcal{W}}(\eta^\beta)$ and $\eta' = \pi_{\mathcal{Q}_\beta,\mathcal{W}'}(\eta^\beta)$. Notice that $\Sigma^{g_\xi}_{\mathcal{Q}_\beta|\eta^\beta}$ has full normalization.

Let now $\mathcal{U}$ and $\mathcal{U}'$ be the trees on $\mathcal{W}|\eta$ and $\mathcal{W}'|\eta'$ that are obtained by the extender-comparison process. Because $\Sigma^{g}_{\mathcal{Q}_\beta|\eta^\beta}$ has full normalization, strategy disagreements do not appear in the above comparison process. Moreover, because $\Sigma^{g}_{\mathcal{Q}_\beta|\eta^\beta}$ has full normalization, $\mathcal{U}$ and $\mathcal{U}'$ have the same last model $\mathcal{M}$ and their main branches do not drop. Finally, it follows from full normalization of $\Sigma^{g_\xi}_{\mathcal{Q}_\beta|\eta^\beta}$ that

$$\pi_{\mathcal{W}|\eta,\mathcal{M}} \circ \pi_{\mathcal{Q}_\beta|\eta^\beta,\mathcal{W}|\eta} = \pi_{\mathcal{W}'|\eta',\mathcal{M}} \circ \pi_{\mathcal{Q}_\beta|\eta^\beta,\mathcal{W}'|\eta'}.$$

Let $\mathcal{X}$ be the normal $\mathcal{Q}_\beta|\eta^\beta$-to-$\mathcal{M}$ iteration, and let $\mathcal{X}_0$ be the id-copy of $\mathcal{X}$ onto $\mathcal{Q}_\beta$. Similarly define $\mathcal{U}_0$ and $\mathcal{U}'_0$. Let $\mathcal{S}$ be the last model of $\mathcal{X}_0$. It follows that $\mathcal{S}$ is the last model of $\mathcal{U}_0$ and $\mathcal{U}'_0$.

Let now $\mathcal{U}_1$ be the $\pi_{\mathcal{W},\mathcal{R}}$-copy of $\mathcal{U}_0$ and $\mathcal{U}'_1$ be the $\pi_{\mathcal{W}',\mathcal{R}'}$-copy of $\mathcal{U}'_0$. Notice that because $\pi_{\mathcal{W},\mathcal{R}} \restriction \eta = id$, $\mathcal{U}_1$ and $\mathcal{U}_0$ have the same extenders and branches. Similarly for $\mathcal{U}'_1$ and $\mathcal{U}'_0$.

Finally, let $\mathcal{X}_1$ be the $\pi_{\mathcal{Q}_\beta,\mathcal{Q}}$-copy of $\mathcal{X}_0$. Again, $\mathcal{X}_1$ and $\mathcal{X}_0$ have the same extenders. Moreover, $\mathcal{X}_1$ is the full normalization of $(\mathcal{T}_{\mathcal{Q},\mathcal{R}})^\frown\mathcal{U}_1$ and $(\mathcal{T}_{\mathcal{Q},\mathcal{R}'})^\frown\mathcal{U}'_1$. It follows that $\mathcal{X}_1$, $\mathcal{U}_1$ and $\mathcal{U}'_1$ have the same last model. Call it $\mathcal{N}$. It follows from full normalization that $\mathcal{N}$ is a Σ^g-iterate of $\mathcal{S}$ via a normal iteration that is above $\pi_{\mathcal{Q}_\beta,\mathcal{S}}(\eta^\beta)$.

Because $\pi^{\mathcal{Q}}_{\mathcal{R},\infty}(y) = \pi^{\mathcal{Q}}_{\mathcal{R}',\infty}(y')$, we have that $\pi_{\mathcal{R},\mathcal{N}}(y) = \pi_{\mathcal{R}',\mathcal{N}}(y')$. Hence, $\pi_{\mathcal{W},\mathcal{S}}(y) = \pi_{\mathcal{W}',\mathcal{S}}(y')$. We now have the following equalities.

$$\begin{aligned}\pi^{\mathcal{P}}_{\mathcal{W},\infty}(y) &= \pi^{\mathcal{P}}_{\mathcal{S},\infty}(\pi_{\mathcal{W},\mathcal{S}}(y)) \\ &= \pi^{\mathcal{P}}_{\mathcal{S},\infty}(\pi_{\mathcal{W}',\mathcal{S}}(y')) \\ &= \pi^{\mathcal{P}}_{\mathcal{W}',\infty}(y').\end{aligned}$$

□

To finish the proof of the lemma, we show that j is onto and Σ_1-elementary. Fix $u \in \mathcal{M}_\infty(\mathcal{P})|\kappa^{\mathcal{P}}_\infty$. We want to see that there is $x \in \mathcal{M}_\infty(\mathcal{Q})|\kappa^{\mathcal{Q}}_\infty$ such that $j(x) = u$. To start with, let $\mathcal{R} \in \mathcal{F}^g_{\mathcal{P}}$ be such that for some $v \in \mathcal{R}$, $\pi^{\mathcal{P}}_{\mathcal{R},\infty}(v) = u$. Let α be such that $\mathcal{T}_{\mathcal{P},\mathcal{R}}$ is based on $\mathcal{P}|\eta_\alpha$ and $v \in \mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\eta_\alpha)$. Let $\mathcal{S}$ be the result of comparing $\mathcal{R}$ with $\mathcal{Q}_\alpha$. It follows from full normalization[3] that $\mathcal{T}_{\mathcal{Q}_\alpha,\mathcal{S}}$ is based on $\mathcal{Q}_\alpha|\eta^\alpha$ and $\mathcal{T}_{\mathcal{R},\mathcal{S}}$ is based on $\mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\eta_\alpha)$. Let $\mathcal{X}$ be $\pi_{\mathcal{Q}_\alpha,\mathcal{Q}}$-copy of $\mathcal{T}_{\mathcal{Q}_\alpha,\mathcal{S}}$. We have that $\mathcal{X}$ and $\mathcal{T}_{\mathcal{Q}_\alpha,\mathcal{S}}$ have the same extenders and branches. Letting $\mathcal{N}$ be the last model of $\mathcal{X}$, we also have that $\mathcal{N}$ is a normal window-based $\Sigma^g_{\mathcal{S}}$-iterate of $\mathcal{S}$ via an iteration that is above $\pi_{\mathcal{Q}_\alpha,\mathcal{S}}(\eta^\alpha)$. Let $y = \pi_{\mathcal{R},\mathcal{S}}(v)$. Notice that $\pi_{\mathcal{S},\mathcal{N}}(y) = y$.

We now easily have that $j(\pi^{\mathcal{Q}}_{\mathcal{N},\infty}(y)) = u$. Indeed,

$$\begin{aligned}j(\pi^{\mathcal{Q}}_{\mathcal{N},\infty}(y)) &= \pi^{\mathcal{P}}_{\mathcal{S},\infty}(y) \\ &= \pi^{\mathcal{P}}_{\mathcal{S},\infty}(\pi_{\mathcal{R},\mathcal{S}}(v)) \\ &= \pi^{\mathcal{P}}_{\mathcal{R},\infty}(v) \\ &= u.\end{aligned}$$

To see that j is Σ_1-elementary, it is enough to notice that given a Σ_1-formula $\phi(...)$, $\mathcal{M}_\infty(\mathcal{Q})|\kappa^{\mathcal{Q}}_\infty \vDash \phi[\vec{x}]$ if and only if there is $\mathcal{R} \in \mathcal{F}^g_{\mathcal{Q}}$ such that for some $\vec{y} \in \mathcal{R}|\kappa$ and some $\alpha > \alpha_{\mathcal{R},\vec{y}}$, $\pi^{\mathcal{Q}}_{\mathcal{R},\infty}(\vec{y}) = \vec{x}$ and

$$\mathcal{W}(\mathcal{R},\vec{y},\alpha)|\pi_{\mathcal{Q}_\alpha,\mathcal{W}(\mathcal{R},\vec{y},\alpha)}(\eta^\alpha) \vDash \phi[\vec{y}].$$

As $j(\vec{x}) = \pi^{\mathcal{P}}_{\mathcal{W}(\mathcal{R},\vec{y},\alpha),\infty}(\vec{y})$, the Σ_1-elementarity follows.

Since j is onto and Σ_1-elementary, it follows that

$$\mathcal{M}_\infty(\mathcal{Q})|\kappa^{\mathcal{Q}}_\infty = \mathcal{M}_\infty(\mathcal{P})|\kappa^{\mathcal{P}}_\infty.$$

The fact that $\mathcal{M}_\infty(\mathcal{Q}) = \mathcal{M}_\infty(\mathcal{P})$ follows from full normalization. □

Set $\kappa_\infty = \kappa^{\mathcal{P}}_\infty$ and $\mathcal{M}_\infty = \mathcal{M}_\infty(\mathcal{P})$. The following useful corollary is an immediate consequence of full normalization and Theorem 1.3.

[3] A similar argument was used in the proof of Claim 2 to conclude that $\mathcal{X}_0$, $\mathcal{U}_0$ and $\mathcal{U}'_0$ have the same last model.

Corollary 1.4 *Suppose $\mathcal{R} \in \mathcal{F}^g_{\mathcal{P}}$ or $\mathcal{R}$ is a window based iterate of $\mathcal{P}$. Suppose $\mathcal{Q}$ is a window-based iterate of $\mathcal{R}$. Then $\mathcal{Q}$ is a window based iterate of $\mathcal{P}$ and*

$$\pi^{\mathcal{R}}_{\mathcal{R},\infty} = \pi^{\mathcal{Q}}_{\mathcal{Q},\infty} \circ \pi_{\mathcal{R},\mathcal{Q}}.$$

The following proposition will be used in the next section.

Proposition 1.5 *Suppose $\mathcal{Q}$ is a genericity iteration of $\mathcal{P}$ and h is a maximal $\mathcal{Q}$-generic. Suppose $X \in \mathcal{V}[g]$ is a countable subset of $\mathcal{M}_\infty(\mathcal{P})|\xi$ for some $\xi < \pi_{\mathcal{P},\infty}(\kappa)$. Then $X \in \mathcal{Q}[h]$.*

Proof. Let $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ witness that $\mathcal{Q}$ is a genericity iteration of $\mathcal{P}$. We can fix $\mathcal{R} \in \mathcal{F}^g_{\mathcal{P}}$, $\alpha < \kappa$, a real $z \in \mathbb{R}_g$ that codes a bijection $f : \omega \to \mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\eta_\alpha)$ and a $y \subseteq \omega$ such that

$$\beta \in X \text{ if and only if there is } i \in y \text{ such that } \pi^{\mathcal{P}}_{\mathcal{R},\infty}(f(i)) = \beta.$$

Let $\mathcal{S}$ be a common iterate of $\mathcal{Q}_\alpha$ and $\mathcal{R}$. We have that $\mathcal{S} \in \mathcal{F}_{\mathcal{Q}_\alpha}$. Let $\mathcal{N}$ be the last model of $\pi_{\mathcal{Q}_\alpha,\mathcal{Q}}$-copy of $\mathcal{T}_{\mathcal{Q}_\alpha,\mathcal{S}}$. Let $\sigma : \mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\eta_\alpha) \to \mathcal{N}|\pi_{\mathcal{Q},\mathcal{N}}(\eta^\alpha)$ be the iteration embedding. We now have that

$$\beta \in X \text{ if and only if there is } i \in y \text{ such that } \pi^{\mathcal{Q}}_{\mathcal{N},\infty}(\sigma(f(i))) = \beta.$$

It follows from Lemma 1.1 that $X \in \mathcal{Q}[z]$. □

2 A Chang model over the derived model

Suppose $\mathcal{V} \vDash \mathsf{ZFC}$ is a hod premouse and κ is a limit of Woodin cardinals of $\mathcal{V}$ such that if $\mathrm{cf}(\kappa) < \kappa$ then $\mathrm{cf}(\kappa)$ is not a measurable cardinal. Let $g \subseteq Coll(\omega, < \kappa)$ be $\mathcal{V}$-generic and let $\mathcal{P} = \mathcal{V}|(\kappa^+)^{\mathcal{V}}$. Set

$$\mathbb{R}^*_g = \cup_{\alpha<\kappa} \mathbb{R}^{\mathcal{V}[g_\alpha]},$$
$$\Gamma^*_g = \{A^g \cap \mathbb{R}^* : \exists \alpha < \kappa (A \in \Gamma^\infty_{g_\alpha})\},$$
$$C(\kappa, g) = L(\mathcal{M}_\infty, \cup_{\xi<\kappa_\infty} \wp_{\omega_1}(\mathcal{M}_\infty|\xi), \Gamma^*_g, \mathbb{R}^*_g)$$

Assume next κ is regular. In this case, $\mathbb{R}^*_g = \mathbb{R}_g$ and $\Gamma^*_g = \Gamma^\infty_g$. Given $\xi < \kappa_\infty$, let $\mu_\xi \in \mathcal{V}[g]$ be the club filter on $\wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. Working in $\mathcal{V}[g]$, set

$$S(\kappa, g) = L(\mathcal{M}_\infty, \cup_{\xi<\kappa_\infty} \wp_{\omega_1}(\mathcal{M}_\infty|\xi), \Gamma^\infty_g, \mathbb{R}_g)[(\mu_\xi : \xi < \kappa_\infty)].$$

More precisely, we let $S(\kappa, g)$ be defined as follows. The definition takes place in $\mathcal{V}[g]$.

1. $S'_0(\kappa, g)$ is the least transitive set containing $\{\mathcal{M}_\infty, \cup_{\xi<\kappa_\infty}\wp_{\omega_1}(\mathcal{M}_\infty|\xi), \Gamma^\infty_g, \mathbb{R}_g\}$[4] and

$$S_0(\kappa, g) = (S'_0(\kappa, g), \{\mu^0_\xi\}_{\xi<\kappa_\infty}, \in)$$

where $\mu^0_\xi = \mu_\xi \cap S'_\alpha(\kappa, g)$.

2. For $\alpha \in Ord \cup \{Ord\}$, $S_\alpha(\kappa, g) = (S'_\alpha(\kappa, g), \{\mu^\alpha_\xi\}_{\xi<\kappa_\infty}, \in)$ where $\mu^\alpha_\xi = \mu_\xi \cap S'_\alpha(\kappa, g)$.

3. For $\alpha \in Ord$, $S'_{\alpha+1}(\kappa, g)$ is the set of all sets that are definable over $S_\alpha(\kappa, g)$ with parameters in $\mathcal{S}'_\alpha(\kappa, g)$.

4. For limit ordinal $\lambda \in Ord$, $S'_\lambda(\kappa, g) = \cup_{\alpha<\lambda} S'_\alpha(\kappa, g)$.

5. $S(\kappa, g) = S_{Ord}(\kappa, g)$.

Theorem 2.1 *$C(\kappa, g) \vDash \mathsf{AD}^+$ and $S(\kappa, g) \vDash \mathsf{AD}^+ + \mathsf{DC}$. Moreover, $S(\kappa, g) \vDash$ "ω_1 is $< \kappa_\infty$-supercompact".*

The proof of the theorem will use the following proposition which establishes an invariance result for the two Chang-style models that we have defined. Suppose $\mathcal{Q}$ is a genericity iterate of $\mathcal{P}$. We then let $\mathcal{V}_\mathcal{Q}$ be the iterate of $\mathcal{V}$ when we apply $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$ to $\mathcal{V}$.

Proposition 2.2 *Suppose $\mathcal{Q}$ is a genericity iterate of $\mathcal{P}$ and h is a maximal $\mathcal{Q}$-generic. Then $C(\kappa, g) = (C(\kappa, h))^{\mathcal{V}_\mathcal{Q}(\mathbb{R}^*_g)}$ and $S(\kappa, g) = (S(\kappa, h))^{\mathcal{V}_\mathcal{Q}[h])}$.*

That $C(\kappa, g) = (C(\kappa, h))^{\mathcal{V}_\mathcal{Q}(\mathbb{R}^*_g)}$ follows immediately from Proposition 1.5. It remains to show that $S(\kappa, g) = (S(\kappa, h))^{\mathcal{V}_\mathcal{Q}(\mathbb{R}^*_g)}$, which we will show after producing invariant definitions of subsets of $\wp_{\omega_1}(\mathcal{M}_\infty|\xi)$ that are in $S(\kappa, g)$. This will be the goal of the next section.

[4]Notice that $\Gamma^\infty_g = \wp(\mathbb{R}^*_g) \cap S_0(\kappa, g)$.

2.1 Invariant definitions

We assume κ is a regular cardinal. We say α is *stable* if whenever $\mathcal{Q}$ is a genericity iteration of $\mathcal{P}$ and h is a maximal $\mathcal{Q}$-generic, for $\beta < \alpha$, $S_\beta(\kappa, h)^{\mathcal{V}_\mathcal{Q}[h]} = S_\beta(\kappa, g)$. It follows that $S'_\alpha(\kappa, h)^{\mathcal{V}_\mathcal{Q}[h]} = S'_\alpha(\kappa, g)$, and it also follows from Proposition 1.5 that 0 is stable.

Notational digression. Suppose that $\mathcal{N}$ is a hod premouse, λ is a regular limit of Woodin cardinals of $\mathcal{N}$, $\iota_0 < \iota_1 < \lambda$, $u_0 <_W u_1$ are windows of $\mathcal{N}$ such that $\delta^{u_1} < \lambda$, $h \subseteq Coll(\omega, < \lambda)$ is $\mathcal{N}$-generic and k is a real coding $(k_0, k_1, k_2, k_3, k_4)$ such that k is generic for $\mathsf{EA}^{\mathcal{N}}_{\mathsf{u}_0}$. Suppose further that

1. k_0 codes a hod pm $\mathcal{X}$,
2. k_1 codes a bijection $l : \omega \to \mathcal{X}$,
3. $k_2 \subseteq \omega$,
4. $k_3 \in \mathbb{R}$,
5. k_4 codes an embedding $i : \mathcal{X} \to \mathcal{N}|(\iota_1^+)^{\mathcal{N}}$ such that $\iota_0 \in rng(i)$,

We then let $\Lambda(\mathcal{N}, k)$ be the i-pullback of $S^{\mathcal{N}[k][h]}_{\mathcal{N}|(\iota_1^+)^{\mathcal{N}}}$ and $Y(\mathcal{N}, k) = \{\pi^{\mathcal{N}|\lambda}_{\mathcal{N}|\lambda,\infty}(l(n)) : n \in k_2\}$.

Fix a stable α and let $\xi < \kappa_\infty$. Let $A \in S'_\alpha(\kappa, g)$ be a subset of $\wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. We say $dt = (\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ is an (α, ξ)-*definability tuple* if

1. $\mathcal{R} \in \mathcal{F}^g_{\mathcal{P}}$ or $\mathcal{R}$ is a window-based iterate of $\mathcal{P}$,
2. $\beta < \alpha$,
3. whenever $\mathcal{S} \in \mathcal{F}^g_{\mathcal{R}}$ or $\mathcal{S}$ is a window-based iterate of $\mathcal{R}$,
$$\pi_{\mathcal{R},\mathcal{S}}(\beta, \xi, \xi_0, ..., \xi_n) = (\beta, \xi, \xi_0, ..., \xi_n),$$
4. $\nu_0 < \nu_1 < \kappa$,
5. $z = (z_0, z_1, z_2, z_3)$ such that z_0 codes $\mathcal{R}|(\nu_1^+)^{\mathcal{R}}$, z_1 codes a bijection $l : \omega \to \mathcal{R}|(\nu_1^+)^{\mathcal{R}}$, $z_2 \subseteq \omega$ and $z_3 \in \mathbb{R}_g$,
6. ψ is a formula,

7. $(\xi_0, ..., \xi_n) \in \kappa_\infty^{<\omega}$.

Suppose

$$dt = (\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$$

is an (α, ξ)-definability tuple. We say $\mathcal{Q}$ is a dt-good genericity iterate of $\mathcal{R}$ if $\mathcal{Q}$ is a genericity iterate of $\mathcal{R}$ such that letting $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ be the windows witnessing that $\mathcal{Q}$ is a genericity iterate of $\mathcal{R}$

1. $\nu_1 < \eta_0$, and
2. letting $z(\mathcal{Q})$ code $\pi_{\mathcal{R},\mathcal{Q}} \restriction (\mathcal{R}|(\nu_1^+)^{\mathcal{R}})$ and setting $k(\mathcal{Q}) = (z_0, z_1, z_2, z_3, z(\mathcal{Q}))$, $k(\mathcal{Q})$ is generic for $\mathsf{EA}^{\mathcal{Q}}_{\pi_{\mathcal{R},\mathcal{Q}}(\mathsf{w}_0)}$.

We say that A has an *invariant definition* if there is an (α, ξ)-definability tuple $dt = (\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ such that whenever $\mathcal{Q}$ is a dt-good genericity iteration of $\mathcal{R}$, for $X \in \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$,

$$X \in A \text{ if and only if } \mathcal{Q}[k(\mathcal{Q})][X] \vDash \psi[\Lambda(\mathcal{Q}, k(\mathcal{Q})), Y(\mathcal{Q}, k(\mathcal{Q})), z_3, \beta, (\xi_0, ..., \xi_n), X].$$

We continue with our stable α.

Lemma 2.3 *Suppose $A \in S'_\alpha(\kappa, g)$. Then A has an invariant definition.*

Proof. Let $\beta < \alpha$ be such that A is definable over $S_\beta(\kappa, g)$. Let $(\phi, Y, \zeta, B, x, (\xi_0, ..., \xi_n))$ be such that

1. ϕ is a formula,
2. $Y \in \wp_{\omega_1}(\mathcal{M}_\infty|\zeta)$,
3. $B \in \Gamma_g^\infty$,
4. $x \in \mathbb{R}_g$,
5. $(\xi_0, ..., \xi_n) \in \kappa_\infty^{<\omega}$,
6. for $X \in \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$, $X \in A$ if and only if $S_\beta(\kappa, g) \vDash \phi[Y, B, x, \mu^\beta_{\xi_0}, ..., \mu^\beta_{\xi_n}, X]$.

Let $\mathcal{R} \in \mathcal{F}^g_{\mathcal{P}}$ be such that whenever $\mathcal{S} \in \mathcal{F}^g_{\mathcal{R}}$, $\pi_{\mathcal{R},\mathcal{S}}(\beta) = \beta$. Moreover, we can pick $\mathcal{R}$ in a way that $Y \cup \{\zeta\} \in rng(\pi^{\mathcal{P}}_{\mathcal{R},\infty})$ and for some $\nu_0 < \kappa$, B is projective in $\Sigma^g_{\mathcal{R}|\nu_0}$. Let $\nu_1 > \max((\pi^{\mathcal{P}}_{\mathcal{R},\infty})^{-1}(\zeta), \nu_0)$ be a regular cardinal of $\mathcal{P}$ such that ν_1 is fixed by $\pi_{\mathcal{P},\mathcal{R}}$ and $\mathcal{T}_{\mathcal{P},\mathcal{R}}$ is based on $\mathcal{P}|\nu_1$. Let $z = (z_0, z_1, z_2, z_3)$ such that

1. z_0 codes $\mathcal{R}|(\nu_1)^+$,

2. z_1 codes a bijection $l : \omega \to \mathcal{R}|(\nu_1)^+$,

3. $z_2 \subseteq \omega$ such that $Y = \{\pi^{\mathcal{P}}_{\mathcal{R},\infty}(l(i)) : i \in z_2\}$,

4. $z_3 = x$.

There is now a formula ϕ_1 such that for $X \in \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$, $X \in A$ if and only if

$$S_\beta(\kappa, g) \vDash \phi_1[z, \pi^{\mathcal{P}}_{\mathcal{R},\infty} \restriction (\mathcal{R}|\nu_1), \mu^\beta_{\xi_0}, ..., \mu^\beta_{\xi_n}, X].$$

Suppose now that $\mathcal{Q}$ is a genericity iterate of $\mathcal{R}$ as witnessed by windows $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ such that $\nu_1 < \eta_0$ and letting $z_4(\mathcal{Q})$ code $\pi_{\mathcal{R},\mathcal{Q}} \restriction (\mathcal{R}|(\nu_1^+)^{\mathcal{R}})$, $k = (z_0, z_1, z_2, z_3, z_4(\mathcal{Q}))$ is generic over $\mathsf{EA}_{\pi_{\mathcal{R},\mathcal{Q}}(\mathsf{w}_0)}$.

Because $\pi^{\mathcal{P}}_{\mathcal{R},\infty} \restriction (\mathcal{R}|\nu_1) = (\pi^{\mathcal{Q}}_{\mathcal{Q},\infty} \circ \pi_{\mathcal{R},\mathcal{Q}}) \restriction (\mathcal{R}|\nu_1)$, we can find a formula ψ independent of $\mathcal{Q}$ such that for $X \in \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$, $X \in A$ if and only if

$$\mathcal{Q}[k][X] \vDash \psi[\Lambda(\mathcal{Q}, k), Y(\mathcal{Q}, k), z_3, \beta, (\xi_0, \xi_1, ..., \xi_n), X].$$

It then follows that $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ is as desired. $\square$

2.2 A characterization of a club filter

Suppose now that α is stable, $\xi < \kappa_\infty$ and $A \in S'_\alpha(\kappa, g)$ is a subset of $\wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. Let $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ is an invariant definition of A. We say that $\mathcal{Q}$ is an A-genericity iterate of $\mathcal{R}$ if $\mathcal{Q}$ is a genericity iterate of $\mathcal{R}$ as witnessed by windows $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ such that $k(\mathcal{Q})$ is generic for $\mathsf{EA}_{\pi_{\mathcal{R},\mathcal{Q}}(\mathsf{w}_0)}$. We will then abuse our notation and write $\mathcal{Q}[k(\mathcal{Q})][X] \vDash$ "$X \in A$" or $\mathcal{Q}[k(\mathcal{Q})][X] \vDash$ "$X \notin A$" where "$X \in A$" is interpreted in the obvious way using Lemma 2.3.

Suppose $\mathcal{R} \in \mathcal{F}^g(\mathcal{P})$ and $\tau < \kappa$. Let $b(\mathcal{R}, \tau) = \pi^{\mathcal{P}}_{\mathcal{R},\infty}[\mathcal{R}|\tau]$ and let $B(\mathcal{R}, \tau) = \{a(\mathcal{S}, \pi_{\mathcal{R},\mathcal{S}}(\tau)) : \mathcal{S} \in \mathcal{F}^g(\mathcal{R})\}$. It is not hard to show that $B(\mathcal{R}, \tau) \in \mu_{\pi^{\mathcal{P}}_{\mathcal{R},\infty}(\tau)}$.

Lemma 2.4 *Suppose α is stable, $\xi < \kappa_\infty$ and $A \in S'_\alpha(\kappa, g)$ is such that $A \subseteq \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. Let $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ is an invariant definition of A such that for some $\tau < \kappa$, $\xi = \pi^{\mathcal{P}}_{\mathcal{R},\infty}(\tau)$. Assume further that $b(\mathcal{R}, \tau) \in A$. Then $B(\mathcal{R}, \tau) \in A$.*

Proof. Let $\mathcal{Q}$ be an A-genericity iterate of $\mathcal{R}$ such that $\mathcal{T}_{\mathcal{R},\mathcal{Q}}$ is above τ. It then follows that $\mathcal{Q}[k(\mathcal{Q}), b(\mathcal{R},\tau)] \vDash$ "$b(\mathcal{R},\tau) \in B$".

Suppose now that $\mathcal{S} \in \mathcal{F}^g(\mathcal{R})$. Pick an A-genericity iterate $\mathcal{Q}$ of $\mathcal{R}$ such that $\mathcal{T}_{\mathcal{R},\mathcal{S}}$ is based on $\mathcal{Q}|\eta_0$ where $w_0 = (\eta_0, \delta_0)$ is the least window used in $\mathcal{T}_{\mathcal{R},\mathcal{Q}}$. Let $\mathcal{W}'$ be the result of applying $\mathcal{T}_{\mathcal{R},\mathcal{S}}$ to $\mathcal{Q}$. Because Σ^g has full normalization, we have that $\mathcal{W}'$ is a normal $\Sigma^g_{\mathcal{S}}$-iterate of $\mathcal{S}$. Let $\mathcal{W}$ be an A-genericity iteration of $\mathcal{W}'$ that is above $\pi_{\mathcal{Q},\mathcal{W}'}(\eta_0)$. It then follows by elementarity and Corollary 1.4 that $\mathcal{W}[k(\mathcal{W}), b(\mathcal{S}, \pi_{\mathcal{R},\mathcal{S}}(\tau))] \vDash$ "$b(\mathcal{S}, \pi_{\mathcal{R},\mathcal{S}}(\tau)) \in A$". Therefore, $b(\mathcal{S}, \pi_{\mathcal{R},\mathcal{S}}(\tau)) \in B$. □

Lemma 2.5 *Suppose α is stable, $\xi < \kappa_\infty$ and $A \in S'_\alpha(\kappa, g)$ is such that $A \subseteq \wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. Then $A \in \mu_\xi$ if and only if for some $\mathcal{R} \in \mathcal{F}^g(\mathcal{P})$ and τ such that $\pi^{\mathcal{P}}_{\mathcal{R},\infty}(\tau) = \xi$, $B(\mathcal{R}, \xi) \subseteq B$.*

Proof. Suppose $B \in \mu_\xi$. Letting $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ is an invariant definition of A with the property that for some $\tau < \kappa$, $\xi = \pi^{\mathcal{P}}_{\mathcal{R},\infty}(\tau)$, we have that either $B(\mathcal{R}, \tau) \subseteq A$ or $B(\mathcal{R}, \tau) \subseteq A^c$. Because $B(\mathcal{R}, \tau)$ contains a club, we must have that $B(\mathcal{R}, \tau) \subseteq B$. □

The following is an easy corollary of Lemma 2.4 and Lemma 2.5.

Corollary 2.6 *Suppose α is stable and $\xi < \kappa_\infty$. Then μ^α_ξ is an ω_1-supercompactness measure over $S'_\alpha(\kappa, g)$.*

Lemma 2.7 *Suppose α is stable. Then $\alpha + 1$ is also stable. Moreover, if $\mathcal{Q}$ is a genericity iteration of $\mathcal{P}$ and h is a maximal $\mathcal{Q}$-generic then*

$$S_\alpha(\kappa, g) = S_\alpha(\kappa, h)^{\mathcal{V}_{\mathcal{Q}}[h]}.$$

Proof. To show the above, it is enough to show that for each $\mathcal{Q}, h$ as above and $\xi < \kappa_\infty$,

$$\mu^\alpha_\xi = (\mu^\alpha_\xi)^{\mathcal{V}_{\mathcal{Q}}[h]}.$$

Fix $(\mathcal{Q}, h, \xi)$ as above and let $A \in S'_\alpha(\kappa, g)$ be a subset of $\wp_{\omega_1}(\mathcal{M}_\infty|\xi)$. Suppose $A \in \mu^\alpha_\xi$. Fix $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \beta)$ that is an invariant definition of A with the property that for some $\tau < \kappa$, $\xi = \pi^{\mathcal{P}}_{\mathcal{R},\infty}(\tau)$. We now have that $B(\mathcal{R}, \tau) \subseteq A$. Let $(w_\gamma = (\eta_\gamma, \delta_\gamma) : \gamma < \kappa)$ be the windows used in $\mathcal{T}_{\mathcal{P},\mathcal{Q}}$. Let ζ be least such that $\mathcal{T}_{\mathcal{P},\mathcal{R}}$ is based on $\mathcal{P}|\eta_\zeta$.

Let $\mathcal{S}$ be the result of comparing $\mathcal{Q}_\zeta$ with $\mathcal{R}$. Let $\mathcal{W}$ be the result of applying $\mathcal{T}_{\mathcal{Q}_\zeta,\mathcal{S}}$ to $\mathcal{Q}$. Let $\nu = \pi_{\mathcal{S},\mathcal{W}}(\pi_{\mathcal{R},\mathcal{S}}(\tau))$.

It now follows from Corollary 1.4 that in $\mathcal{Q}[h]$, $B(\mathcal{W}, \nu) \subseteq A$. Hence, $A \in (\mu_\xi^\alpha)^{\mathcal{V}_\mathcal{Q}}$. The claim now follows from Corollary 2.6. ☐

Finally, we can prove that $S(\kappa, g)$ is invariant, thus finishing the proof of Proposition 2.2

Corollary 2.8 *Suppose $\mathcal{Q}$ is an genericity iterate of $\mathcal{P}$ and h is a maximal $\mathcal{Q}$-generic. Then $S(\kappa, g) = S(\kappa, h)^{\mathcal{Q}[h]}$.*

Proof. It is enough to show that each α is stable. We have that 0 is stable. We also clearly have that if α is a limit ordinal and every $\beta < \alpha$ is stable then α is also stable. The claim now follows from Lemma 2.7. ☐

The following is an easy corollary of Corollary 2.6 and Corollary 2.8.

Corollary 2.9 $S(\kappa, g) \vDash$ *"ω_1 is $< \kappa_\infty$-supercompact".*

3 AD^+ in C and S models

In this section, we finish the proof of Theorem 2.1. As it is harder to show that $S(\kappa, g) \vDash \mathsf{AD}^+$, we will do this and leave the C case to the reader.

We continue with $\mathcal{V}, g$ and $\mathcal{P}$ as defined in the previouse sections. Suppose $A \in S(\kappa, g) \cap \wp(\mathbb{R}_g)$. It is enough to show that $A \in \Gamma^\infty_g$. To show this, it is enough to show that for some window w, A is projective in $\Sigma^g_{\mathcal{P}|\delta^w}$.

Fix α least such that A is definable over $S_\alpha(\kappa, g)$. We can think of A as a subset of $\wp_{\omega_1}(\mathcal{M}_\infty|\omega)$. As such, A has an invariant definition as shown by Lemma 2.3. Let then $(\mathcal{R}, (\nu_0, \nu_1), z, \psi, (\xi_0, ..., \xi_n), \alpha)$ be an invariant definition of A.

Let $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \kappa)$ be a $<_W$-incresing sequence of windows of $\mathcal{P}$ such that $\mathcal{T}_{\mathcal{P},\mathcal{R}}$ is based on $\mathcal{P}|\eta_0$. The following lemma finishes the proof.

Lemma 3.1 *A is projective in $\Sigma^g_{\mathcal{P}|(\delta_1^+)^\mathcal{P}}$.*

Proof. Let $\mathcal{W}$ be an iterate of $\mathcal{R}$ based on $\pi_{\mathcal{P},\mathcal{R}}(w_0)$ such that $k(\mathcal{W})$ is generic for $\mathsf{EA}^\mathcal{W}_{\pi_{\mathcal{P},\mathcal{W}}(w_0)}$. Let $\theta[x, k(\mathcal{W}), \alpha, (\xi_0, ..., \xi_n)]$ be the formula that expresses "$x \in A$" over A-genericity iterates of $\mathcal{W}$ that are obtained by iterating above $\pi_{\mathcal{P},\mathcal{W}}(\delta_0)$.

We now claim that $x \in A$ if and only if whenever $\mathcal{S} \in \mathcal{F}^g_\mathcal{W}$ is an iterate of $\mathcal{W}$ based on the window $\pi_{\mathcal{P},\mathcal{W}}(w_1)$ such that x is generic for $\mathsf{EA}^\mathcal{S}_{\pi_{\mathcal{P},\mathcal{S}}(w_1)}$,

$$\mathcal{S}[k(\mathcal{S}), x] \vDash \theta[x, k(\mathcal{S}), \alpha, (\xi_0, ..., \xi_n)].$$

First notice that for any such iterate $k(\mathcal{S}) = k(\mathcal{W})$.

To see that the equivalence holds, fix $(x, \mathcal{S})$ as above, and let $\mathcal{Q}$ be a genericity iterate of $\mathcal{S}$ using the windows $(\pi_{\mathcal{P},\mathcal{S}}(w_\alpha) : \alpha \in [2, \kappa))$. We then have the following equivalences:

$$\begin{aligned} x \in A &\leftrightarrow \mathcal{Q}[k(\mathcal{Q}), x] \vDash \theta[x, k(\mathcal{Q}), \alpha, (\xi_0, ..., \xi_n)] \\ &\leftrightarrow \mathcal{S}[k(\mathcal{S}), x] \vDash \theta[x, k(\mathcal{S}), \alpha, (\xi_0, ..., \xi_n)]. \end{aligned}$$

Set $u = \pi_{\mathcal{P},\mathcal{W}}(w_1)$, $u = (\eta, \delta)$ and $\mathcal{X} = \mathcal{W}|(\delta^+)^{\mathcal{W}}$. Let $\tau \in (\mathcal{W}[k(\mathcal{W})])^{Coll(\omega,\delta)}$ be such that whenever $G \subseteq Coll(\omega, \delta)$ is $\mathcal{W}[k(\mathcal{W})]$-generic, $\tau_G = \{x \in \mathcal{W}[k(\mathcal{W})][G] : \mathcal{W}[k(\mathcal{W}), x] \vDash \theta[x, k(\mathcal{S}), \alpha, (\xi_0, ..., \xi_n)]$. It then follows that the following are equivalent:

1. $x \in A$.
2. Whenever $\mathcal{Y}$ is a $\Sigma^g_{\mathcal{X}}$-iterate of $\mathcal{X}$ based on u such that x is generic for $\mathsf{EA}^{\mathcal{Y}}_{\pi_{\mathcal{X},\mathcal{Y}}(u)}$, $\mathcal{Y}[k(\mathcal{W}), x] \vDash\!\!\!\Vdash_{Coll(\omega, \pi_{\mathcal{X},\mathcal{Y}}(\delta))} \check{x} \in \tau$.

It follows that A is projective in $\Sigma^g_{\mathcal{X}}$, and since $\mathcal{X}$ is a $\Sigma^g_{\mathcal{P}|(\delta_1^+)^{\mathcal{P}}}$-iterate of $\mathcal{P}|(\delta_1^+)^{\mathcal{P}}$, the claim follows. □

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