

Covering with universally Baire operators^{*†}

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Abstract

We introduce a covering conjecture and show that it holds below $AD_{\mathbb{R}} + “\Theta$ is regular”.

We then use it to show that in the presence of mild large cardinal axioms, PFA implies that there is a transitive model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + “\Theta$ is regular”.

The method used to prove the Main Theorem of this paper is the core model induction. The paper contains the first application of the core model induction that goes significantly beyond the region of $AD^+ + \theta_0 < \Theta$.

One of the central themes in set theory is to identify canonical inner models which compute successor cardinals correctly. A prototype of such results is Jensen’s famous covering theorem which in particular implies that provided $0^\#$ doesn’t exist, for every cardinal $\kappa \geq \omega_2$, $\text{cf}((\kappa^+)^L) \geq \kappa$ where L is the constructible universe.

Clearly “canonical inner model” is open for interpretations. For an inner model theorist, the canonical objects of a set theoretic universe are the sets coded by a mixture of *fine extender sequences* and the *universally Baire sets*. Recall that a set of reals is universally Baire if its continuous preimages in all compact Hausdorff spaces have the property of Baire¹.

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¹A set of reals is said to have the property of Baire if it is different from an open set by a meager set.

In more set theoretic terms, a set of reals A is κ -universally Baire (or simply κ -uB) if there are trees T and S on $\kappa \times \omega$ such that $p[T] = A$ and for every partial ordering \mathbb{P} of size $< \kappa$ and for every generic $g \subseteq \mathbb{P}$, $V[g] \models "p[T] = (p[S])^c"$. A set of reals A is called universally Baire if it is κ -uB for every κ . Universally Baire sets are considered to be canonical because they have canonical interpretations in all generic extensions of V , namely if A, T and S are as above then $p[T]$ computed in various generic extensions is the canonical interpretation of A . Moreover, a simple absoluteness argument shows that this interpretation is independent of the choice of T and S . Such interpretations can be used to show that the information coded by universally Baire sets persists to generic extensions. For instance, if all Σ_3^1 sets are universally Baire then Σ_3^1 generic absoluteness holds, i.e., for every Σ_3^1 sentence ϕ , ϕ is true in some generic extension if and only if it is true in V .

The fine extender sequences that have been considered in order to construct canonical inner models exhibiting covering properties come from the K^c constructions. A K^c construction is a construction that produces a model denoted by K^c with the property that given any sufficiently robust embedding $j : M \rightarrow N$, then for any $\eta \in Ord$, $j \upharpoonright (K^c|\eta) \in K^c$. We have some freedom over how robust $j : M \rightarrow N$ should be. Weaker robustness conditions yield bigger models. The K^c construction of [14] requires, among other things, that for some inaccessible κ , $V_\kappa \subseteq M$, $cp(j) = \kappa$ and M and N are countably closed. By calibrating the robustness conditions we get many different K^c constructions. Some of these constructions are trivial or too weak as there may never be an embedding $j : M \rightarrow N$ which has our desired robustness condition or there may be stronger embeddings present in the universe which our robustness condition just ignores. We informally say that the K^c construction is maximal if the robustness condition essentially covers all possible (and reasonable) embeddings. Showing that one of these maximal K^c constructions converges to a transitive inner model containing the ordinals is one of the central open problems in inner model theory. If a maximal K^c construction converges to a transitive inner model containing the ordinals then the resulting model has covering properties. For instance, the authors of [2] introduced a K^c construction and showed that if it converges to a transitive inner model containing the ordinals and κ is an inaccessible cardinal then, assuming there is no inner model with a superstrong cardinal, $cf(Ord \cap \mathcal{S}(K^c|\kappa)) \geq \kappa$ where \mathcal{S} is the stack of all countably iterable mice extending $K^c|\kappa$ and projecting to κ .

In most cases, to show that a K^c construction converges it is enough to show that the countable submodels of the models produced during the K^c construction are countably iterable. This is the content of the iterability conjecture of [19] (see Conjecture 6.5 of [19]). The best partial result is that a K^c construction converges provided there is no non-domestic mouse (see [1]). One of the modern new techniques in inner model theory is the core model

induction and the work in this paper started by asking how the core model induction can be used to prove instances of the iterability conjecture.

We remark that from now on by “ K^c construction” we mean the K^c construction introduced in [1].

Naturally, an inner model theorist will conjecture that transitive inner models that are closed under all the universally Baire sets of the universe and also are closed under robust embeddings have covering properties, i.e., if M is this hypothetical universe then for many cardinals κ , $cf((\kappa^+)^M) \geq \kappa$. Taking this intuition seriously, when we try to prove the iterability conjecture via core model induction, instead of proving instances of iterability conjecture, we arrive at a natural covering conjecture which we call the *UB-Covering Conjecture*. As far as applications go, such a covering conjecture is all one needs (see Theorem 0.3).

In intuitive terms, the UB-Covering Conjecture says the following. First fix a cardinal η such that there is a sufficiently rich large cardinal structure below η and η itself is measurable. Then there is some uB set A such that (i) the K^c construction which is done in a way that the resulting model is closed under A reaches height η or (ii) if μ is the sup of the ordinals that are bigger than η and are coded by those subsets of η that appear in A -closed mice then $cf(\mu) \geq \eta$. As is usual with such conjectures, we have to add an anti large cardinal axiom simply because the theory of mice is only fully developed below superstrong cardinals. We now explain the above intuition in more formal terms. Below, we will use iteration strategies as our source of uB sets. We will explain the reason behind this move after we state the conjecture.

We say a name $\dot{F} \in V^{Coll(\omega, \nu)}$ is *symmetric* if whenever $g, h \subseteq Coll(\omega, \nu)$ are such that $V[g] = V[h]$ then $\dot{F}_g = \dot{F}_h$. For instance, the canonical name for the set of reals is a symmetric name.

Suppose η is an inaccessible cardinal. Let (\mathcal{M}, Σ) be such that \mathcal{M} is a countable transitive model of some fragment of *ZFC* and Σ is η or (η, η) -iteration strategy for \mathcal{M} with hull condensation. Suppose F is either

1. a Σ -mouse operator defined on V_η , i.e., there is some formula ϕ in the language of Σ -mice such that F is a function with the property that for some set X , $dom(F) = \{Y : X \in L_1(Y)\}$ and for every $Y \in dom(F)$, $F(Y)$ is the minimal Σ -mouse over Y satisfying $\phi[Y]$ (the set X is called a base for F) or
2. an iteration strategy for some transitive \mathcal{N} such that for some $b \in HC$, \mathcal{N} is a sound Σ -mouse over b such that $\rho(\mathcal{N}) = b$.

Notice that F could just be Σ . We then say “ $(K^{c,F})^{V_\eta}$ exists” if the $K^{c,F}$ construction done inside V_η converges. Here, $K^{c,F}$ is the K^c constructions done relative to F (see Definition

1.3.10 of [12] or Chapter 1 of [14]). In order to make sense of $K^{c,\Sigma}$ construction it is important that Σ has *hull condensation*. This is a property that guarantees that Skolem hulls of iteration trees that are according to Σ remain according to Σ . It is used to show that the models appearing in the $K^{c,\Sigma}$ construction that have been obtained by a core (or a Skolem hull) of some other model appearing in the $K^{c,\Sigma}$ construction are still Σ -mice. Below we will discuss K^c constructions relative to two functions F_1 and F_2 . We will use the terminology isolated above in that context as well. The following will later become the first clause of the UB-Covering Conjecture.

Definition 0.1. *We say “some symmetric hybrid K^c construction below η converges” if there is $\nu < \eta$ and symmetric names $\dot{F}_0, \dot{F}_1 \in V^{\text{Coll}(\omega, \nu)}$ such that whenever $g \subseteq \text{Coll}(\omega, \nu)$ is generic, the following holds in $V[g]$:*

1. for $i < 2$, $(\dot{F}_i)_g = (\mathcal{M}_i, \Sigma_i, F_i)$ is such that

(a) \mathcal{M}_i is a countable transitive model of some fragment of ZFC,

(b) Σ_i is an η -uB iteration strategy for \mathcal{M}_i with hull condensation,

(c) F_i is a Σ_i -mouse operator defined on $V_\eta[g]$ or for some $b \in HC$ and some sound Σ_i -mouse \mathcal{N} over b with the property that $\rho(\mathcal{N}) = b$, F_i is an η - or (η, η) -iteration strategy for \mathcal{N} ,

and

2. $(K^{c, F_0, F_1})^{V_\eta[g]}$ exists.

Next we introduce the second clause of the UB-Covering Conjecture. Recall that $Lp^\Sigma(B)$ is the stack of all sound Σ -mice over B which project to $\text{sup}(B)$ and whose countable submodels are $\omega_1 + 1$ -iterable.

Definition 0.2 (Covering with lower parts). *Suppose η is an inaccessible cardinal. We say covering with lower parts holds at η if there is a transitive $\mathcal{P} \in H_{\eta^+}$ satisfying some fragment of ZFC such that \mathcal{P} has a η^+ - or (η^+, η^+) -iteration strategy Σ with hull condensation and with the property that for some $B \subseteq \eta$, $\text{cf}(\text{Ord} \cap Lp^\Sigma(B)) \geq \eta$.*

Finally, given $\kappa < \eta$ and $A \subseteq \eta$, we say κ is A -reflecting if for every $\nu \geq \eta$, there is $j : V \rightarrow M$ witnessing that κ is ν -strong and $j(A) \cap V_\eta = A$.

UB-Covering Conjecture. *Suppose η is a measurable limit of strong cardinals that are A -reflecting where $A = \{\nu < \eta : \nu \text{ is strong}\}$. Then one of the following holds:*

1. *Some symmetric hybrid K^c construction below η converges.*

2. *Covering with lower parts holds at η .*

3. *There is a mouse with a superstrong cardinal.*

It must be clear that “UB” in the name of the conjecture stands for “universally Baire”. It is called “UB-Covering Conjecture” because iteration strategies and hybrid mouse operators are the prime examples of uB sets. In fact, assuming the existence of proper class of Woodin cardinals, all uB sets can be Wadge reduced to uB iteration strategies. The aforementioned fact is an unpublished result due to Woodin. However, the readers familiar with the methods of descriptive inner model theory can dovetail a proof of it by combining the Derived Model Theorem of [18] with Theorem 10.42 of [16] or Theorem 1.2.9 of [7].

As is well-known, the $K^{c,\Sigma}$ construction may not converge for various fine structural reasons. However, as mentioned above, if all countable submodels of the models appearing in the $K^{c,\Sigma}$ construction are $\omega_1 + 1$ -iterable then the $K^{c,\Sigma}$ construction converges (see [19]). Relativizing the results of [1] to Σ , we get that $K^{c,\Sigma}$ construction converges provided there is no non-domestic Σ -mouse. In particular, $K^{c,\Sigma}$ construction converges provided $\mathcal{M}_\omega^{\#, \Sigma}$ doesn't exist. The later statement just says that there is no sound, active Σ -mouse which has ω Woodin cardinals and is $\omega_1 + 1$ -iterable. Readers who feel the need to know more about the intricate details of the theory of hybrid mice and $K^{c,\Sigma}$ constructions should consult [7] and [12] (hull condensation is introduced in Definition 1.1.7 of [7]).

Finally, the last clause of the UB-Covering Conjecture is needed because the theory of mice is currently only fully understood in the region of superstrong cardinals and somewhat past them. We do not know if an equivalent conjecture could be expected to be true in the context of supercompact cardinals. However, UB-Covering Conjecture should be compared with Woodin's HOD Conjecture (see [21] and [22]).

As stated only clause 2 of the UB-Covering Conjecture says something about covering. However, it is shown in [14] (see the proof of Theorem 1.4 of [14]) that if K^c exists below a measurable cardinal then either for measure one many η , $(\eta^+)^{K^c} = \eta^+$ or K^c has a Shelah cardinal. Moreover, it is shown in [2] (see Theorem 3.4 of [2]) that if K^c exists and η is an inaccessible cardinal then either K^c has a superstrong cardinal or $\text{cf}(\mathcal{S}(K^c|\eta) \cap \text{Ord}) \geq \eta$ where $\mathcal{S}(K^c|\eta)$ is the stack over K^c (the projecting mice are allowed to have extenders overlapping η). These examples show that the existence of K^c can be viewed as a covering principle.

The reason the UB-Covering Conjecture is important to us is because it can be used to prove theorems like the following.

Theorem 0.3. *Suppose η witnesses that the UB-Covering Conjecture holds. If there is no mouse with a superstrong cardinal then \square_η holds. In particular, if PFA holds then there is a*

mouse with a superstrong cardinal.

Proof. We assume that there is no mouse with a superstrong cardinal. Then either clause 1 or clause 2 of the UB-Covering Conjecture is true. Suppose first that clause 2 holds. Let (\mathcal{M}, Σ) and B be as in clause 2. By the relativized results of [10] and [11], either $\square(\eta)$ or \square_η holds depending on whether $\text{cf}(Lp^\Sigma(B) \cap \text{Ord}) = \eta$ or $Lp^\Sigma(B) \cap \text{Ord} = \eta^+$. But since η is a measurable cardinal, we have that $\neg\square(\eta)$. It then follows that \square_η holds.

Suppose then clause 1 of the UB-Covering Conjecture is true. Let $\nu < \eta$ and for $i = 0, 1$ let $\dot{F}_i \in V^{\text{Coll}(\omega, \nu)}$ be as in Definition 0.1. Let $g \subseteq \text{Coll}(\omega, \nu)$ be generic and let $(\mathcal{M}_i, \Sigma_i, F_i) = (\dot{F}_i)_g$. Let $\kappa \in (\nu, \eta)$ be an inaccessible cardinal and let $\mathcal{S} = \mathcal{S}(K^{c, F_0, F_1} | \kappa)$ be the stack over $K^{c, F_0, F_1} | \kappa$. It follows from Theorem 3.4 of [2] that $\text{cf}(\mathcal{S}(K^{c, F_0, F_1} | \eta) \cap \text{Ord}) \geq \eta$. Again, it follows from the results of [10] and [11] that either $\square(\kappa)$ or \square_κ holds in $V[g]$. However, because \dot{F}_i is a symmetric name, it follows from the homogeneity of the collapse that the \mathcal{S} -least square sequence is in V . We then have that $\square(\kappa)$ or \square_κ holds in V . Because η is a measurable cardinal and κ was an arbitrary inaccessible cardinal, we get that either $\square(\kappa)$ or \square_κ holds on a measure one set of κ . It then follows that either $\square(\eta)$ or \square_η holds. Again, since η is a measurable cardinal, we must have that \square_η holds.

The last claim of the theorem follows from the well-known theorem of Todorćević that PFA implies $\neg\square_\eta$. \square

The proof of Theorem 0.3 shows the reason behind the need to restrict to operators which have a symmetric name. Without such a condition we do not know if Theorem 0.3 is still true. We have to warn the reader that the need to restrict to operators with symmetric names causes several difficulties and adds some technicalities to the proofs. The main problem comes from defining fullness preservation. The main problem in doing so is that with our current hypothesis we cannot in general show that if g is a $< \eta$ -generic and \mathcal{M} is a mouse over some set $a \in V_\eta[g]$ such that $\rho(\mathcal{M}) = a$ and \mathcal{M} is η -iterable in all $< \eta$ -generic extensions then \mathcal{M} has an η -strategy Λ such that $L^\Lambda(\mathbb{R}) \models AD^+$. We can only establish such a fact for Λ that has a symmetric name. It might be instructive to compare Definition 4.4 with Definition 2.4 of [9].

Our main theorem shows that the conjecture is true when clause 3 is weakened to $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$.

Main Theorem. *Suppose η is a measurable limit of strong cardinals that are A -reflecting where $A = \{\nu < \eta : \nu \text{ is strong}\}$. Then one of the following holds:*

1. *Some symmetric hybrid K^c construction below η converges.*
2. *Covering with lower parts holds at η .*

3. *There is a transitive inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} +$ “ Θ is regular”.*

The following then is an easy corollary:

Corollary 0.4. *Assume PFA and suppose there is η as in the hypothesis of UB-Covering Conjecture. Then there is a transitive inner model containing the reals and satisfying $AD_{\mathbb{R}} +$ “ Θ is regular”.*

The proof of the Main Theorem will be presented in several sections. The background material we need has been presented in [6], [7], [9], [12], and [15]. Because of this instead of reviewing the preliminaries we will recall the notions as they come up in the proof. We will mainly follow the terminology developed in the preliminary sections of [6] and [9]. Also, we will use basic theory of K^c constructions as developed in [14]. This theory generalizes to $K^{c,F}$. We refer the reader to Section 1.3 of [12] for more details.

Acknowledgments. I arrived at the conjecture by trying to understand how core model induction can be used to prove the iterability conjecture of [19] (see Conjecture 6.5 of [19]). Some initial fruitful conversations with Itay Neeman suggested that one may be able to prove a covering conjecture instead of the iterability conjecture. I am indebted to Neeman for those conversations. The current statement of the conjecture owes great deal to the conversations that I had with my visitors while I was a Leibniz fellow at Mathematisches Forschungsinstitut Oberwolfach. I thank Paul Larson, Nam Trang, Trevor Wilson, Martin Zeman and Yizheng Zhu for those conversations. I am indebted to the referee for a long list of important corrections. Also, John Steel and Nam Trang have pointed out some mistakes in earlier versions of this paper and Andres Caicedo has made some very useful comments on the exposition of this paper. I am very grateful to them for those comments. Finally I would like to thank Martin Zeman for very enlightening conversations on the topic of this paper during his two visits to Rutgers in the Fall of 2012 and during the Set Theory workshop in Luminy in the Fall of 2012.

1 How to read this paper

In this section we explain what to expect in the next few sections. An honest confession is that the paper assumes familiarity with the techniques of descriptive inner model theory. The minimum background one would need to read this paper is familiarity with [19] and some ideas from [12] and [7]. Unfortunately, we feel that it is an unreasonable task to make this paper self contained.

The proof of the Main Theorem is via the core model induction. However, most core model induction applications that are currently available in written form only reach the level $AD^+ + \theta_0 < \Theta$ and perhaps a little bit past that (see [6], [9], [12] and [15]). Our first task must then be to set up the core model induction apparatus. Here we took a different route than the authors of [12]. Instead of introducing *model operators* we introduce *core model induction operators* (cmi operators). This will be done in Section 7. However, we have to rely on [12] for the proof of Theorem 7.4 as proving it here is simply beyond the scope of this paper.

Recall that while doing core model induction, our main task is to show that cmi operators are closed under $\mathcal{M}_1^\#$ operator, i.e., given such an operator F , we need to show that $\mathcal{M}_1^{\#,F}$ exists. This step is done by induction on the complexity of cmi operators. The bottom level of this induction consists of operators that have certain *self-determining* properties (see Definition 3.5). If Σ is an iteration strategy that is self-determining then we can have a reasonable notion of Σ -mice over the reals². Let then Σ be a self-determining iteration strategy. We can then define *the maximal model* (see the discussion before Proposition 3.4) relative to Σ . This is the stack of all sound *self-iterable* (see Section 7) Σ -mice over the reals that project to reals. The cmi operators defined relative to Σ are the operators that give a description of those levels of the maximal model relative to Σ where a new Σ_1 fact about Σ and some real becomes true (see Definition 7.2). The ideas behind analyzing the maximal model this way are due to Martin, Steel and Woodin who have done it for $L(\mathbb{R})$. Later Steel also analyzed the maximal model relative to the \emptyset predicate (i.e., the strategy is just the trivial strategy). For more on this work see [4], [13], [17], and [12].

The first major task of any core model induction application is to show that the maximal model satisfies AD^+ . In Section 7, we introduce the principle $Proj(\eta)$ and isolate Theorem 7.4. The main point of Theorem 7.4 is that $Proj(\eta)$ implies that the maximal model computed relative to various strategies indeed satisfies AD^+ . However, as mentioned before, the proof of Theorem 7.4 is beyond the scope of this paper and the interested readers should consult [12] for its proof in the case when Σ is trivial. The general proof is only notationally more complicated.

The next major task of any core model induction application is to construct a canonical set of reals which is beyond the maximal model. Readers familiar with core model induction applications probably know that this is the part of the induction where the hypothesis must be used rather heavily as unlike in the first case, we do not have the description of the new operator for free. In this paper, we proceed as follows. We introduce two useful

²Recall that there are issues defining hybrid mice over reals concerning with the fact that the set of reals isn't in general self well-ordered set, i.e., there may not be a well-ordering of \mathbb{R} in $L(\mathbb{R})$.

principles, *covering with lower parts* (see Definition 0.2) and *limit derived model hypothesis* (see Definition 8.1). The logical steps that lead to the proof of the Main Theorem are outlined below.

First, in Section 9, we show that the negation of the first and the third clauses imply that the limit derived model hypothesis holds at η of the Main Theorem. Then, in Section 11, we show that provided clause 3 fails, the limit derived model hypothesis at η implies that covering with lower parts holds at η , which is the second clause. Thus, the Main Theorem is proved by showing that the negation of clause 3 of the Main Theorem implies clause 1 or clause 2 of the Main Theorem.

The construction of the next set of reals beyond the maximal model appears in the step when we prove that the negation of clause 1 and clause 3 of the Main Theorem implies that the limit derived model hypothesis holds at η of the Main Theorem. This step is summarized in Lemma 9.3. Many of the ideas involved in proving Lemma 9.3 are due to Ketchersid and Woodin and first appeared in [3]. Our exposition here has a few new ideas due to the fact that we are proving a relativized version of it. What we end up showing is that, just like in [3], the next set of reals beyond the maximal model is just an iteration strategy Σ for some hod premouse \mathcal{P} with the property that the direct limit of all Σ iterates of \mathcal{P} is essentially the HOD of the maximal model. To implement this step we need to review the theory of hod mice which we do in Section 2. Another collection of results that is important for this step is presented in Section 5. Here we prove that the lower part stacks computed in various maximal models are the same, a result that eventually allows us to prove that the strategy mentioned above is fullness preserving.

The above step finishes the proof that the failure of clause 1 and clause 3 of the Main Theorem implies that the limit derived model hypothesis holds at η of the Main Theorem. Next, as we mentioned above, we show that the limit derived model hypothesis implies covering with lower parts provided there is no transitive inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. The proof is presented as follows. Suppose η is as in the hypothesis of the Main Theorem. First we take the direct limit of all hod pairs below η . Call this \mathcal{P} . Our goal is to show that \mathcal{P} has a strategy which is fullness preserving. That this is indeed the case is shown in Section 11. One of the main ingredients of the proof is the fact that if $j : V \rightarrow M$ is an embedding witnessing the measurability of η , then j has *weak condensation* (see Section 10). This is done by first showing that hod pairs below η have *canonical witnesses* (see Section 6) which then is used in the proof of Lemma 10.4 to show that the failure of weak condensation can be seen by a countable iteration. The importance of this is that we can now use the fact that j acts on universal models extending \mathcal{P} to conclude that it has to have weak condensation (see the part where j is applied to \bar{Q}

in the last part of the proof of Theorem 10.3, after the claim).

Weak condensation of j is used to show that if Λ is the strategy constructed for \mathcal{P} in Section 11 then Λ is fullness preserving, a result that eventually leads to the fact that in M , (\mathcal{P}, Λ) is a hod pair below η . It then follows that the direct limit of all Λ iterates of \mathcal{P} that are in M converges to a proper initial segment of $j(\mathcal{P})$. We then show that this later fact implies that \mathcal{P} has a regular limit of Woodin cardinals, which finishes the proof as explained below.

Many of the proofs of this paper use the fact that there is no inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. This is the negation of clause 3 of the Main Theorem. However, what we actually need is the failure of clause 3 in all $< \eta$ homogeneous generic extensions of V . Having such a failure of clause 3 is implied by the following principle which essentially says that the sharp of the minimal model of $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ exists. Below η is an inaccessible cardinal.

$\#_{\Theta\text{-reg}}(\eta)$: There is a pair (\mathcal{P}, Σ) such that for some η , there is a sequence of hod pairs $((\mathcal{P}(\alpha), \Sigma_\alpha) : \alpha < \eta)$ such that

1. for $\alpha < \beta$, $\mathcal{P}_\alpha \triangleleft_{\text{hod}} \mathcal{P}_\beta$ and $(\Sigma_\beta)_{\mathcal{P}(\alpha)} = \Sigma_\alpha$,
2. for some δ , $\mathcal{P}|\delta = \cup_{\alpha < \eta} \mathcal{P}_\alpha$, $\mathcal{P} = (\mathcal{P}|\delta)^\#$, and in \mathcal{P} , $\delta^{\mathcal{P}}$ is a regular limit of Woodin cardinals and
3. Σ is a (η, η) -iteration strategy for \mathcal{P} which is (η, η) -extendable and such that $\Sigma_{\mathcal{P}(\alpha)} = \Sigma_\alpha$.

We also let

$\dagger_{\Theta\text{-reg}}$: There is a transitive proper class model M containing the reals and ordinals such that $\mathcal{P}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})^M$ and $M \models \text{“}AD_{\mathbb{R}} + \Theta \text{ is regular”}$.

Clearly, $\dagger_{\Theta\text{-reg}}$ is essentially clause 3 of the Main Theorem. We then have that

Lemma 1.1. $\#_{\Theta\text{-reg}}(\eta) \rightarrow \dagger_{\Theta\text{-reg}}$.

Proof. To see this fix (\mathcal{P}, Σ) as in $\#_{\Theta\text{-reg}}(\eta)$. Let then \mathcal{R} be countable and such that there is $\sigma : \mathcal{R} \rightarrow \mathcal{P}$. Let $\Phi = (\sigma\text{-pullback of } \Sigma)$. Then it follows from Theorem 2.3 that Φ has branch condensation. Let now $M = L(\Gamma(\mathcal{R}, \Phi), \mathbb{R})$. Because $(\mathcal{R}|\delta^{\mathcal{R}})^\# = \mathcal{R}$, we have that $M \models AD^+$ and $\mathcal{P}(\mathbb{R}) \cap M = \Gamma(\mathcal{R}, \Phi)$ (it follows from Theorem 2.6.4 of [7] that M is the derived model of \mathcal{R} computed using Φ). It now follows from Theorem 3.3.7 of [7] that $M \models \text{“}AD_{\mathbb{R}} + \Theta \text{ is regular”}$. Moreover, as Σ defines a surjective map of \mathbb{R} onto Θ^M , we have that $\Sigma \notin M$. \square

We also have that the following holds.

Lemma 1.2. *Suppose $\kappa < \eta$ are such that η is an inaccessible cardinal and κ is a strong cardinal. Suppose further that there is a set $X \in V$ such that whenever $g \subseteq \text{Coll}(\omega, < \kappa)$ is generic, in $V[g]$, there is a transitive model M containing the reals and ordinals such that M is ordinal definable from X , $M \models AD^+$ and $M \models \dagger_{\Theta\text{-reg}}$. Then $\#_{\Theta\text{-reg}}(\eta)$ holds.*

Proof. Fix $g \subseteq \text{Coll}(\omega, < \kappa)$ and M as in the hypothesis. There is then a set of reals $A \in M$ such that letting $\Gamma = \{B \subseteq \mathbb{R} : B \leq_W A\}$ then $L(\Gamma, \mathbb{R}) \models \text{“}AD_{\mathbb{R}} + \Theta \text{ is regular”}$. In $V[g]$, let $A \in M$ be a set of reals such that among all sets of reals witnessing the aforementioned fact in M it has a minimal Wadge rank. Let in $V[g]$, $\Gamma = \{B \subseteq \mathbb{R} : B \leq_W A\}$. Then $L(\Gamma, \mathbb{R}) \models \neg \dagger_{\Theta\text{-reg}}$. Using Theorem 2.2, we can find $(\mathcal{P}, \Sigma) \in M$ such that in M , Σ is fullness preserving, has branch condensation and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$. Let $\mathcal{Q} = (\mathcal{M}_{\infty}(\mathcal{P}, \Sigma))^M$, i.e., \mathcal{Q} is the direct limit of all iterates of \mathcal{P} via Σ . It follows from Theorem 2.2 that \mathcal{Q} is ordinal definable in M , and therefore, $\mathcal{Q} \in V$. Fix now $j : V \rightarrow N$ witnessing that κ is a measurable cardinal and let $h \subseteq \text{Coll}(\omega, < j(\kappa))$ be V -generic such that $g \subseteq h$. We let $j^+ : V[g] \rightarrow N[h]$ be the lift of j . It then follows that $j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_{\eta} \in V$ (see the proof of Lemma 7.10). Let then $\Lambda = j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_{\eta}$ and $\mathcal{S} = (\mathcal{Q}|\delta^{\mathcal{Q}})^{\#}$. It follows from the proof of Theorem 2.2 that $\mathcal{S} \in \mathcal{Q}$ (see Theorem 2.7.6 of [7]). We then have that $(\mathcal{S}, \Lambda_{\mathcal{S}})$ is as in $\#_{\Theta\text{-reg}}(\eta)$. \square

Throughout this paper we assume that $\#_{\Theta\text{-reg}}(\eta)$ fails. Therefore, also the hypothesis of Lemma 1.2 fails. As is demonstrated by Lemma 1.1, $\#_{\Theta\text{-reg}}(\eta)$ is stronger than clause 3 of the Main Theorem.

2 Review of the theory of hod pairs

The purpose of this section is to review some of the material developed in [7]. When we say “recall” we mean recall from [7]. Here, to analyze stacks on hod premice, we use *terminal nodes* (see Definition 2.1) instead of the essential components, which is what we used in [7]. The terminal nodes of a stack are essentially the models that player I can use at the beginning of a new round in the iteration game. This notion is the only new contribution of this section.

The precise definition of a hod premouse can be found in [7] (see Definition 2.1.2 of [7]). Recall that a hod premouse $\mathcal{P} = \mathcal{J}_{\alpha}^{\vec{E}^{\mathcal{P}}, S^{\mathcal{P}}}$ has two predicates: $\vec{E}^{\mathcal{P}}$ is as usual an extender sequence, and $S^{\mathcal{P}}$ codes the strategy of a certain carefully chosen initial segment of \mathcal{P} . The initial segments whose strategy is being coded into $S^{\mathcal{P}}$ are determined by the layers of \mathcal{P} . These are the Woodin cardinals of \mathcal{P} and the limit of Woodin cardinals of \mathcal{P} . Recall that $\lambda^{\mathcal{P}}$ is the order type of the layers of \mathcal{P} , and if $\alpha < \lambda^{\mathcal{P}}$, $\delta_{\alpha}^{\mathcal{P}}$ is the α th layer of \mathcal{P} . Also, $\mathcal{P}(\alpha)$ is

the ω -stack of $\bigoplus_{\beta < \alpha} S_\beta^{\mathcal{P}}$ -mice over $\mathcal{P}|\delta_\alpha^{\mathcal{P}}$ that are $o(\mathcal{P})$ -iterable (this definition takes place in \mathcal{P}). It is called the α th hod-initial segment of \mathcal{P} . Now if $\alpha + 1 \leq \lambda^{\mathcal{P}}$ then the α th strategy $S_\alpha^{\mathcal{P}}$ is a strategy for $\mathcal{P}(\alpha)$. Recall that it is required that in \mathcal{P} , for every $\alpha < \lambda^{\mathcal{P}}$, $S_\alpha^{\mathcal{P}}$ has *branch condensation* and *hull condensation* (see Definition 1.1.4 and Definition 1.1.7 of [7]).

Recall that a pair (\mathcal{P}, Σ) is called a hod pair if \mathcal{P} is a hod premouse and Σ is an (ω_1, ω_1) -iteration strategy for \mathcal{P} with hull condensation. The definition is most useful in AD^+ context. However, while doing core model induction, we are ought to consider situations when we need this notion in the ZFC context. Our current situation is one such situation.

In this paper, all hod mice that we will use do not have measurable limits of Woodin cardinals. We will not, however, explicitly state this.

If $n < \omega$ and $f : HC^n \rightarrow HC$ is a function then we let $Code(f)$ be the set of reals coding f under some standard way of coding countable sets with reals. More precisely, given a real x which is a code of a countable set, we let M_x be the structure coded by x and let $\pi_x : M_x \rightarrow N_x$ be the transitive collapse of M_x . We let WF be the set of reals which code countable sets. Suppose then for some $n < \omega$, $f : HC^n \rightarrow HC$ is a function. Then $Code(f)$ is the set of triples $(x, n, m) \in \mathbb{R} \times \omega \times \omega$ such that $x \in WF$, $\pi_x(n) \in dom(f)$ and $\pi_x(m) \in f(\pi_x(n))$. If $A \subseteq \mathbb{R} \times \omega \times \omega$ then we let f_A be the function, if exists, such that $Code(f_A) = A$.

Next recall that if (\mathcal{P}, Σ) is a hod pair then

$$I(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{P} \text{ according to } \Sigma \text{ with last model } \mathcal{R} \text{ such that } \pi^{\vec{\mathcal{T}}}$$

exists},

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \exists \mathcal{Q}((\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \triangleleft_{hod} \mathcal{R}^3)\},$$

When $A \subseteq X \times Y$ then we write $pA = \{y \in Y : \exists x \in X(x, y) \in A\}$ for the projection of A onto the second coordinate. In the sequel, we will write $pI(\mathcal{P}, \Sigma)$ and $pB(\mathcal{P}, \Sigma)$ for the projections of these sets onto their second coordinates.

Recall that if $\lambda^{\mathcal{P}}$ is limit then

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A \leq_W Code(\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}))\}.$$

where, given two sets of reals B and C , we write $B \leq_W C$ if B is a continuous preimage of C or the complement of C . We do not need hull condensation to make sense of the above definitions. Also, in [7], we defined $\Gamma(\mathcal{P}, \Sigma)$ in the case $\lambda^{\mathcal{P}}$ is a successor (see page 127 of [7]).

Suppose M is a transitive structure and \mathcal{T} is an iteration tree on M^4 . Let \mathcal{S} be a node in \mathcal{T} . Then we write $\mathcal{T}_{\geq \mathcal{S}}$ for the component of \mathcal{T} that comes after stage \mathcal{S} and $\mathcal{T}_{\leq \mathcal{S}}$ for the

³I.e., there is $\alpha < \lambda^{\mathcal{Q}}$ such that $\mathcal{Q}(\alpha) = \mathcal{R}$.

⁴Recall that all trees are normal.

component of \mathcal{T} up to stage \mathcal{S} . We say \mathcal{T} is *reducible* if there is a node \mathcal{S} in \mathcal{T} such that $\mathcal{T}_{\geq \mathcal{S}}$ is a tree on \mathcal{S} . Otherwise we say \mathcal{T} is *irreducible*. We say \mathcal{T} has a *last irreducible component* if there is a node \mathcal{S} in \mathcal{T} such that $\mathcal{T}_{\geq \mathcal{S}}$ is an irreducible tree on \mathcal{S} .

Suppose now that \mathcal{P} is a hod premouse and $\vec{\mathcal{T}}$ is a stack on \mathcal{P} with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$. Recall that the definition of a stack on a hod premouse \mathcal{P} is such that it guarantees that for every $\alpha < \eta$, $\pi_{0,\alpha}^{\vec{\mathcal{T}}} : \mathcal{M}_0 \rightarrow \mathcal{M}_\alpha$ exists.

Definition 2.1. We say \mathcal{R} is a terminal node in $\vec{\mathcal{T}}$ if for some $\alpha < \eta$ and $\beta < lh(\mathcal{T}_\alpha)$, $\mathcal{R} = \mathcal{M}_\beta^{\mathcal{T}_\alpha}$ and $\pi_{0,\beta}^{\mathcal{T}_\alpha}$ exists. We say \mathcal{R} is a non-trivial terminal node if letting (α, β) be as in the previous sentence, $E_\beta^{\mathcal{T}_\alpha}$ is applied to \mathcal{R} .

If \mathcal{R} is non-trivial terminal node then $\xi^{\vec{\mathcal{T}}, \mathcal{R}}$ is the least ξ such that $E_\beta^{\mathcal{T}_\alpha} \in \mathcal{R}(\xi + 1)$. We also let $\vec{\mathcal{T}}_{\mathcal{R}}$ be the largest initial segment of $\vec{\mathcal{T}}_{\geq \mathcal{R}}$ that can be regarded as a stack on $\mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1)$. Also let $\pi_{\mathcal{R}}^{\vec{\mathcal{T}}}$ be the iteration embedding from \mathcal{P} -to- \mathcal{R} and set

$$\begin{aligned} \text{tn}(\vec{\mathcal{T}}) &= \{\mathcal{R} : \mathcal{R} \text{ is a terminal node in } \vec{\mathcal{T}}\} \\ \text{ntn}(\vec{\mathcal{T}}) &= \{\mathcal{R} : \mathcal{R} \text{ is a non-trivial terminal node in } \vec{\mathcal{T}}\}. \end{aligned}$$

Notice that if $\mathcal{R} \in \text{tn}(\vec{\mathcal{T}})$ then player I can legitimately start a new round on \mathcal{R} . Next, given two $\mathcal{Q}, \mathcal{R} \in \text{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ if, in $\vec{\mathcal{T}}$, \mathcal{Q} -to- \mathcal{R} iteration embedding exists. If $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ then we let $\pi_{\mathcal{Q}, \mathcal{R}}^{\vec{\mathcal{T}}} : \mathcal{Q} \rightarrow \mathcal{R}$ be the iteration embedding given by $\vec{\mathcal{T}}$. Again given two $\mathcal{Q}, \mathcal{R} \in \text{tn}(\vec{\mathcal{T}})$ we let $\mathcal{Q} \preceq^{\vec{\mathcal{T}}, s} \mathcal{R}$ if⁵ $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ and if $\vec{\mathcal{U}}$ is the part of $\vec{\mathcal{T}}$ between \mathcal{Q} and \mathcal{R} then $\vec{\mathcal{U}}$ is an iteration of \mathcal{Q} . We then let $\vec{\mathcal{T}}_{\mathcal{Q}, \mathcal{R}}$ stand for the part of $\vec{\mathcal{T}}$ between \mathcal{Q} and \mathcal{R} .

Suppose now that $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ is a stack on \mathcal{P} and $C \subseteq \text{tn}(\vec{\mathcal{T}})$. We say C is linear if it is linearly ordered by $\preceq^{\vec{\mathcal{T}}}$. We say $C \subseteq \text{tn}(\vec{\mathcal{T}})$ is strongly linear if C is linearly ordered by $\preceq^{\vec{\mathcal{T}}, s}$. Suppose C is strongly linear and $(\mathcal{R}_\alpha : \alpha < \eta)$ is a $\preceq^{\vec{\mathcal{T}}, s}$ -increasing enumeration of C . We let $lh(C) = \gamma$. Suppose further that γ is a limit ordinal. Then we let $\mathcal{R}_C^{\vec{\mathcal{T}}}$ be the direct limit of the \mathcal{R}_α s under the iteration embeddings $\pi_{\mathcal{R}_\alpha, \mathcal{R}_\beta}^{\vec{\mathcal{T}}}$. We then say $C \subseteq \text{tn}(\vec{\mathcal{T}})$ is closed if it is strongly linear and for every limit $\alpha < lh(C)$, $\mathcal{R}_{C|\alpha}^{\vec{\mathcal{T}}} \in C$. Notice that strong linearity implies that for each limit $\alpha < lh(C)$, $\mathcal{R}_{C|\alpha}^{\vec{\mathcal{T}}}$ is a node in $\vec{\mathcal{T}}$. We say C is cofinal if for every node \mathcal{S} of $\vec{\mathcal{T}}$ either $\mathcal{S} \in C$ or there are $\mathcal{R} \preceq^{\vec{\mathcal{T}}, s} \mathcal{Q} \in C$ such that \mathcal{S} is a node in $\vec{\mathcal{T}}_{\mathcal{R}, \mathcal{Q}}$. Notice that if C is closed and cofinal and $\mathcal{S} \notin C$ then there is $\preceq^{\vec{\mathcal{T}}, s}$ -largest $\mathcal{R} \in C$ such that for any $\mathcal{Q} \in C$ such that $\mathcal{R} \preceq^{\vec{\mathcal{T}}, s} \mathcal{Q}$, \mathcal{S} is a node in $\vec{\mathcal{T}}_{\mathcal{R}, \mathcal{Q}}$.

Notice also that if $\vec{\mathcal{T}}$ doesn't have a last model but there is a strongly linear closed and unbounded $C \subseteq \text{tn}(\vec{\mathcal{T}})$ then C uniquely identifies the branch of $\vec{\mathcal{T}}$. Indeed, let $D = \{\mathcal{S} \in \text{tn}(\vec{\mathcal{T}}) : \exists \mathcal{R}, \mathcal{Q} \in C (\mathcal{R} \preceq^{\vec{\mathcal{T}}} \mathcal{S} \preceq^{\vec{\mathcal{T}}} \mathcal{Q})\}$. Let $\mathcal{R} \in D$ be $\preceq^{\vec{\mathcal{T}}}$ -minimal member of D and let b be

⁵“s” stands for “strongly”.

the set of indices of the nodes of $\vec{\mathcal{T}}$ between \mathcal{P} and \mathcal{R} . Then the union of b with the indices of the nodes of D constitute a branch b_C of $\vec{\mathcal{T}}$. Its not hard to see that we have $\mathcal{M}_{b_C}^{\vec{\mathcal{T}}} = \mathcal{R}_C^{\vec{\mathcal{T}}}$.

Suppose now that $\vec{\mathcal{T}}$ doesn't have a last model and there is no strongly linear closed and cofinal $C \subseteq tn(\vec{\mathcal{T}})$. It follows that γ must be a successor ordinal. Let $\alpha = \gamma - 1$ and $\mathcal{T} = \mathcal{T}_\alpha$. It then follows that there is $\mathcal{S} \in tn(\mathcal{T})$ such that $\mathcal{T}_{\geq \mathcal{S}}$ is an irreducible tree on \mathcal{S} or there is $\mathcal{W} \notin tn(\mathcal{T})$, $\mathcal{T}_{\geq \mathcal{W}}$ is a tree on \mathcal{W} . Let then $D = \{\mathcal{S} \in tn(\mathcal{T}) : \mathcal{T}_{\geq \mathcal{S}} \text{ is a tree on } \mathcal{S}\}$. It follows from our discussion that D has a $\preceq^{\vec{\mathcal{T}},s}$ -largest element. We then let $\mathcal{S}_{\vec{\mathcal{T}}}$ be this largest element. Such an analysis of stacks is very useful because we have that $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is a normal tree based on some window of $\mathcal{S}_{\vec{\mathcal{T}}}$.

Next we recall the notion of *super fullness preservation* from [7]⁶. Suppose M is a transitive model of AD^+ containing the reals and (\mathcal{P}, Σ) is a hod pair such that \mathcal{P} is countable, Σ is an (ω_1, ω_1) -strategy with branch condensation, Σ is M -fullness preserving and for any $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}} \in M$. Suppose that $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and suppose $\alpha + 1 \leq \lambda^{\mathcal{R}}$. Let in M

$U_{\mathcal{R}(\alpha), \Sigma} = \{(x, y) : x \in \mathbb{R} \text{ codes a countable set } a \text{ and } y \text{ codes a sound } \Sigma_{\mathcal{R}(\alpha)}\text{-mouse } \mathcal{M} \text{ over } a \text{ such that } \rho(\mathcal{M}) = a\}$ and

$W_{\mathcal{R}(\alpha), \Sigma} = \{(x, y, z) : (x, y) \in U_{\mathcal{R}(\alpha), \Sigma} \text{ and } z \text{ codes an iteration tree on the mouse } \mathcal{M} \text{ coded by } y \text{ that is according to the unique strategy of } \mathcal{M}\}$.

We then say Σ is *M -super fullness preserving*, if whenever $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$, then the fragment of $\Sigma_{\mathcal{R}(\alpha+1)}$ acting on trees that are above $\delta_\alpha^{\mathcal{R}}$ respects both $U_{\mathcal{R}(\alpha), \Sigma}$ and $W_{\mathcal{R}(\alpha), \Sigma}$, i.e., whenever $(\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{R}(\alpha + 1), \Sigma_{\mathcal{R}(\alpha+1)})$,

$$\pi^{\vec{\mathcal{U}}}(\tau_{U_{\mathcal{R}(\alpha), \Sigma}}^{\mathcal{R}(\alpha+1), \Sigma}) = \tau_{U_{\mathcal{Q}(\alpha), \Sigma}}^{\mathcal{Q}, \Sigma}$$

where $\tau_{U_{\mathcal{R}(\alpha), \Sigma}}^{\mathcal{R}(\alpha+1), \Sigma}$ is the join of the ω terms that capture $U_{\mathcal{R}(\alpha), \Sigma}$ over \mathcal{R} at $((\delta^{\mathcal{R}(\alpha+1)})^{+n})^{\mathcal{R}}$ ($\tau_{U_{\mathcal{Q}(\alpha), \Sigma}}^{\mathcal{Q}, \Sigma}$ is defined similarly). Also, similar equality holds for $W_{\mathcal{R}(\alpha), \Sigma}$.

Continuing with M and (\mathcal{P}, Σ) of the previous paragraph, we say that Σ is *correctly M -guided* if for every $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and for every $\alpha < \lambda^{\mathcal{Q}}$, there is a sequence $\vec{B} = (B_i : i < \omega) \subseteq (\mathbb{B}[\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}])^M$ such that $\Sigma_{\mathcal{Q}(\alpha+1)}$ is guided by \vec{B} . Recall the definition of $(\mathbb{B}[\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}])^M$ from Section 3.1 of [7]. More precisely, working in M , we let $[\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}]$ be the Dodd-Jensen equivalence class of $(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)})$, i.e., the set of all hod pairs (\mathcal{R}, Ψ) such that $\Gamma(\mathcal{R}, \Psi) = \Gamma(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)})$. We then let $\mathbb{B}[\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}]$ be the set of all $B \subseteq [\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}] \times \mathbb{R} \times \mathbb{R}$ which are *OD* and for any $(\mathcal{R}, \Psi) \in [\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}]$, the first coordinate of $B_{(\mathcal{R}, \Psi)}$ consists of the reals coding \mathcal{R} . Each such B is term captured over $\mathcal{Q}(\alpha + 1)$ (this

⁶The definition of this notion in [7] contains a mistake which has been corrected in the later versions of that paper.

follows from the strong mouse capturing, see Lemma 3.1.2 of [7]). We then say that $\Sigma_{\mathcal{Q}(\alpha+1)}$ is guided by \vec{B} if it chooses the unique branch which moves the canonical term relations capturing members of \vec{B} correctly. For more details see Definition 3.1.11 of [7]. If $\Sigma \in M$ then we write $M \models$ “ Σ is correctly guided” to mean that Σ is correctly M -guided.

Next, we state some useful theorems from [7]. The following is essentially the combination of Lemma 3.2.4 and Theorem 3.4.1 of [7].

Theorem 2.2. *Assume $AD^+ + \neg \dagger_{\Theta\text{-reg}}$. Fix α such that $\theta_\alpha < \Theta$. Then there is a hod pair (\mathcal{P}, Σ) such that Σ has branch condensation, is super fullness preserving, and is correctly guided. Moreover, $\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$ and letting $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the direct limit of all Σ -iterates of \mathcal{P} ,*

$$\mathcal{M}_\infty|_{\theta_\alpha} = (V_{\theta_\alpha}^{\text{HOD}}, \vec{E}^{\mathcal{M}_\infty|_{\theta_\alpha}}, S^{\mathcal{M}_\infty|_{\theta_\alpha}}, \in)$$

where $S^{\mathcal{M}_\infty|_{\theta_\alpha}}$ is the strategy predicate of \mathcal{M}_∞ .

The next theorem is essentially Theorem 2.7.6 of [7].

Theorem 2.3. *Assume AD^+ and suppose (\mathcal{P}, Σ) is a hod pair except that Σ may not have hull condensation. Suppose $\lambda^{\mathcal{P}}$ is a limit ordinal. Furthermore, suppose that whenever $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{P}, \Sigma)$ there is a hod pair (\mathcal{Q}, Λ) and an embedding $\pi : \mathcal{R} \rightarrow \mathcal{Q}$ such that $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}} = (\pi\text{-pullback of } \Lambda)$ and Λ has branch condensation, is super fullness preserving and is correctly guided. Then for any $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}}$ has branch condensation and is $\Gamma(\mathcal{R}, \Sigma_{\mathcal{R}, \vec{\mathcal{T}}})$ -fullness preserving. Moreover, for every $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$, and for any $\alpha < \beta < \lambda^{\mathcal{R}}$, $L(\Gamma(\mathcal{R}(\beta), \Sigma_{\mathcal{R}(\beta), \vec{\mathcal{T}}})) \models$ “ $\Sigma_{\mathcal{R}(\alpha)}$ is super fullness preserving and is correctly guided”.*

The next theorem is essentially Theorem 2.8.1 of [7].

Theorem 2.4. *Assume AD^+ and suppose (\mathcal{P}, Σ) is a hod pair except that Σ may not have hull condensation. Suppose $\text{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is a measurable cardinal in \mathcal{P} and suppose that whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has branch condensation. Then there is $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has branch condensation.*

Finally recall the statement of *strong mouse capturing* (SMC). Assume AD^+ and suppose (\mathcal{P}, Σ) is a hod pair such that Σ has branch condensation. Then we say mouse capturing relative to Σ holds ($\text{MC}(\Sigma)$) if for all $x, y \in \mathbb{R}$, $x \in OD_{\Sigma, y}$ if and only if there is a Σ -mouse \mathcal{M} over y such that $x \in \mathcal{M}$. We say strong mouse capturing holds if for every hod pair (\mathcal{P}, Σ) , mouse capturing relative to Σ holds. The following is the main theorem of [7] (see Chapter 3 of [7] for its proof).

Theorem 2.5. *Assume $AD^+ + \neg\uparrow_{\Theta\text{-reg}}$. Then SMC holds.*

Our next theorem is a useful tool in core model induction applications. We will use it throughout the paper. Unfortunately its proof is unpublished.

Theorem 2.6 (S-Steel). *Assume AD^+ . Suppose $V = L(\mathcal{P}(\mathbb{R}))$ and SMC holds. Suppose (\mathcal{P}, Σ) is a hod pair such that Σ has branch condensation and is fullness preserving. Then there is a (Θ, Θ) -iteration strategy Σ^* with branch condensation such that Σ^* extends Σ and*

$$\{A \subseteq \mathbb{R} : A \text{ is ordinal definable from a real and } \Sigma\} = Lp^{\Sigma^*}(\mathbb{R}).$$

A standard Skolem hull argument using hull condensation of Σ shows that there is only one such iteration strategy Σ^* . Because of this in the sequel we will write $Lp^{\Sigma}(\mathbb{R})$ for $Lp^{\Sigma^*}(\mathbb{R})$. Also, see the discussion at the end of Section 3 on hybrid mice over the reals.

3 Extendable strategies

In this section we isolate a class of iteration strategies that can be canonically interpreted in various generic extensions. We will use the ideas used in the *tree production lemma* to make this notion precise (specifically see Lemma 4.1 of [18]). In the next section, we will use the material of this section to introduce the most important object of this paper, *hod pairs below η* .

For the duration of this section, we fix an uncountable cardinal $\lambda \geq \omega_2$. Suppose Z is some set and $\nu \geq \eta$ is such that $Z \in H_{\nu^+}$. Let ϕ and ψ be some formulae in the language of set theory and let $X \in \mathcal{P}_{\omega_1}(H_{\nu^+})$ be such that $Z \in X$. Let $\pi : M \rightarrow X \prec H_{\nu^+}$ be the transitive collapse of X and π be the inverse of the collapse. Let $\bar{Z} = \pi^{-1}(Z)$ and $\bar{\eta} = \pi^{-1}(\eta)$. Following the terminology of Section 4 of [18], we then say that (X, ψ) is (ϕ, a, η) -generically correct if whenever $\mathbb{P} \in H_{\bar{\eta}}^M$ is a poset and g is M -generic over \mathbb{P} then for any $x \in \mathbb{R}$, $M[g] \models \psi[\bar{Z}, x]$ if and only if $H_{\nu^+} \models \phi[Z, x]$.

We then say (ϕ, ψ) are η -generically complementing as witnessed by Z if for club many $X \in \mathcal{P}_{\omega_1}(H_{\nu^+})$, (X, ψ) is (ϕ, Z, η) -generically correct. Let $A_\phi = \{x \in \mathbb{R} : H_{\nu^+} \models \phi[Z, x]\}$. It follows from Lemma 4.1 of [18] that A_ϕ is η -uB. If T is the tree constructed in [18] with the property that $p[T] = A_\phi$ then T is essentially a tree whose branches are pairs (x, X) where $x \in \mathbb{R}$ and $X \in \mathcal{P}_{\omega_1}(H_{\nu^+})$ belongs to the club witnessing that (ϕ, ψ) are η -generically complementing.

We use the idea behind η -generically complementing formulae to introduce η -extendable strategies. The main difficulty here is that the strategy isn't a strategy for a countable structure. Otherwise, saying that the strategy is η -uB is enough for our purposes. Below we

state the definition of only an η -extendable strategies. Trivial modification of the definition can be used to define (η, η) -extendable strategies. Similarly all the lemmas and theorems that we will state for η -extendable strategies generalize easily to (η, η) -extendable strategies.

Definition 3.1 (η -extendable strategies). *Suppose (\mathcal{M}, Σ) is such that $\mathcal{M} \in H_\eta$ is a transitive model of some fragment of ZFC and for some $\alpha, \beta \leq \eta$, Σ is a (α, β) -iteration strategy for \mathcal{M} with hull condensation.*

1. *We say Σ is weakly η -extendable if there is an η -iteration strategy Λ such that $\Lambda \upharpoonright \text{dom}(\Sigma) = \Sigma$ and Λ has hull condensation.*
2. *We say Σ is η -extendable if there are formulae (ϕ, ψ) and a set Z such that whenever $g \subseteq \text{Coll}(\omega, |\mathcal{M}|)$ is generic, in $V[g]$,*
 - (a) *(ϕ, ψ) are η -generically complementing as witnessed by Z ,*
 - (b) *for any partial ordering $\mathbb{P} \in V[g]$ of size $< \eta$ and any $V[g]$ -generic $h \subseteq \mathbb{P}$, $V[g * h] \models$ “ A_ϕ is a code set of an ω_1 -iteration strategy Λ for \mathcal{M} such that Λ has hull condensation and*

$$\Lambda \upharpoonright (\text{dom}(\Sigma) \cap \text{HC}^{V[g*h]}) = \Sigma \upharpoonright (\text{dom}(\Sigma) \cap \text{HC}^{V[g*h]}).$$

We say that $r = (\phi, \psi, Z)$ witnesses that Σ is η -extendable. We also say that (\mathcal{M}, Σ) is an η -extendable pair.

The following is an easy lemma which we leave to the reader.

Lemma 3.2. *Suppose (\mathcal{M}, Σ) is as in the hypothesis of Definition 3.1. Then the following holds.*

1. *Suppose \mathcal{M} is countable. Then Σ is η -extendable if and only if Σ is η -uB.*
2. *Suppose Σ is η -extendable. Then Σ is weakly extendable.*

For the rest of this paper, when we say that (\mathcal{M}, Σ) is η -extendable then, because Σ has a canonical extension to an η -iteration strategy (modulo an extendability witness), we will tacitly assume that Σ is already an η -iteration strategy.

Suppose (\mathcal{M}, Σ) is as in the hypothesis of Definition 3.1 and let $r = (\phi, \psi, Z)$ witness that Σ is η -extendable. Suppose g is a generic for some partial ordering \mathbb{P} of size $< \eta$ and $h \subseteq \text{Coll}(\omega, \max(|\mathbb{P}|, |\mathcal{M}|))$ is $V[g]$ -generic. We then let, in $V[g * h]$, letting Λ be the strategy coded by A_ϕ , $\Sigma^{r,g} =_{\text{def}} \Lambda \upharpoonright H_\eta[g]$. It follows from the homogeneity of the collapse that $\Sigma^{r,g} \in V[g]$ and moreover, $V[g] \models$ “ $\Sigma^{r,g}$ is η -extendible as witnessed by r ”. Often times

r will be clear from the context and in those situations we will drop it from our notation. In particular, if \mathcal{M} is countable then for any two η -extendability witnesses r and s and for any $< \eta$ -generic g , $\Sigma^{r,g} = \Sigma^{s,g}$. In such cases too we will drop r from our notation.

There might be some confusion with our current notation and already established notation Σ^π . The later refers to π -pullback of Σ . We will refrain from using “ Σ^π ” in this paper. Since we are discussing pullbacks, we mention that pullbacks of extendable pairs are also extendable. More precisely, suppose (\mathcal{M}, Σ) is an η -extendable pair and $i : \mathcal{N} \rightarrow \mathcal{M}$ is an elementary embedding. Let Λ be the i -pullback of Σ . Then clearly we have that

Lemma 3.3. *(\mathcal{N}, Λ) is an η -extendable pair.*

Just like in [9], given an η extendable pair (\mathcal{M}, Σ) we can introduce the various stacks relative to Σ . First fix r which witnesses that Σ is η -extendable and let g be a generic for some poset of size $< \eta$. Given $a \in H_\eta[g]$, recall that we say that a Σ -mouse \mathcal{M} over a is countably ξ -iterable if whenever $\pi : \mathcal{N} \rightarrow \mathcal{M}$ is a countable hull of \mathcal{M} then letting $\Lambda = (\pi$ -pullback of $\Sigma)$, \mathcal{N} is a ξ -iterable Λ -mouse over $\pi^{-1}(a)$. When $\xi = \omega_1 + 1$ then we just say that \mathcal{M} is *countably iterable*. Let then in $V[g]$,

- (1) $Lp^{\Sigma^{r,g}}(a)$ be the stack of all sound $\Sigma^{r,g}$ -mice projecting to a which are countably iterable,
- (2) $\mathcal{W}^{\eta, \Sigma^{r,g}}(a)$ be the stack of all sound $\Sigma^{r,g}$ -mice projecting to a which are η -iterable, and
- (3) $\mathcal{K}^{\eta, \Sigma^{r,g}}(a)$ be the stack of all sound $\Sigma^{r,g}$ -mice projecting to a which are countably η -iterable.

The above definitions make sense when $\lambda = \omega_1$ but ultimately are not very useful unless ω_1 -iterability implies $\omega_1 + 1$ -iterability. The aforementioned fact holds under *AD*. When $\lambda = \omega_1$ then we will omit it from our notation.

Next we would like to define *the maximal model*. Continuing with the above notation, recall from [9] that if $\kappa \leq \eta$ is uncountable, $\mathcal{M} \in H_\kappa$ and $g \subseteq \text{Coll}(\omega, < \kappa)$ or $g \subseteq \text{Coll}(\omega, \kappa)$ is V -generic then $\mathcal{S}^{\eta, \Sigma^{r,g}} =_{\text{def}} L(\mathcal{K}^{\eta, \Sigma^{r,g}}(\mathbb{R}^{V[g]}))$ is the maximal model. In the case \mathcal{M} is already countable, we let $\mathcal{S}^{\eta, \Sigma^{r,g}} =_{\text{def}} L(\mathcal{K}^{\eta, \Sigma^{r,g}}(\mathbb{R}))$.

Proposition 3.4 (Proposition 2.2 of [9]). *For any $< \eta$ generic g and any $a \in H_\eta[g]$, in $V[g]$, $\mathcal{W}^{\eta, \Sigma^{r,g}}(a) \trianglelefteq \mathcal{K}^{\eta, \Sigma^{r,g}}(a) \trianglelefteq Lp^{\Sigma^{r,g}}(a)$. Moreover, if $a \in H_\eta$ then $\mathcal{W}^{\eta, \Sigma^{r,g}}(a) \trianglelefteq \mathcal{W}^{\eta, \Sigma}(a)$, $\mathcal{K}^{\eta, \Sigma^{r,g}}(a) \trianglelefteq \mathcal{K}^{\eta, \Sigma}(a)$ and $(Lp^{\Sigma^{r,g}}(a))^{V[g]} \trianglelefteq (Lp^\Sigma(a))^V$.*

However, there is a slight problem with the above definition of the maximal model. Because the maximal model is a hybrid premouse over the set of reals, which in general isn't a self-wellordered set, one needs to be careful when defining $\mathcal{S}^{\eta, \Sigma^{r,g}}$. We do not know

how to define it in general but only for a large class of strategies (the definition given below works for arbitrary strategies with hull condensation but we do not know how to show that in general $\Sigma \upharpoonright \mathbb{R} \in \mathcal{S}^{\eta, \Sigma^g}$). We use the ideas of Section 2.10 of [7] to describe the strategies for which we can reasonably define $\mathcal{S}^{\eta, \Sigma^{r,g}}$.

Suppose (\mathcal{P}, Σ) is an η -extendable pair as witnessed by r . For $n \leq \omega$, we let $\mathcal{M}_n^{\#, \Sigma}$ be the minimal active sound Σ -mouse with n Woodin cardinals. We let \mathcal{M}_n^Σ be the result of iterating the last active measure of $\mathcal{M}_n^{\#, \Sigma}$ through the ordinals. We let $\mathcal{N}_\omega^{\#, \Sigma}$ be the minimal active hybrid mouse such that it has ω many Woodin cardinals and letting δ be the sup of the Woodin cardinals of $\mathcal{N}_\omega^{\#, \Sigma}$ then $\mathcal{N}_\omega^{\#, \Sigma}|\delta$ is a Σ -mouse and letting $\mathcal{N}_\omega^\Sigma$ be the result of iterating the last measure of $\mathcal{N}_\omega^{\#, \Sigma}$ through the ordinals, $\mathcal{N}_\omega^\Sigma = \mathcal{J}[\mathcal{N}_\omega^\Sigma|\delta]$ ⁷. For $n \leq \omega$, we say “ $\mathcal{M}_n^{\#, \Sigma}$ exists” if it exists as a set. We also say $\mathcal{M}_\omega^{\#, \Sigma, r}$ exists and is η -iterable via an η -extendable strategy if $\mathcal{M}_\omega^{\#, \Sigma}$ exists and has an η -extendable strategy Φ with the property that whenever g is generic for a poset of size $< \eta$ then Φ^g -iterates of $\mathcal{M}_\omega^{\#, \Sigma}$ are $\Sigma^{r,g}$ -mice. Sometimes when r is clear from context, we will drop it from our notation. In particular, if \mathcal{P} is countable and Σ is η -uB then, since $\Sigma^{r,g}$ is independent of the choice of r , we will drop it from our notation. We will use similar expressions for $\mathcal{N}_\omega^{\#, \Sigma}$.

Definition 3.5. *Suppose (\mathcal{P}, Σ) is an η -extendable pair as witnessed by r . We say (\mathcal{P}, Σ) is a self-determining pair below η or that Σ is self-determining below η as witnessed by r if for some $n \in [1, \omega)$, $\mathcal{M}_n^{\#, \Sigma, r}$ exists and is η -iterable via an η -extendable strategy Φ such that there is a formula $\phi(u, v)$ in the language of Σ -mice such that whenever $\nu < \eta$ is an uncountable cardinal and $g \subseteq \text{Coll}(\omega, < \nu)$ is generic then in $V[g]$, for every Φ -iterate \mathcal{M} of $\mathcal{M}_n^{\#, \Sigma^{r,g}, r}$ that is obtained by iterating below the first Woodin of $\mathcal{M}_n^{\#, \Sigma^{r,g}, r}$ and is such that the iteration embedding $i : \mathcal{M}_n^{\#, \Sigma^{r,g}, r} \rightarrow \mathcal{M}$ exists, letting δ be the first Woodin cardinal of \mathcal{M} , whenever $\vec{T} \in V_\eta[g]$ is a stack according to $\Sigma^{r,g}$ such that it has a last normal component of limit length and it is generic for the extender algebra of \mathcal{M} at δ then*

$$\Sigma^{r,g}(\vec{T}) = b \iff b \in \mathcal{M}[\vec{T}] \wedge \mathcal{M}[\vec{T}] \models \phi[\vec{T}, b].$$

We say r witnesses that (\mathcal{P}, Σ) is a self-determining pair below η or that Σ is self-determining below η . We let $n_{r, \Sigma}$ be the least integer n as above.

It is shown in [7] that under AD^+ , if (\mathcal{P}, Σ) is a hod pair such that Σ is super fullness preserving and has branch condensation then Σ is self-determining below ω_1 (see Section 2.9

⁷The difference between $\mathcal{N}_\omega^{\#, \Sigma}$ and $\mathcal{M}_\omega^{\#, \Sigma}$ is that the later is closed under Σ up to its last measure. Another way to see the distinction between $\mathcal{N}_\omega^{\#, \Sigma}$ and $\mathcal{M}_\omega^{\#, \Sigma}$ is to compare their derived models. The first gives rise to $L(\Sigma, \mathbb{R})$ and the second to $L^\Sigma(\mathbb{R})$. The advantage of using $\mathcal{N}_\omega^{\#, \Sigma}$ over $\mathcal{M}_\omega^{\#, \Sigma}$ is that if $\mathcal{N}_\omega^{\#, \Sigma}$ exists and $\Lambda \in L(\Sigma, \mathbb{R})$ is a strategy for some hybrid structure with hull condensation, then one can show by using full backgrounded constructions, that $\mathcal{N}_\omega^{\#, \Lambda}$ also exists (we will use this observation while proving Lemma 4.11). We conjecture that the equivalent result also holds for $\mathcal{M}_\omega^{\#, \Sigma}$.

and Theorem 2.10.3). Next we show that hybrid mice projecting to their base have self-determining strategies. However, we first have to make this more precise. This is because if \mathcal{Q} is a Σ -mouse with iteration strategy Φ and \mathcal{R} is a Φ -mouse with an iteration strategy Ψ it may not be the case that whenever \mathcal{M} is a Φ -mouse it is closed under Σ (i.e., $\Sigma \upharpoonright \mathcal{M} \subseteq \mathcal{M}$). Having such a closure is key to our proofs that certain strategies are self-determining. To take care of this difficulty, we need to modify Definition 3.5 for relativized pairs.

Definition 3.6. *Suppose (\mathcal{P}, Σ) is a self-determining pair below η as witnessed by r . Let $n = n_{r, \Sigma}$ and Φ be the iteration strategy of $\mathcal{M}_n^{\#, \Sigma, r}$ and suppose (\mathcal{Q}, Λ) is an η -extendable pair as witnessed by s such that \mathcal{Q} is a Σ -mouse. We say (\mathcal{Q}, Λ) is a self-determining pair below η or that Λ is self-determining below η as witnessed by s if for some $n \in [1, \omega)$, $\mathcal{M}_n^{\#, \Lambda, \Phi, s}$ exists and is η -iterable via an η -extendable strategy Ψ such that there is a formula $\phi(u, v)$ in the language of (Λ, Φ) -mice such that whenever $\nu < \eta$ is an uncountable cardinal and $g \subseteq \text{Coll}(\omega, < \nu)$ is generic then in $V[g]$, whenever \mathcal{M} is a Ψ -iterate of $\mathcal{M}_n^{\#, \Lambda, \Phi, s}$ obtained by iterating below the first Woodin of $\mathcal{M}_n^{\#, \Lambda, \Phi, s}$ and such that the iteration embedding $i : \mathcal{M}_n^{\#, \Lambda, \Phi, s} \rightarrow \mathcal{M}$ exists then letting δ be the first Woodin cardinal of \mathcal{M} , whenever $\vec{\mathcal{T}} \in V_\eta[g]$ is a stack according to $\Lambda^{s, g}$ such that it has a last normal component of limit length and it is generic for the extender algebra of \mathcal{M} at δ then*

$$\Lambda^{s, g}(\vec{\mathcal{T}}) = b \iff b \in \mathcal{M}[\vec{\mathcal{T}}] \wedge \mathcal{M}[\vec{\mathcal{T}}] \models \phi[\vec{\mathcal{T}}, b].$$

We say s witnesses that (\mathcal{Q}, Λ) is a self-determining pair below η or that Λ is self-determining below η . We let $n_{s, \Lambda}$ be the least integer n as above.

We can now prove the following lemma.

Lemma 3.7. *Suppose (\mathcal{P}, Σ) is a self-determining pair below η as witnessed by r . Let Φ be the η -extendable strategy of $\mathcal{M}_n^{\#, \Sigma, r}$ where $n = n_{r, \Sigma}$. Suppose \mathcal{M} is a sound Σ -premouse over some set a such that $\rho(\mathcal{M}) = a$ and suppose that \mathcal{M} has an η -extendable strategy Ψ . Suppose $\mathcal{M}_2^{\#, \Psi, \Phi}$ exists and is η -iterable via an η -extendable strategy. Then (\mathcal{M}, Ψ) is self-determining below η .*

Proof. The proof is very much like the proof of Theorem 2.5.10 of [7]. Because of this, we only sketch the argument. Let Λ be the strategy of $\mathcal{N} =_{\text{def}} \mathcal{M}_2^{\#, \Psi, \Phi}$. We will give the description of Ψ in generic extensions of \mathcal{N} and leave it to the reader to verify that it works in generic extensions of the iterates of \mathcal{N} . Let δ be the least Woodin cardinal of \mathcal{N} . Let $g \subseteq \text{Coll}(\omega, \delta)$ be V -generic. As Φ is an η -strategy, we only need to consider iteration trees. Let then $\mathcal{T} \in \mathcal{N}[g]$ be a tree of limit length and according to Φ . We need a uniform way of identifying $\Phi(\mathcal{T})$. Notice that because \mathcal{N} is closed under Ψ , we have that $\Sigma^{r, g} \upharpoonright \mathcal{N}[g]$ is definable over $\mathcal{N}[g]$. Let then $\Sigma^* = \Sigma^{r, g} \upharpoonright \mathcal{N}[g]$.

Let γ be the second Woodin of \mathcal{N} . Now, working in \mathcal{N} notice that it follows from the universality of the fully backgrounded constructions (see Lemma 1.1.24-1.1.26 of [7]) that there is some limit cardinal $\kappa \in (\delta, \gamma)$ and ordinal ξ such that letting \mathcal{N}_ξ be the ξ th model of the the fully backgrounded construction of $\mathcal{N}|\kappa$ done above δ with respect to Σ over a , $\mathcal{C}(\mathcal{N}_\xi) = \mathcal{M}$. It then follows that in $\mathcal{N}[g]$, the iterability of \mathcal{M} reduces to the iterability of \mathcal{N}_ξ . It follows from the results of Chapter 12 of [5] that in $\mathcal{N}[g]$, the iterability of \mathcal{N}_ξ reduces to the iterability of $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ above δ for non-dropping countable trees.

Let then Λ^* be the fragment of Λ which acts on non-dropping trees on $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ that are above δ . Notice that $\Lambda^* \upharpoonright \mathcal{N}|(\gamma^+)^{\mathcal{N}} \in \mathcal{N}$. Let \mathcal{S} be the output of the fully backgrounded construction of $\mathcal{N}|\eta$ done with respect to (Ψ, Φ) , over $\mathcal{N}|\delta$ and using extenders with critical points $> \kappa^+$. Let then $\mathcal{U} \in \mathcal{N}[g]$ be a countable non-dropping tree on $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ which is above δ and is of limit length. It follows that $\mathcal{Q}(\mathcal{U})$ -exists and its just a matter of identifying it uniformly inside $\mathcal{N}[g]$.

It follows from the stationarity of the fully backgrounded constructions (see Lemma 1.1.24 of [7]) that in the comparison of $\mathcal{M}(\mathcal{U})$ with \mathcal{S} , \mathcal{S} side doesn't move. Moreover, the comparison can be done inside $\mathcal{N}[g]$ by using the \mathcal{Q} -structures given by the initial segments of \mathcal{S} . More precisely, if \mathcal{W} is a tree constructed via the comparison process and it has limit length then $\mathcal{Q}(\mathcal{W})$ is the largest initial segment of \mathcal{S} in which $\delta(\mathcal{W})$ is Woodin. Let now ξ be such that $\mathcal{M}(\mathcal{U})$ iterates to $\mathcal{S}|\xi$ via the above process and let $\pi : \mathcal{M}(\mathcal{U}) \rightarrow \mathcal{S}|\xi$ be the iteration embedding. Let $\mathcal{R} \trianglelefteq \mathcal{S}$ be the largest initial segment of \mathcal{S} in which ξ is Woodin. We then have that if E is the (δ, ξ) -extender derived from π then $Ult(\mathcal{Q}(\mathcal{U}), E) = \mathcal{R}$. It then follows by a simple absoluteness argument that if $h \subseteq Coll(\omega, \mathcal{M}(\mathcal{U}))$ then there is $\mathcal{Q} \in \mathcal{N}[g][h]$ such that $Ult(\mathcal{Q}, E) = \mathcal{R}$. We must now have that $\mathcal{Q} = \mathcal{Q}(\mathcal{U})$ implying that $\mathcal{Q} \in \mathcal{N}[g]$ and in $\mathcal{N}[g]$, it is the unique (Ψ, Φ) -mouse \mathcal{Q}^* with the property that $Ult(\mathcal{Q}^*, E) = \mathcal{R}$. It is then easy to find the desired formula ϕ which witnesses that (\mathcal{M}, Φ) is self-determining below η . \square

Let now (\mathcal{P}, Σ) be a self-determining pair below η (possibly a relativized pair) as witnessed by r . Let $n = n_{r, \Sigma}$ and let Φ be the η -extendable strategy of $\mathcal{M}_n^{\#, \Sigma, r}$. We then say \mathcal{M} is a Σ -premouse over \mathbb{R} (or any non self-wellordered set a) if it is a Φ -premouse over \mathbb{R} in the sense of Definition 2.10.2 of [7]. In that definition, one defines Φ -premouse as one in which the strategy predicate describes branches for only two kinds of iterations. These are generic genericity iterations done below the first Woodin of $\mathcal{M}_n^{\#, \Sigma, r}$ or generic comparisons which too are done below the first Woodin of $\mathcal{M}_n^{\#, \Sigma, r}$. For more details see Definition 2.10.2 of [7]. It clearly follows that under our current definition of Σ -premouse over \mathbb{R} , $\Sigma \in \mathcal{S}^{\eta, \Sigma^{r, g}}$. Finally, the discussion above applies not just to \mathbb{R} but to any nonself-wellordered set a . See Section 2.10 of [7] for more details.

Definition 3.8. Let $SD(\eta)$ be the set of pairs (\mathcal{Q}, Λ) which are self-determining below η as defined by Definition 3.5 and Definition 3.6. Given $(\mathcal{Q}, \Lambda) \in SD(\eta)$ we let $W(\eta, \Lambda) = \{r : r \text{ witnesses that } \Lambda \text{ is self-determining below } \lambda\}$.

4 Hod pairs below η

In [9], we introduced the notion of a hod pair below a cardinal. Here we introduce the same notion in a different form as the one introduced in [9] cannot be used in our current context. In particular, the *fullness preservation* condition used here is stronger than in [9]. We need the stronger condition to prove Theorem 6.2, which is a crucial ingredient in the proof of one of the most important technical theorems of this paper, Theorem 10.3. To introduce hod pairs, we need to introduce η -stability, η -super fullness preservation and branch condensation below η . The first ensures that the maximal models computed at various cardinals below η satisfy AD^+ . The second says that the strategies we will consider are correctly guided with respect to various maximal models and are super fullness preserving in these maximal models. The third says that the extensions of the strategy have branch condensation. We start with stability.

For the purpose of the next few definitions, we fix an inaccessible cardinal η which is a limit of strong cardinals and a pair $(\mathcal{P}, \Sigma) \in SD(\eta)$ such that $\mathcal{P} \in H_\eta$ and Σ is an iteration strategy for \mathcal{P} (we are not assuming that \mathcal{P} is a hod premouse).

Definition 4.1 (η -stable). We say Σ is η -stable if there is $r \in W(\eta, \Sigma)$ such that whenever $\nu < \eta$ is a strong cardinal with the property that $\mathcal{P} \in H_\nu$ and $h \subseteq \text{Coll}(\omega, < \nu)$ is V -generic, in $V[h]$, $\Sigma^{r,h}$ has hull condensation and $\mathcal{S}^{\eta, \Sigma^{r,h}} \models AD^+$.

Definition 4.2 (η -super fullness preservation). Suppose now that (\mathcal{P}, Σ) is a hod pair. We say Σ is η -super fullness preserving if there is $r \in W(\eta, \Sigma)$ such that whenever $\nu < \eta$ is a strong cardinal with the property that $\mathcal{P} \in H_\nu$ and $h \subseteq \text{Coll}(\omega, < \nu)$ is V -generic then in $V[h]$, $\Sigma^{r,h}$ has hull condensation and whenever $\mathcal{Q} \in (pI(\mathcal{P}, \Sigma))^V \cap HC^{V[h]}$ and $\alpha < \lambda^{\mathcal{Q}}$ then $\Sigma_{\mathcal{Q}(\alpha+1)}^{r,h} \upharpoonright HC$ is $\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}(\alpha)}^{r,h}}$ -super fullness preserving and is correctly $\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}(\alpha)}^{r,h}}$ -guided.

Definition 4.3. Again suppose that (\mathcal{P}, Σ) is a hod pair. Then we say Σ has branch condensation below η if there is $r \in W(\eta, \Sigma)$ such that whenever $g \subseteq \text{Coll}(\omega, < \eta)$ is generic then $\Sigma^{r,g}$ has branch condensation.

We are now in a position to define hod pairs below η . First given a transitive model M such that $\delta \in M$ is a limit of Woodin cardinals, we write “ $D(M, \delta) \models \phi$ ” to mean that the

derived model of $M^{Coll(\omega, < \delta)}$ satisfies ϕ . More precisely, whenever $g \subseteq Coll(\omega, < \delta)$ is M -generic, letting $\mathbb{R}^* = \cup_{\alpha < \delta} (\mathbb{R}^{M[g \cap Coll(\omega, \alpha)]})$, $L(Hom^*, \mathbb{R}^*) \models \phi$. If δ is the sup of the Woodin cardinals of M then we just write “ $D(M) \models \phi$ ”.

Definition 4.4 (Hod pairs below η). *A hod pair $(\mathcal{P}, \Sigma) \in SD(\eta)$ is called a hod pair below η if there is $r \in W(\eta, \Sigma)$ such that r witnesses that*

1. $\mathcal{N}_\omega^{\#, \Sigma, r}$ exists and is η -iterable via an η -extendable strategy,
2. Σ has branch condensation below η ,
3. $D(\mathcal{N}_\omega^\Sigma) \models$ “ Σ is super fullness preserving”.
4. for every $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}}$ is η -stable,
5. Σ is η -super fullness preserving.

We say that r witnesses that (\mathcal{P}, Σ) is a hod pair below η .

Notice that given an iteration strategy Ψ which has hull condensation, we could also define the notion of a Ψ -hod pair below η . Here by a Ψ -hod premouse we mean a hod premouse which is relative to Ψ . Such hod premice are pure Ψ -mice below their first Woodin cardinal.

Below as well as throughout the paper, if (\mathcal{P}, Σ) is a hod pair below η as witnessed by r , $\nu < \eta$ is a strong cardinal and $g \subseteq Coll(\omega, < \nu)$ is generic then we use $pB(\mathcal{P}, \Sigma)$ and $pI(\mathcal{P}, \Sigma)$ for the interpretation of these sets in V and we use $pB(\mathcal{P}, \Sigma^{r, g})$ and $pI(\mathcal{P}, \Sigma^{r, g})$ for the interpretation of these sets in $V[g]$. The next lemma will be used throughout this paper.

Lemma 4.5. *Suppose η is a limit of strong cardinals and (\mathcal{P}, Σ) is a hod pair below η as witnessed by $r \in W(\eta, \Sigma)$. Suppose that $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$ and $\nu < \eta$ is a strong cardinal such that $\mathcal{P}, \mathcal{Q} \in H_\nu$ and $g \subseteq Coll(\omega, < \nu)$ is V -generic. Then in $V[g]$, $\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}}^{r, g}} \subseteq L(\Sigma^{r, g}, \mathbb{R})$.*

Proof. Its enough to show that in $V[g]$, $(\mathcal{P}(\mathbb{R}))^{\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}}^{r, g}}} \subseteq L(\Sigma^{r, g}, \mathbb{R})$. We work in $V[g]$. Let $\mathcal{S} \in pI(\mathcal{P}, \Sigma)$ be such that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{S}$. Let $\alpha < \lambda^{\mathcal{S}}$ be such that $\mathcal{S}(\alpha) = \mathcal{Q}$. Then we have that $\Sigma_{\mathcal{S}(\alpha+1)}^{r, g}$ is $\mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, g}}$ -fullness preserving, is correctly $\mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, g}}$ -guided and $\Sigma_{\mathcal{S}(\alpha+1)}^{r, g} \in L(\Sigma^{r, g}, \mathbb{R})$. Let then $A \in (\mathcal{P}(\mathbb{R}))^{\mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, g}}}$. Since $\Sigma_{\mathcal{S}(\alpha+1)}^{r, g}$ is correctly $\mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, g}}$ -guided, we can fix $(B_i : i \in \omega) \subseteq (\mathbb{B}[\mathcal{S}(\alpha), \Sigma_{\mathcal{S}(\alpha)}^{r, g}])^{\mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, g}}}$ which guides $\Sigma_{\mathcal{S}(\alpha+1)}^{r, g} \upharpoonright HC^V[g]$. It is now not hard to see that $(B_i : i \in \omega) \subseteq L(\Sigma^{r, g}, \mathbb{R})$ because for each i , B_i can be recovered in $L(\Sigma^{r, g}, \mathbb{R})$ from the triple $(\Sigma_{\mathcal{S}(\alpha+1)}^{r, g}, \tau_{B_i}^{\mathcal{S}(\alpha+1), \Sigma_{\mathcal{S}(\alpha)}^{r, g}}, \mathcal{S}(\alpha+1))$ via genericity iterations (recall that $\tau_{B_i}^{\mathcal{S}(\alpha+1), \Sigma_{\mathcal{S}(\alpha)}^{r, g}}$ is the term relation capturing B_i over $\mathcal{S}(\alpha+1)$). Since there is i such that $A \leq_W (B_i)_{\mathcal{S}(\alpha), \Sigma_{\mathcal{S}(\alpha)}^{r, g}}$, we get that $A \in L(\Sigma^{r, g}, \mathbb{R})$. \square

In this paper, we will need to work with a collection of iteration strategies that properly contains the iteration strategies of hod mice. The reason for this lies behind our proof of Theorem 10.3. In this proof, for some (\mathcal{R}, Ψ) , which is a “pullback” of a hod pair below η , we need to consider Ψ -hod pairs (\mathcal{Q}, Λ) . In order for our proofs to go through, we need to also investigate such strategies. The main problem is that these strategies may give rise to “unwanted” mice. Fortunately, under mild conditions this doesn’t happen (for instance, see Lemma 5.3 and Lemma 5.4).

We now work towards introducing this larger collection of iteration strategies that we will need to study. We continue with our fixed η and define *short hod pairs*. These are essentially hod pairs obtained by taking a union of an increasing sequence of hod pairs.

Definition 4.6. *We say (\mathcal{S}, Φ) is a short hod pair if there is a limit ordinal γ denoted by $\lambda^{\mathcal{S}}$ and there is a sequence $((\mathcal{S}_\alpha, \Phi_\alpha) : \alpha < \gamma)$ such that*

1. $(\mathcal{S}_\alpha, \Phi_\alpha)$ is a hod pair,
2. for $\alpha < \beta$, $\mathcal{S}_\alpha \triangleleft_{\text{hod}} \mathcal{S}_\beta$ and $\lambda^{\mathcal{S}_{\alpha+1}} = \lambda^{\mathcal{S}_\alpha} + 1$,
3. for all limit $\alpha < \gamma$, $\mathcal{S}_\alpha \upharpoonright \delta^{\mathcal{S}_\alpha} = \bigcup_{\beta < \alpha} \mathcal{S}_\beta$,
4. for $\alpha < \beta$, $(\Phi_\beta)_{\mathcal{S}_\alpha} = \Phi_\alpha$,
5. $\mathcal{S} = \bigcup_{\alpha < \gamma} \mathcal{S}_\alpha$ and $\Phi = \bigoplus_{\alpha < \gamma} \Phi_\alpha$.

We then let $\mathcal{S}(\alpha) = \mathcal{S}_\alpha$ and $\delta^{\mathcal{S}} = \sup_{\alpha < \gamma} \delta^{\mathcal{S}_\alpha}$. Notice that if (\mathcal{S}, Φ) is a short hod pair then its height is $\delta^{\mathcal{S}}$. We say (\mathcal{S}, Φ) is a short hod pair below η if (\mathcal{S}, Φ) is a short hod pair and for all $\alpha < \lambda^{\mathcal{S}}$, $(\mathcal{S}(\alpha), \Phi_{\mathcal{S}(\alpha)})$ is a hod pair below η .

Suppose now that (\mathcal{Q}, Λ) is a hod pair below η or a short hod pair below η and $\lambda^{\mathcal{Q}}$ is a limit ordinal. We then say $(\mathcal{Q}^*, \Lambda^*)$ is a *pullback* of (\mathcal{Q}, Λ) if $(\mathcal{Q}^*, \Lambda^*)$ is a short hod pair and there is $\pi : \mathcal{Q}^* \rightarrow_{\Sigma_1} \mathcal{Q} \upharpoonright \delta^{\mathcal{Q}}$ such that for every $\alpha < \lambda^{\mathcal{Q}^*}$, $\Lambda^*_{\mathcal{Q}^*(\alpha)} = (\pi\text{-pullback of } \Lambda_{\mathcal{Q}(\pi(\alpha))})$. Notice that if r witnesses that (\mathcal{Q}, Λ) is η -extendable then it is easy to see that (r, π) codes some u witnessing that (\mathcal{Q}, Λ) is η -extendable (u). Moreover, it follows from the results of Section 2.5, Section 2.6 and Section 2.9 of [7] that

Lemma 4.7. *if $r \in W(\eta, \Sigma)$, $n = n_{r, \Sigma}$, Φ is the η -extendable strategy of $\mathcal{M}_n^{\#, \Sigma, r}$ and $\mathcal{M}_2^{\#, \Phi}$ exists and is η -iterable via an η -extendable strategy then $(\mathcal{Q}, \Lambda) \in SD(\eta)$ and $u \in W(\eta, \Lambda)$.*

Proof. The main point is that it follows from Theorem 2.3 that Λ is $\Gamma(\mathcal{Q}, \Lambda)$ -super fullness preserving. The proof is very much like the proof of Lemma 3.7 and Theorem 6.2. Because of this, we omit the proof. \square

Our collection of iteration strategies consist of iteration strategies for hod premice, short hod premice, pullbacks of hod premice and the relativized versions of the previous three classes. More precisely

Definition 4.8. We let $IS(\eta)^8$ be the set of (\mathcal{Q}, Λ) such that

1. (\mathcal{Q}, Λ) is a hod pair below η or a short hod pair below η , or
2. (\mathcal{Q}, Λ) is a pullback of a hod pair below η or a pullback of a short hod pair below η , or
3. for some (\mathcal{R}, Ψ) which is as in 1 or 2 above, (\mathcal{Q}, Λ) is a Ψ -hod pair below η .

We say r is a witness for (\mathcal{R}, Ψ) if r witnesses that $(\mathcal{R}, \Psi) \in IS(\eta)$.

Suppose $(\mathcal{R}, \Psi) \in IS(\eta)$. Then we let $\mu_{\mathcal{R}, \Psi}$ be the least strong cardinal μ such that $\mathcal{R} \in H_\mu$. Let now $h \subseteq Coll(\omega, < \eta)$ be generic. We let in $V[h]$,

$$D(\eta, h) = \{A \subseteq \mathbb{R} : \exists (\mathcal{Q}, \Lambda) \in IS(\eta) \text{ for some } r \text{ witnessing that } (\mathcal{Q}, \Lambda) \in IS(\eta), \\ A \in L(\Lambda^{r, h}, \mathbb{R})\}.$$

Definition 4.9. Work in $V[h]$. Suppose $(\mathcal{Q}, \Lambda) \in D(\eta, h)$ is a pair such that \mathcal{Q} is a transitive structure and Λ is an ω_1 -iteration strategy for \mathcal{Q} with hull condensation. Suppose $a \in HC^{V[h]}$. We let, in $V[h]$, $\mathcal{V}^{\eta, \Lambda}(a)$ be the union of all sound Λ -mice \mathcal{M} over a such that $\rho(\mathcal{M}) = a$ and \mathcal{M} has an ω_1 -iteration strategy in $D(\eta, h)$ ⁹.

Given an uncountable cardinal $\nu < \eta$ we let $h_\nu = h \cap Coll(\omega, < \nu)$ (this will be a standard notation for us and often we will use it without reintroducing it). The next few lemmas show that $D(\eta, h)$ behaves nicely. We first show that $D(\eta, h)$ is closed under $\mathcal{N}_\omega^\#$ -operator. Our first observation is that $D(\eta, h)$ can be defined by using hod pairs below η .

Lemma 4.10. Work in $V[h]$. Then $D(\eta, h)$ consists of the set of reals A such that for some $(\mathcal{Q}, \Lambda) \in IS(\eta)$ which is a hod pair below η (possibly a relativized hod pair) as witnessed by r , $A \in L(\Lambda^{r, h}, \mathbb{R})$.

Proof. Since pullbacks of hod pairs below η are Wadge reducible to hod pairs below η , it is easy to see that there is some (possibly relativized) $(\mathcal{P}, \Lambda) \in IS(\eta)$ such that (\mathcal{P}, Λ) is a hod pair below η and $A \in L(\Lambda, \mathbb{R})$. □

⁸ IS stands for “iteration strategies”.

⁹Here, to make things simple, we omit h from our notation of $\mathcal{V}^{\eta, \Lambda}(a)$.

Lemma 4.11. *Suppose (\mathcal{P}, Σ) is a hod pair below η as witnessed by r , $\nu < \eta$, and in $V[h_\nu]$, $(\mathcal{M}, \Lambda) \in L(\Sigma^{r, h_\nu}, \mathbb{R})$ is a pair such that \mathcal{M} is a transitive set and Λ is an ω_1 -iteration strategy with hull condensation. Then in $V[h_\nu]$, Λ is η -uB and $\mathcal{N}_\omega^{\#, \Lambda, \Sigma^{r, h_\nu}}$ exists and is η -iterable via an η -extendable strategy¹⁰.*

Proof. We work in $V[h_\nu]$. Let $\mathcal{N} = \mathcal{N}_\omega^{\#, \Sigma, r}$, Φ be its η -iteration strategy and let $M = L(\Sigma^{r, h_\nu}, \mathbb{R})$. Let x be a real such that it codes \mathcal{M} and for some $n < \omega$, letting s be the sequence of the first n -indiscernibles of M , $M \models$ “ Λ is definable from Σ^{r, h_ν}, x, s ”. Let \mathcal{S} be an iterate of \mathcal{N} via Φ such that x is generic over \mathcal{S} for the extender algebra of \mathcal{S} at its first Woodin cardinal. It is now easy to see that $(r, \Sigma^{r, h_\nu}, \Phi_{\mathcal{S}}, x)$ codes some u which witnesses that Λ is η -extendable (essentially u is the witness we get by asking what is true in the derived model of iterates of $\mathcal{S}[x]$). It then follows that in $V[h_\nu]$, Λ is η -uB.

Let now δ be the sup of the Woodin cardinals of \mathcal{S} . We have that $\Lambda \upharpoonright (\mathcal{S}|\delta)[x] \in \mathcal{S}[x]$. Let \mathcal{W} be the output of the fully backgrounded construction of $(\mathcal{S}|\delta)[x]$ done relative to $(\Lambda \oplus \Sigma^{r, h_\nu}) \upharpoonright ((\mathcal{S}|\delta)[x])$. Standard arguments now show that $\mathcal{J}[\mathcal{W}]$ has no level projecting across δ and that in \mathcal{W} , δ is a limit of ω Woodin cardinals. Let now E be the last active extender of \mathcal{S} and let α be the index of E in $\vec{E}^{\mathcal{S}}$. Then $\mathcal{W}^+ = (\mathcal{J}_\alpha[\mathcal{W}], E)$ inherits iterability from \mathcal{S} . It then follows that $\mathcal{N}_\omega^{\#, \Lambda, \Sigma^{r, h_\nu}}$ is the Σ_1 -hull of \mathcal{W}^+ . Because Φ is η -extendable, it follows that $\mathcal{N}_\omega^{\#, \Lambda, \Sigma^{r, h_\nu}}$ inherits an η -extendable strategy from Φ . \square

Corollary 4.12. *Suppose $h \subseteq \text{Coll}(\omega, < \eta)$ is V -generic and in $V[h]$, $(\mathcal{M}, \Sigma) \in D(\eta, h)$ is a pair such that \mathcal{M} is a transitive set and Σ is an ω_1 -strategy with hull condensation. Then there is a strong cardinal $\nu < \eta$ such that $\mathcal{M} \in HC^{V[h_\nu]}$, $\Sigma \upharpoonright V_\eta[h_\nu] \in V[h_\nu]$ and letting $\Psi = \Sigma \upharpoonright HC^{V[h_\nu]}$, in $V[h_\nu]$, Ψ is η -uB, $\Sigma = \Psi^h$ and $\mathcal{N}_\omega^{\#, \Psi}$ exists and is η -iterable via an η -extendable strategy.*

Proof. The proof borrows a result from the next section, namely, Lemma 5.1. Let (\mathcal{R}, Φ) be (possibly relativized) hod pair below η as witnessed by r such that in $V[h]$, $(\mathcal{M}, \Sigma) \in L(\Phi^{r, h}, \mathbb{R})$. Let $\mu_{\mathcal{R}, \Phi} \leq \nu$ be such that there is a real $x \in \mathbb{R}^{V[h_\nu]}$ such that in $V[h]$, for some n letting s be the sequence of the first n indiscernibles of $L(\Phi^{r, h}, \mathbb{R})$ then in $L(\Phi^{r, h}, \mathbb{R})$, Σ is definable from $(\Phi^{r, h}, x, s)$. It follows from Lemma 5.1 that $\mathcal{N}_\omega^{\#, \Phi, r}$ is η^+ -iterable via a η^+ -extendable strategy. It then follows from the derived model theorem and genericity iterations that $(\mathcal{M}, \Sigma \upharpoonright HC^{V[h_\nu]}) \in L(\Phi^{r, h_\nu}, \mathbb{R}^{V[h_\nu]})$. We can now apply Lemma 4.11. Let $\Psi = \Sigma \upharpoonright HC^{V[h_\nu]}$. We have that in $V[h_\nu]$, $\Psi^h = \Sigma$ because we can choose an extendability witness for Ψ with the property that $\Psi^{u, h} = \Sigma$ and also since Ψ is η -uB, $\Psi^{u, h}$ is independent of u . The u that gives $\Psi^{u, h} = \Sigma$ can be constructed by asking what is true in the derived model of the iterates of $\mathcal{N}_\omega^{\#, \Phi, r}$. \square

¹⁰We do not have to put r in our superscript as Σ^{r, h_ν} is η -uB in $V[h_\nu]$.

Corollary 4.13. *Suppose $\nu < \eta$ is a strong cardinal, $a \in HC^{V[h_\nu]}$ and $(\mathcal{Q}, \Lambda) \in D(\eta, h)$ is such that Λ has hull condensation. Then there is a strong cardinal $\mu \in [\nu, \eta)$ such that setting $\Psi = \Lambda \upharpoonright HC^{V[h_\mu]}$, $\mathcal{V}^{\eta, \Lambda}(a) \in V[h_\mu]$, $\Psi \in V[h_\mu]$ and in $V[h_\mu]$, Ψ is η -uB and such that $\Psi^h = \Lambda$. Moreover, if $a \in V$ and $(\mathcal{Q}, \Lambda) \in V$ is an η -extendable pair as witnessed by some r such that $\Lambda^{r, h} \in D(\eta, h)$ then $\mathcal{V}^{\eta, \Lambda}(a) \in V$.*

Proof. Both claims follow from Corollary 4.12 and the homogeneity of the collapse. \square

The closure under $\mathcal{N}_\omega^\#$ operators isn't as powerful as one might think. For instance, it doesn't give us the tools to show that whenever $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda) \in IS(\eta)$ as witnessed by r and s , $\nu < \eta$ is a strong cardinal such that $\mathcal{P}, \mathcal{Q} \in H_\nu$ and $g \subseteq Coll(\omega, < \eta)$ is generic then $L(\Sigma^{r, g}, \Lambda^{s, g}, \mathbb{R}) \models AD^+$. Our next hypothesis implies the aforementioned closure property.

Definition 4.14 (Closure under hybrid \mathcal{N}_ω -operators). *We say η is closed under hybrid \mathcal{N}_ω -operators if*

1. *whenever (\mathcal{Q}, Λ) and (\mathcal{R}, Ψ) are members of $IS(\eta)$ as witnessed by r and s respectively then $\mathcal{N}_\omega^{\#, \Lambda, \Psi, r, s}$ exists and is η -iterable via an η -extendable strategy,*
2. *for any $(\mathcal{Q}, \Lambda) \in IS(\eta)$, for any r witnessing that $(\mathcal{Q}, \Lambda) \in IS(\eta)$, for any strong cardinal $\mu \in [\mu_{\mathcal{Q}, \Lambda}, \eta)$, for any V -generic $g \subseteq Coll(\omega, < \mu)$, for any $a \in HC^{V[g]}$ and for any $\mathcal{M} \trianglelefteq (Lp^{\Lambda^{r, g}}(a))^{\mathcal{S}^{\eta, \Lambda^{r, g}}}$ such that $\rho(\mathcal{M}) = a$ letting $\Phi \in \mathcal{S}^{\eta, \Lambda^{r, g}}$ be the ω_1 -strategy of \mathcal{M} , in $V[g]$, $\mathcal{N}_\omega^{\#, \Lambda^{r, g}, \Phi}$ exists and is η -iterable via an η -extendable strategy, and*
3. *for any η -extendable hod pair (\mathcal{P}, Σ) such that Σ has branch condensation and whenever $\mathcal{R} \in pB(\mathcal{P}, \Sigma)$, $(\mathcal{R}, \Sigma_{\mathcal{R}}) \in IS(\eta)$, for any r witnessing η -extendability of Σ , $\mathcal{N}_\omega^{\#, \Sigma, r}$ exists and is η -iterable via an η -extendable strategy.*

In Section 7, we will show (see Theorem 7.8) that if no symmetric K^c construction below η converges then η is closed under hybrid \mathcal{N}_ω operators. In the next section, we will show that if η is closed under hybrid \mathcal{N}_ω -operators then hod pairs below η are $D(\eta, h)$ -fullness preserving. One consequence of the closure under hybrid \mathcal{N}_ω -operators is that it implies a very crucial property, namely *lower part persistence below η* (see Lemma 5.3). To introduce this notion, we continue with our fixed η and h . Let (\mathcal{P}, Σ) be a pair such that $\mathcal{P} \in H_\eta$ and Σ is η -extendable.

Definition 4.15 (Lower part persistence). *We say Σ is lower part persistent below η if there is r witnessing that (\mathcal{P}, Σ) is η -extendable and such that for any $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, for any two $\mu < \nu$ strong cardinals such that $\mathcal{Q} \in H_\mu$ and for any $a \in HC^{V[h_\mu]}$,*

$$(Lp^{\Sigma_{\mathcal{Q}}^{r, h_\mu}}(a))^{\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\mu}}} = (Lp^{\Sigma_{\mathcal{Q}}^{r, h_\nu}}(a))^{\mathcal{S}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\nu}}}.$$

Definition 4.16 (Lower part persistence below η). *We say lower part persistence holds below η and write $lpp(\eta)$ if for every $(\mathcal{Q}, \Lambda) \in IS(\eta)$, Λ is lower part persistent below η .*

5 Proving persistence

One of the main goals of this section is to prove that closure under hybrid \mathcal{N}_ω -operators implies persistence. To do this, we need to reproduce a useful lemma from [9] (in particular, see Lemma 2.5 and Corollary 2.6 of [9]). For the purposes of the next lemma, suppose $\mu < \eta$ are such that μ is a strong cardinal and η is inaccessible. Let $j : V \rightarrow M$ be an embedding witnessing that μ is η^+ -strong and let $h \subseteq Coll(\omega, < j(\mu))$ be a V -generic. Let $j^+ : V[h_\mu] \rightarrow M[h]$ be the lift of j .

Suppose first that (\mathcal{P}, Σ) is a μ -extendable pair and let r witness this. Its not hard to see that (\mathcal{P}, Σ) is in fact an η -extendable pair. Indeed we have that in M , $(\phi, \psi, j(Z))$ witnesses that (\mathcal{P}, Σ) is $j(\mu)$ -extendable. It now follows that in M , $(\phi, \psi, j(Z))$ witnesses that (\mathcal{P}, Σ) is η -extendable. But because $H_{\eta^+} \in M$, we have that $(\phi, \psi, j(Z))$ also witnesses that (\mathcal{P}, Σ) is η -extendable. The moral is that $j(r)$ witnesses that (\mathcal{P}, Σ) is η -extendable.

Lemma 5.1. *Suppose $(\mathcal{P}, \Sigma) \in SD(\mu)$, $r \in W(\mu, \Sigma)$ and $a \in V_\eta[h_\mu]$. Then (\mathcal{P}, Σ) is an η -extendable pair as witnessed by $j(r)$ and*

$$\begin{aligned} (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} &= (\mathcal{W}^{j(\eta), \Sigma^{j(r), h}}(a))^{M[h]} = (\mathcal{K}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} \\ &= (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} = (Lp^{\Sigma^{j(r), h}}(a))^{M[h]}. \end{aligned}$$

Moreover, if $a \in HC^{V[h_\mu]}$, $X \in V$ and in $V[h_\mu]$, N is a transitive inner model containing the reals and ordinals and such that N is OD_X and $\Sigma^{r, h_\mu} \in N$ then $(Lp^{\Sigma^{r, h_\mu}}(a))^N \trianglelefteq Lp^{\Sigma^{r, h_\mu}}(a)$. In particular, $(Lp^{\Sigma^{r, h_\mu}}(\mathbb{R}))^N \trianglelefteq \mathcal{S}^{\eta, \Sigma^{r, h_\mu}}$.

Proof. The arguments that follow are rather standard. To prove the equalities, we first show that $(\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} = (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$. Work in $V[h_\mu]$. Clearly $(\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} \trianglelefteq (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$. Let then $\mathcal{M} \trianglelefteq (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$ be such that $\rho(\mathcal{M}) = a$. We want to see that $\mathcal{M} \trianglelefteq (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$. To see this, notice that by a standard absoluteness argument, there is $\sigma : \mathcal{M} \rightarrow j^+(\mathcal{M})$ such that $\sigma \in M[h]$ and $\sigma \upharpoonright \mathcal{P} = id$. Because $\Sigma^{j(r), h}$ has hull condensation, we have that, in $M[h]$, \mathcal{M} is $\omega_1 + 1$ -iterable $\Sigma^{j(r), h}$ -mouse. Let in $M[h]$, Λ be the unique $\omega_1 + 1$ -iteration strategy of \mathcal{M} witnessing that \mathcal{M} is a $\Sigma^{j(r), h}$ -mouse. It follows from the homogeneity of the collapse and the uniqueness of Λ that $\Lambda \upharpoonright V_\eta[h_\mu] \in V[h_\mu]$. Hence, $\mathcal{M} \trianglelefteq (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$.

To see that $(\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} = (\mathcal{W}^{j(\eta), \Sigma^{j(r), h}}(a))^{M[h]}$, first suppose

$$\mathcal{M} \trianglelefteq (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}.$$

Then, in $M[h]$, $j(\mathcal{M}) \trianglelefteq \mathcal{W}^{j(\eta), j^+(\Sigma^{r, h_\mu})}(j^+(a))$. Again by a standard absoluteness argument, there is $\sigma : \mathcal{M} \rightarrow j^+(\mathcal{M})$ such that $\sigma \in M[h]$ and $\sigma \upharpoonright \mathcal{P} = id$. It follows that in $M[h]$, $\mathcal{M} \trianglelefteq \mathcal{W}^{j(\eta), j^+(\Sigma^{r, h_\mu})}(a)$. Next, suppose $\mathcal{M} \trianglelefteq (\mathcal{W}^{j(\eta), j^+(\Sigma^{r, h_\mu})}(a))^{M[h]}$ is such that $\rho(\mathcal{M}) = a$. It follows from the homogeneity of the collapse and the uniqueness of the strategy of \mathcal{M} that $\mathcal{M} \in V[h_\mu]$ and that $\mathcal{M} \trianglelefteq (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}$.

We thus have that

$$(1) (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} = (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} = (\mathcal{W}^{j(\eta), j^+(\Sigma^{r, h_\mu})}(a))^{M[h]}.$$

Notice we have that

$$(2) (\mathcal{W}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} \trianglelefteq (\mathcal{K}^{\eta, \Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]} \trianglelefteq (Lp^{\Sigma^{j(r), h_\mu}}(a))^{V[h_\mu]}.$$

Applying the first equality of (1) inside M and noticing that a is countable in $M[h]$ we get that,

$$(3) (\mathcal{W}^{j(\eta), \Sigma^{j(r), h}}(a))^{M[h]} = (Lp^{\Sigma^{j(r), h}}(a))^{M[h]}.$$

(1), (2) and (3) now easily imply the first part of the lemma. To see the second part of the lemma, fix a, N as in the statement of the lemma. Let $\mathcal{M} \trianglelefteq (Lp^{\Sigma^{r, h_\mu}}(a))^N$ be such that $\rho(\mathcal{M}) = a$ and let $\Pi \in N$ be such that $N \models \text{“}\Pi \text{ is the unique } \omega_1 + 1\text{-strategy of } \mathcal{M}\text{”}$. Then it follows from the homogeneity of the collapse that $j^+(\Pi) \upharpoonright V_\eta[h_\mu] \in V[h_\mu]$. Hence, \mathcal{M} is $\omega_1 + 1$ -iterable in $V[h_\mu]$ implying that $\mathcal{M} \trianglelefteq Lp^{\Sigma^{r, h_\mu}}(a)$. The last part of the lemma is an immediate consequence of the first two parts of the lemma. \square

We can now state and prove the main lemma of this section. For the rest of this section we fix an inaccessible cardinal η which is a limit of strong cardinals and a V -generic $h \subseteq Coll(\omega, < \eta)$. The following lemma is the main technical lemma of this section. *In what follows we will have situations where M is a model of AD^+ and Σ is a (η, η) -strategy with the property that $\Sigma \upharpoonright HC \in M$. In this case, we will write $M \models \phi[\Sigma, \dots]$ instead of $M \models \phi[\Sigma \upharpoonright HC, \dots]$. We will use this convention throughout the paper.*

Lemma 5.2. *Suppose $(\mathcal{P}, \Sigma) \in IS(\eta)$ as witnessed by r . Suppose that $\nu < \eta$ is a strong cardinal such that $\mathcal{P} \in H_\nu$ and $a \in HC^{V[h_\nu]}$. Suppose in $V[h_\nu]$, (\mathcal{M}, Π) is such that \mathcal{M} is a sound Σ^{r, h_ν} -mouse over a such that $\rho(\mathcal{M}) = a$, Π is an ω_1 -strategy for \mathcal{M} with hull conden-*

sation and $\mathcal{N}_\omega^{\#, \Pi, \Sigma^{r, h_\nu}}$ exists and is η -iterable via an η -extendable strategy. Then whenever $\mu \in [\nu, \eta)$ is a strong cardinal, $\mathcal{M} \trianglelefteq (Lp^{\Sigma^{r, h_\mu}}(a))^{\mathcal{S}^{\eta, \Sigma^{r, h_\mu}}}$.

Proof. Suppose that ν, a and (\mathcal{M}, Π) are as in the hypothesis of the lemma. Let $\mathcal{N} = \mathcal{N}_\omega^{\#, \Pi, \Sigma^{r, h_\nu}}$. First notice that because \mathcal{N} has an η -extendable strategy, if $\kappa < \nu$ is an inaccessible cardinal such that $\mathcal{M}, \mathcal{P} \in HC^{V[h_\kappa]}$ then letting $\Phi = \Pi \upharpoonright HC^{V[h_\kappa]}$, $\Phi \in V[h_\kappa]$, $V[h_\kappa] \models \text{“}\Phi \text{ is } \eta\text{-uB”}$ and $\Phi^{h_\nu} = \Pi$. Fix then such a cardinal κ .

Let now $\mu \in [\nu, \eta)$ be a strong cardinal. Because \mathcal{N} is η -iterable via an η -extendable strategy, it follows that in $V[h_\mu]$, letting $N = L(\Sigma^{r, h_\mu} \upharpoonright HC, \Pi^{h_\mu} \upharpoonright HC, \mathbb{R})$, $N \models AD^+$. We have that in $V[h_\mu]$, N is *OD* from $(\Sigma^{r, h_\mu}, \Phi)$. Because $N \models \text{“every set is ordinal definable from a real and } \Sigma^{r, h_\mu}\text{”}$, it follows from Theorem 2.6 that in $V[h_\mu]$,

$$N \models \mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp^{\Sigma^{r, h_\mu}}(\mathbb{R}).$$

Therefore, in $V[h_\mu]$, $\Pi^{h_\mu} \in Lp^{\Sigma^{r, h_\mu}}(\mathbb{R})$. It now follows from the last part of Lemma 5.1 (applied with V changed to $V[h_\mu]$) that $(Lp^{\Sigma^{r, h_\mu}}(\mathbb{R}))^{V[h_\mu]} \trianglelefteq \mathcal{S}^{\eta, \Sigma^{r, h_\mu}}$. Hence, $\Pi^{h_\mu} \in \mathcal{S}^{\eta, \Sigma^{r, h_\mu}}$. We then get that $\mathcal{M} \trianglelefteq (Lp^{\Sigma^{r, h_\mu}}(a))^{\mathcal{S}^{\eta, \Sigma^{r, h_\mu}}}$. \square

We now list some very useful corollaries of Lemma 5.2. The following is an immediate corollary of Lemma 5.2 (recall Definition 4.14).

Corollary 5.3 (Criteria for persistence). *Suppose η is closed under hybrid \mathcal{N}_ω -operators. Then $lpp(\eta)$ holds.*

Lemma 5.4. *Suppose (\mathcal{P}, Σ) is a hod pair below η as witnessed by r . Then in $V[h]$, $\Sigma^{r, h}$ is $D(\eta, h)$ -fullness preserving.*

Proof. Recall that since (\mathcal{P}, Σ) is a hod pair below η , letting $\nu < \eta$ be a strong cardinal such that $\mathcal{P} \in H_\nu$, in $V[h_\nu]$, $\Sigma^{r, h_\nu} \upharpoonright HC^{V[h_\nu]}$ is $L(\Sigma^{r, h_\nu} \upharpoonright HC, \mathbb{R})$ -fullness preserving. It follows that its enough to show that

(1) whenever $\mathcal{Q} \in pB(\mathcal{P}, \Sigma^{r, h}) \cap HC^{V[h]}$ and $\nu < \eta$ is a strong cardinal such that $\mathcal{P}, \mathcal{Q} \in HC^{V[h_\nu]}$ then for every $a \in HC^{V[h_\nu]}$, $\mathcal{V}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\nu}}(a) = (Lp_{\mathcal{Q}}^{\Sigma_{\mathcal{Q}}^{r, h_\nu}}(a))^{L(\Sigma^{r, h_\nu}, \mathbb{R}^{V[h_\nu]})}$.

Fix then \mathcal{Q} , ν and a be as in (1). Let $\mathcal{Q}^+ \in pI(\mathcal{P}, \Sigma^{r, h_\nu}) \cap HC^{V[h_\nu]}$ be such that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{Q}^+$. We claim that we can find $\mathcal{S} \in pI(\mathcal{P}, \Sigma) \cap HC^{V[h_\nu]}$ such that $\mathcal{S} \in pI(\mathcal{Q}^+, \Sigma_{\mathcal{Q}^+}^{r, h_\nu})$. Let $\mathcal{M} = (\mathcal{M}_\infty(\mathcal{P}, \Sigma^{r, h_\nu}))^{L(\Sigma^{r, h}, \mathbb{R})}$. Then it follows from the homogeneity of the collapse that $\mathcal{M} \in V$. Let now $\pi : H[h_\kappa] \rightarrow H_{\nu^+}[h_\nu]$ be an elementary embedding such that κ is an inaccessible cardinal, $cp(\pi) = \kappa$, $H \in V$ and $(\mathcal{P}, \Sigma), \mathcal{Q}^+ \in rng(\pi)$. We then let $\mathcal{S} = \pi^{-1}(\mathcal{M})$.

Clearly $\mathcal{S} \in V$ and $H[h_\kappa] \models \text{“}\mathcal{S} \in pI(\mathcal{Q}^+, \Sigma_{\mathcal{Q}^+}^{r, h_\kappa})\text{”}$. It then follows that \mathcal{S} is as desired. Let $j = \pi_{\mathcal{Q}^+, \mathcal{S}}^{\Sigma^{r, h_\nu}}$. Let $\alpha < \lambda^{\mathcal{S}}$ be such that $\mathcal{S}(\alpha) = j(\mathcal{Q})$.

Let now $\mathcal{M} \trianglelefteq \mathcal{V}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\nu}}(a)$ be such that $\rho(\mathcal{M}) = a$. It follows from Lemma 4.11 that in $V[h_\nu]$, if Π is the unique strategy of \mathcal{M} then $\mathcal{N}_\omega^{\#, \Pi, \Sigma_{\mathcal{Q}}^{r, h_\nu}}$ exists and is η -iterable via an η -extendable strategy. It then follows that in $V[h_\nu]$, $L(\Pi, \Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC, \mathbb{R}) \models AD^+$. Let in $V[h_\nu]$, $M = L(\Pi, \Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC, \mathbb{R})$. Notice that in $V[h_\nu]$, $M = L(\Pi, \Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu} \upharpoonright HC, \mathbb{R})$ implying that $M \models V = OD_{\Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu}, \mathbb{R}}$. It then follows from Theorem 2.6 that in $V[h_\nu]$,

$$(2) \ (\mathcal{P}(\mathbb{R}))^M = \mathcal{P}(\mathbb{R}) \cap (Lp_{\Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu}}(\mathbb{R}))^M$$

implying that $\Pi \in (Lp_{\Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu}}(\mathbb{R}))^M$. But it follows from the last part of Lemma 5.1 that $(Lp_{\Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu}}(\mathbb{R}))^M \trianglelefteq \mathcal{S}^{\eta, \Sigma_{\mathcal{S}(\alpha)}^{r, h_\nu}}$ (just like in the proof of Lemma 5.2 we need to apply Lemma 5.1 with V changed to some $V[h_\kappa]$). It follows from Lemma 4.5 and (2) that in $V[h_\nu]$, $\Pi \in L(\Sigma^{r, h_\nu} \upharpoonright HC, \mathbb{R}^{V[h_\nu]})$ implying that $\mathcal{M} \trianglelefteq (Lp_{\mathcal{Q}}^{\Sigma^{r, h_\nu}}(a))^{L(\Sigma^{r, h_\nu} \upharpoonright HC, \mathbb{R}^{V[h_\nu]})}$.

It remains to show that in $V[h_\nu]$, $(Lp_{\mathcal{Q}}^{\Sigma^{r, h_\nu}}(a))^{L(\Sigma^{r, h_\nu} \upharpoonright HC, \mathbb{R})} \trianglelefteq \mathcal{V}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\nu}}(a)$. We work in $V[h_\nu]$. Let $\mathcal{M} \trianglelefteq (Lp_{\mathcal{Q}}^{\Sigma^{r, h_\nu}}(a))^{L(\Sigma^{r, h_\nu} \upharpoonright HC, \mathbb{R})}$ be such that $\rho(\mathcal{M}) = a$. It follows from Lemma 4.11 that if $\Pi \in L(\Sigma^{r, h_\nu}, \mathbb{R}^{V[h_\nu]})$ is the ω_1 -iteration strategy of \mathcal{M} then $\mathcal{N}_\omega^{\#, \Pi, \Sigma^{r, h_\nu}}$ exists and is η -iterable via an η -extendable strategy. It then follows that in $V[h]$, $\Pi^h \in L(\Sigma^{r, h}, \mathbb{R})$ implying that $\mathcal{M} \trianglelefteq \mathcal{V}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\nu}}(a)$. \square

The following is an immediate corollary of the proof of Lemma 5.4 (consult the last paragraph of that proof).

Corollary 5.5. *Suppose (\mathcal{P}, Σ) is a hod pair below η as witnessed by r and that $(\mathcal{Q}, \Lambda) \in IS(\eta)$ as witnessed by u . Suppose that $\nu < \eta$ is a strong cardinal such that in $V[h_\nu]$, $\Sigma^{r, h_\nu} \upharpoonright HC \in L(\Lambda^{u, h_\nu} \upharpoonright HC, \mathbb{R})$. Then in $V[h_\nu]$, $L(\Lambda^{u, h_\nu} \upharpoonright HC, \mathbb{R}) \models \text{“}\Sigma^{r, h_\nu} \text{ is fullness preserving”}$.*

6 Canonical witnesses

The question that concerns us here is whether there is a canonical r witnessing that (\mathcal{P}, Σ) is a hod pair below η . Here we give one useful example of such an r . In Section 10, we will use it to show that certain embeddings witnessing measurability have *weak condensation*, a result that is crucial for the results of Section 11.

We start by fixing an inaccessible cardinal η which is a limit of strong cardinals and a V -generic $h \subseteq Coll(\omega, < \eta)$. Suppose (\mathcal{P}, Σ) is a hod pair below η as witnessed by r . Recall that $pI(\mathcal{P}, \Sigma)$ and $pB(\mathcal{P}, \Sigma)$ are sets in V , and if $\nu < \eta$ then $pI(\mathcal{P}, \Sigma^{r, h_\nu})$ and $pB(\mathcal{P}, \Sigma^{r, h_\nu})$ are sets in $V[h_\nu]$.

Suppose $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$ is such that $\lambda^{\mathcal{Q}}$ is a successor. Let $\kappa \in (|\mathcal{Q}|, \eta)$ be a strong cardinal and let in $V[h_\kappa]$, $M_{\kappa, \mathcal{Q}} = \mathcal{S}^{\eta, \Sigma_{\mathcal{Q}}^{r, h_\kappa}}(\mathbb{R})$. Also, let $\mathcal{S}_\kappa^{\mathcal{Q}}$ be the result of generically comparing all $\Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\kappa}$ -premise \mathcal{R} such that $\mathcal{R} \in HC^{V[h_\kappa]}$ and $M_{\kappa, \mathcal{Q}} \models \text{“}\mathcal{R} \text{ is } \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\kappa}\text{-suitable and } \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\kappa}\text{-short tree iterable”}$ ¹¹. Let also $\mathcal{W}_\kappa^{\mathcal{Q}}$ be the result of iterating \mathcal{Q} above $\delta_{\lambda^{\mathcal{Q}-1}}^{\mathcal{Q}}$ according to $\Sigma_{\mathcal{Q}}$ to make H_κ^V generically generic (the effect of such an iteration is that whenever $m \subseteq \text{Coll}(\omega, H_\kappa^V)$ is generic and x_m is the canonical real in $V[m]$ coding H_κ^V then x_m is generic over the extender algebra $\mathbb{B}_\delta^{\mathcal{W}_\kappa^{\mathcal{Q}}}$ where $\delta = \delta^{\mathcal{W}_\kappa^{\mathcal{Q}}}$). Let $\pi_\kappa^{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{S}_\kappa^{\mathcal{Q}}$ and $\sigma_\kappa^{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{W}_\kappa^{\mathcal{Q}}$ be the iteration embeddings.

Recall that if \mathcal{M} is a hybrid premouse and $\xi < o(\mathcal{M})$ then $\mathcal{O}_\xi^{\mathcal{M}} = \cup\{\mathcal{N} \trianglelefteq \mathcal{M} : \mathcal{M} \upharpoonright \xi \trianglelefteq \mathcal{N} \wedge \xi \text{ is a cutpoint in } \mathcal{N}\}$. Also recall the sets U and W defined in Section 2. Also recall the definition of a *fatal drop* from [6]. If \mathcal{M} is a hybrid premouse and \mathcal{U} is a tree on \mathcal{M} then we say \mathcal{U} has a fatal drop at (γ, ξ) if the pair is the lexicographically least (α, β) such that $\mathcal{T}_{\geq \alpha}$ is a tree on $\mathcal{O}_\beta^{\mathcal{M}_\alpha^{\mathcal{U}}}$ which is above β .

Lemma 6.1. *Suppose $\mathcal{T} \in HC^{V[h]}$ is a normal tree on \mathcal{Q} that is above $\delta_{\lambda^{\mathcal{Q}-1}}^{\mathcal{Q}}$. Then \mathcal{T} is according to $\Sigma_{\mathcal{Q}}^{r, h}$ if and only if for every limit $\alpha < lh(\mathcal{T})$ letting $\mathcal{U} = \mathcal{T} \upharpoonright \alpha$ and b be the branch of \mathcal{U} in \mathcal{T} , b is the unique branch of \mathcal{U} such that whenever $\kappa < \eta$ is a strong cardinal such that $\mathcal{U} \in HC^{V[h_\kappa]}$, one of the following holds:*

1. \mathcal{U} doesn't have fatal drops, $\mathcal{Q}(b, \mathcal{U})$ exists, and $\mathcal{W}_\kappa^{\mathcal{Q}}[\mathcal{Q}(b, \mathcal{U})] \models \text{“whenever } m \subseteq \text{Coll}(\omega, \delta^{\mathcal{W}_\kappa^{\mathcal{Q}}}) \text{ is generic and } x, y \in \mathbb{R}^{\mathcal{W}_\kappa^{\mathcal{Q}}[\mathcal{Q}(b, \mathcal{U})][m]} \text{ are such that } x \text{ codes } \mathcal{M}(\mathcal{U}) \text{ and } y \text{ codes } \mathcal{Q}(b, \mathcal{U}), \text{ then } (x, y) \in \sigma_\kappa^{\mathcal{Q}}(\tau_{\mathcal{U}(\lambda^{\mathcal{Q}-1), \Sigma}^{\mathcal{Q}})}^{\mathcal{Q}})\text{”}$.
2. \mathcal{U} has a fatal drop at (γ, ξ) and $\mathcal{W}_\kappa^{\mathcal{Q}}[\mathcal{U}, b] \models \text{“whenever } m \subseteq \text{Coll}(\omega, \delta^{\mathcal{W}_\kappa^{\mathcal{Q}}}) \text{ is generic and } x, y, z \in \mathbb{R}^{\mathcal{W}_\kappa^{\mathcal{Q}}[\mathcal{U}, b][m]} \text{ are such that } x \text{ codes } \mathcal{M}(\mathcal{U}), y \text{ codes } \mathcal{O}_\xi^{\mathcal{M}_\gamma^{\mathcal{U}}}, \text{ and } z \text{ codes } (\mathcal{U}_{\geq \gamma}) \setminus \{\mathcal{M}_b^{\mathcal{U}}\} \text{ then } (x, y, z) \in \sigma_\kappa^{\mathcal{Q}}(\tau_{\mathcal{W}(\lambda^{\mathcal{Q}-1), \Sigma}^{\mathcal{Q}})}^{\mathcal{Q}})\text{”}$.
3. $\pi_b^{\mathcal{U}}$ exists and there is $\sigma : \mathcal{M}_b^{\mathcal{U}} \rightarrow \mathcal{S}_\kappa^{\mathcal{Q}}$ such that $\pi_\kappa^{\mathcal{Q}} = \sigma \circ \pi_b^{\mathcal{U}}$.

Proof. Suppose first that $\Sigma_{\mathcal{Q}}^{r, h}(\mathcal{U}) = b$. Suppose that $\mathcal{Q}(b, \mathcal{U})$ exists. We need to show that clause 1 or clause 2 holds. We present the argument only for clause 2 as the argument for clause 1 is very similar. Fix two strong cardinals $\kappa < \nu < \eta$ such that $\mathcal{U} \in HC^{V[h_\kappa]}$. Notice that we have that \mathcal{U} is generic over $\mathcal{W}_\kappa^{\mathcal{Q}}$. Suppose then \mathcal{U} has a fatal drop at (γ, ξ) . Let $\theta(u, v, w, x)$ be the formula displayed in clause 2. Thus, we need to show that $\mathcal{W}_\kappa^{\mathcal{Q}}[\mathcal{U}, b] \models \theta[\mathcal{U}, b, (\gamma, \xi), \sigma_\kappa^{\mathcal{Q}}(\tau_{\mathcal{W}(\lambda^{\mathcal{Q}-1), \Sigma}^{\mathcal{Q}})}^{\mathcal{Q}})]$. Because (\mathcal{P}, Σ) is a hod pair below η , we have that in $V[h_\nu]$, $\Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC$ is $M_{\nu, \mathcal{Q}}$ -super fullness preserving¹². It then follows that

¹¹The idea of generic comparisons goes back to Woodin. The form that is needed here was worked out by the author in [7], see Section 2.9 of [7].

¹²Recall that since ν is a strong cardinal, ω_1 -iterable structures in $M_{\nu, \mathcal{Q}}$ are η -iterable, which implies that $\Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC$ is $M_{\nu, \mathcal{Q}}$ -super fullness preserving.

$M_{\nu, \mathcal{Q}} \models \text{“}\mathcal{W}_\kappa^\mathcal{Q} \text{ is } \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\nu}\text{-suitable”}$. We also have that $\sigma_\kappa^\mathcal{Q}(\tau_{W_{\mathcal{Q}(\lambda^{\mathcal{Q}-1), \Sigma}}^\mathcal{Q}}^\mathcal{Q}) = \tau_{W_{\mathcal{Q}(\lambda^{\mathcal{Q}-1), \Sigma}}^{\mathcal{W}_\kappa}}^\mathcal{W}_\kappa$. Therefore, $\mathcal{W}_\kappa^\mathcal{Q}[\mathcal{U}, b] \models \theta[\mathcal{U}, b, (\gamma, \xi), \sigma_\kappa^\mathcal{Q}(\tau_{W_{\mathcal{Q}(\lambda^{\mathcal{Q}-1), \Sigma}}^\mathcal{Q}}^\mathcal{Q})]$.

Conversely, suppose b is a branch such that clause 2 holds. We need to show that $\Sigma_{\mathcal{Q}}^{r, h}(\mathcal{U}) = b$. Again let $\kappa < \nu < \eta$ be strong cardinals such that $\mathcal{U} \in HC^{V[h_\kappa]}$ and let $\theta(u, v, w, x)$ be as above. Because $\mathcal{W}_\kappa^\mathcal{Q}[\mathcal{U}, b] \models \theta[\mathcal{U}, b, (\gamma, \xi), \pi_{\mathcal{Q}, \mathcal{S}_\kappa^\mathcal{Q}}^\Sigma(\tau_{W_{\mathcal{Q}(\lambda^{\mathcal{Q}-1), \Sigma}}^\mathcal{Q}}^\mathcal{Q})]$ and because in $V[h_\nu]$, $\Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC$ is $M_{\nu, \mathcal{Q}}$ -fullness preserving, we have that $M_{\nu, \mathcal{Q}} \models \text{“the phalanx obtained from } (\mathcal{U}_{\geq \gamma})^\frown \mathcal{M}_b^\mathcal{U} \text{ is } \omega_1\text{-iterable”}$. Because $M_{\nu, \mathcal{Q}} \models AD^+$ and $\Sigma_{\mathcal{Q}}^{r, h_\nu} \upharpoonright HC$ is $M_{\nu, \mathcal{Q}}$ -fullness preserving, it follows that we must have that $\Sigma_{\mathcal{Q}}^{r, h_\nu}(\mathcal{U}) = b$.

Suppose now that $\Sigma_{\mathcal{Q}}^{r, h}(\mathcal{U}) = b$ but $\mathcal{Q}(b, \mathcal{U})$ doesn't exist. Let $\kappa < \eta$ be a strong cardinal such that $\mathcal{U} \in HC^{V[h_\kappa]}$. Because in $V[h_\kappa]$, $\Sigma_{\mathcal{Q}}^{r, h_\kappa}$ is $M_{\kappa, \mathcal{Q}}$ -fullness preserving it follows that $M_{\kappa, \mathcal{Q}} \models \text{“}\mathcal{M}_b^\mathcal{U} \text{ is } \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\kappa}\text{-suitable and } \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}-1})}^{r, h_\kappa}\text{-short tree iterable”}$. It then follows that $\mathcal{S}_\kappa^\mathcal{Q}$ is an iterate of $\mathcal{M}_b^\mathcal{U}$ and hence, letting $\sigma : \mathcal{M}_b^\mathcal{U} \rightarrow \mathcal{S}_\kappa^\mathcal{Q}$ be the iteration embedding we have that $\pi_\kappa^\mathcal{Q} = \sigma \circ \pi_b^\mathcal{U}$. Conversely, if b is a branch such that there is $\sigma : \mathcal{M}_b^\mathcal{U} \rightarrow \mathcal{S}_\kappa^\mathcal{Q}$ with the property that $\pi_\kappa^\mathcal{Q} = \sigma \circ \pi_b^\mathcal{U}$ then because $\Sigma_{\mathcal{Q}}^{r, h}$ has branch condensation, we have that $\Sigma_{\mathcal{Q}}^{r, h}(\mathcal{U}) = b$. \square

We let $\phi(u, v, w)$ be the formula that expresses clauses 1-3 of Lemma 6.1 where u stands for \mathcal{Q} , v stands for \mathcal{T} and w stands for (Σ, r) . Next we state the general result.

Theorem 6.2. *Suppose that $\vec{\mathcal{T}} \in HC^{V[h]}$ is a stack on \mathcal{P} . Then $\vec{\mathcal{T}}$ is according to $\Sigma^{r, h}$ if and only if whenever \mathcal{R} is a non-trivial terminal node of $\vec{\mathcal{T}}$ there is $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ and an embedding $\sigma : \mathcal{R} \rightarrow \mathcal{Q}$ such that $\pi_{\mathcal{P}, \mathcal{Q}}^\Sigma = \sigma \circ \pi_{\mathcal{R}}^{\vec{\mathcal{T}}}$ and $\phi[\mathcal{Q}(\sigma(\xi^{\vec{\mathcal{T}}, \mathcal{R}}) + 1), \sigma \vec{\mathcal{T}}_{\mathcal{R}}]$ holds.*

Proof. We only prove the forward direction as the reverse direction is an easy application of branch condensation and Lemma 6.1. The proof of the forward direction is by induction on the Dodd-Jensen order on $pI(\mathcal{P}, \Sigma^{r, h}) \cup pB(\mathcal{P}, \Sigma^{r, h})$. We let $\psi[\mathcal{P}, \Sigma^{r, h}]$ express the conclusion of the forward direction of the theorem. Thinking of the proof as induction, it is enough to show that whenever $\mathcal{R} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$ is such that for every $\mathcal{R}^* \in pB(\mathcal{R}, \Sigma)$, $\psi[\mathcal{R}^*, \Sigma_{\mathcal{R}^*}^{r, h}]$ holds then $\psi[\mathcal{R}, \Sigma_{\mathcal{R}}^{r, h}]$ holds. Fix then such an \mathcal{R} .

Suppose first that $\lambda^{\mathcal{R}}$ is limit. It is not difficult to see that it follows from the inductive hypothesis that

(1) whenever $(\vec{\mathcal{U}}, \mathcal{R}^*) \in I(\mathcal{R}, \Sigma_{\mathcal{R}^*}^{r, h})$ then there is $(\vec{\mathcal{W}}, \mathcal{Q}) \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$ and an embedding $\sigma : \mathcal{R}^* \rightarrow \mathcal{Q}$ such that $\pi^{\vec{\mathcal{W}}} = \sigma \circ \pi^{\vec{\mathcal{U}}}$.

To establish (1) one just needs to successively use the copying construction and dovetail them all into one iteration. The reader can find more details by consulting the proof of Lemma 3.35 of [7]. Fix then $\vec{\mathcal{U}}$ on \mathcal{R} which is according to $\Sigma_{\mathcal{R}}^{r, h}$ and let \mathcal{S} be a non-trivial

terminal node in \vec{U} . Let \vec{U}^* be the initial segment of \vec{U} up to the stage where \mathcal{S} first appears in \vec{U} . Using (1) we can find $(\vec{W}, \mathcal{Q}) \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$ such that $\pi^{\vec{W}} = \sigma \circ \pi^{\vec{U}^*}$. Because $(\vec{W}, \mathcal{Q}) \in V$, it follows from Lemma 6.1 that $\phi[\mathcal{Q}(\xi^{\vec{U}, \mathcal{S}} + 1), \sigma \vec{U}_{\mathcal{S}}]$ holds.

Suppose next that $\lambda^{\mathcal{R}}$ is a successor ordinal. Fix \vec{U} on \mathcal{R} and let \mathcal{S} be a non-trivial terminal node in \vec{U} . If $\xi^{\vec{U}, \mathcal{S}} + 1 < \lambda^{\mathcal{S}}$ then the claim follows from the inductive hypothesis. Thus we assume that $\xi^{\vec{U}, \mathcal{S}} + 1 = \lambda^{\mathcal{S}}$. The proof now is by induction on the stage where \mathcal{S} appears in \vec{U} . Suppose the claim is true for any other non-trivial terminal node in \vec{U} that appears before \mathcal{S} . It is then again easy to see that our two inductive hypotheses imply that there is $(\vec{W}, \mathcal{Q}) \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$ and an embedding $\sigma : \mathcal{S} \rightarrow \mathcal{Q}$ such that $\pi^{\vec{W}} = \sigma \circ \pi^{\vec{U}}$. Because $(\vec{W}, \mathcal{Q}) \in V$, it follows from Lemma 6.1 that $\phi[\mathcal{Q}(\xi^{\vec{U}, \mathcal{S}} + 1), \sigma \vec{U}_{\mathcal{S}}]$ holds. \square

Remark 6.3. *We remark that Theorem 6.2 holds true if we just assume that (\mathcal{P}, Σ) is an η -extendable hod pair such that whenever $h \subseteq \text{Coll}(\omega, < \eta)$, Σ is $D(\lambda, h)$ -super fullness preserving and correctly $D(\lambda, h)$ -guided. The proof of this fact is almost word-by-word the same as the proof of Theorem 6.2.*

Suppose now that (\mathcal{P}, Σ) is a hod pair below η . It is easy to see that the proof of Theorem 6.2 gives an s which is a witness to the η -extendability of (\mathcal{P}, Σ) . We leave the details of the exact conversion of the proof of Theorem 6.2 to s to the readers. Notice that s is independent of the initial choice of the extendability witness. We then let r_{Σ} be the witness given by Theorem 6.2. Suppose now that (\mathcal{P}, Σ) is a short hod pair below η . In this case we let $r_{\Sigma} = \bigoplus_{\alpha < \lambda^{\mathcal{P}}} r_{\Sigma_{\mathcal{P}(\alpha)}}$. Suppose then $(\mathcal{Q}, \Lambda) \in IS(\eta)$ as witnessed by u and (\mathcal{P}, Σ) is a Λ -hod pair below η . We then let $r_{\Sigma, u}$ be the witness given by the relativized version of Theorem 6.2. Putting all these witnesses together we obtain the collection of “canonical” witnesses for members of $IS(\eta)$.

Definition 6.4. *Suppose $(\mathcal{Q}, \Lambda) \in IS(\eta)$. We let $R(\eta, \Lambda)$ be the set of $r \in W(\eta, \Lambda)$ such that one of the following holds:*

1. (\mathcal{Q}, Λ) is a hod pair or a short hod pair and $r = r_{\Lambda}$.
2. (\mathcal{Q}, Λ) is a pullback of a hod pair or a short hod pair (\mathcal{R}, Ψ) as witnessed by $\pi : \mathcal{Q} \rightarrow \mathcal{R} \mid \delta^{\mathcal{R}}$ and r is the witness of η -extendability given by the pair $(\pi, r_{\mathcal{R}, \Psi})$.
3. For some pair $(\mathcal{S}, \Phi) \in IS(\eta)$ as in clause 1 and 2 above, (\mathcal{Q}, Λ) is a Φ -hod pair below η and for some $u \in R(\eta, \Phi)$, $r_{\Lambda, u} = r$.

One nice consequence of Theorem 6.2 is that when dealing with hod pairs or short hod pairs we can omit the witness from our notation. Thus, in what follows, we will drop the superscript r whenever our pair is a hod pair below η which is a limit of strong cardinals. In particular, we will write Σ^g instead of $\Sigma^{r, g}$.

7 The core model induction

In this section, we review some material that is needed for doing core model induction under our hypothesis. Suppose that $|\mathbb{R}|^+ = \nu$, and $(\mathcal{M}, \Sigma) \in SD(\nu)$ is such that \mathcal{M} is countable and Σ has hull condensation. Following [9], we introduce *core model induction operators* associated with Σ . We repeat the definition here for convenience and also because it is somewhat different than the one given in [9]. Recall that if \mathcal{M} is a hybrid premouse and $\alpha \leq o(\mathcal{M})$, then $\mathcal{M}||\alpha$ is \mathcal{M} up to α including the predicates indexed at α while $\mathcal{M}|\alpha$ is \mathcal{M} up to stage α without the predicates indexed at α .

Suppose now that $\theta(v)$ is a Σ_1 -formula. As in [15] (see page 6) we can associate a canonical sequence $(\theta^k : k < \omega)$ such that θ^k is Σ_k and for any Σ -mouse \mathcal{M} over \mathbb{R} and a real x ,

$$\mathcal{J}_1(\mathcal{M}) \models \theta[x] \iff \exists k < \omega \mathcal{M} \models \theta^k[x].$$

We say a sound Σ -premouse \mathcal{M} over \mathbb{R} is *self-iterable* if $\mathcal{M} \models AD^+$ and whenever $\alpha < o(\mathcal{M})$ and $\pi : \mathcal{Q} \rightarrow \mathcal{M}||\alpha$ is such that \mathcal{Q} is countable and π is elementary then \mathcal{Q} has an iteration strategy in \mathcal{M} .

Suppose now that \mathcal{M} is a Σ -premouse over some real z . Following Definition 1.8 of [15], we say that \mathcal{M} is a (θ, z) -witness if \mathcal{M} is a sound $(\omega, \omega_1, \omega_1)$ -iterable z -mouse in which there are $\delta_0 < \dots < \delta_9$, S and T such that \mathcal{M} satisfies the formula expressing

1. ZFC,
2. $\delta_0, \dots, \delta_9$ are Woodin cardinals,
3. S and T are trees on some $\omega \times \alpha$ which are absolutely complementing in $V^{Coll(\omega, \delta_9)}$, and
4. for some $k < \omega$, letting \mathcal{N} be the least self-iterable \mathbb{R} -premouse such that $\mathcal{N} \models \theta^k[z]$, $p[T]$ is the Σ_{k+3} -theory (in the language with names for each real) of \mathcal{N} .

The following lemma is essentially the generalization of Lemma 1.10 of [15].

Lemma 7.1. *Suppose there is a (θ, z) witness. Then $\mathcal{S}^{\nu, \Sigma} \models \theta[z]$.*

Suppose now that $\phi(v, w)$ is a Σ_1 -formula. Following [15], we let

$$\begin{aligned} \phi_n^*(v) = \exists \mathcal{M} \trianglelefteq \mathcal{S}^{\nu, \Sigma} (\mathcal{M} \text{ is self-iterable and if } \mathcal{M} = \mathcal{J}_\alpha^{\vec{E}, S} \text{ then} \\ \mathcal{M} \models \text{“}\alpha + \omega n \text{ exists”} \wedge \forall i < \omega (i > 0 \implies \phi((v)_0, (v)_1))) \end{aligned}$$

We can now define Σ -cmi operators.

Definition 7.2. We say $F : HC \rightarrow HC$ is a Σ core model induction operator or just Σ -cmi operator if one of the following holds:

1. For some $\alpha \in \text{Ord}$, letting $M = \mathcal{S}^{\nu, \Sigma} \parallel \alpha$, $M \models AD^+ + MC(\Sigma)$ ¹³ and one of the following holds:
 - (a) There is a Σ_1 -formula $\phi(u, v)$ such that F is a Σ -mouse operator with the property that for every $X \in \text{dom}(F)$, $F(X)$ is the least Σ -mouse such that whenever $g \subseteq \text{Coll}(\omega, X)$ is generic then for every n there is γ such that $F(X) \parallel \gamma$ is a (ϕ_n^*, z) witness where z is the canonical real coding X in $F(X)[g]$.¹⁴
 - (b) For some $b \in HC$ and some Σ -premouse $\mathcal{Q} \in HC^V$ over b , F is an (ω_1, ω_1) -iteration strategy for \mathcal{Q} which is M -fullness preserving, has branch condensation and is guided by some $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD_{b, \Sigma, x}^M$ for some $x \in b$. Moreover, α ends either a weak or a strong gap in the sense of [20].
 - (c) For some $H : HC \rightarrow HC$ satisfying a or b above and for some $n < \omega$, F is $x \rightarrow \mathcal{M}_n^{\#, H}(x)$ operator or for some $b \in HC$, F is the ω_1 -iteration strategy of $\mathcal{M}_n^{\#, H}(b)$.
2. For some $\alpha \in \text{Ord}$ such that $\rho(\mathcal{S}^{\nu, \Sigma} \parallel \alpha) = \mathbb{R}$ and some countable $\pi : \mathcal{N} \rightarrow \mathcal{S}^{\nu, \Sigma} \parallel \alpha$, such that if Λ is \mathcal{N} 's unique ν -strategy then $(\mathcal{N}, \Lambda) \in SD(\nu)$, one of the above conditions hold for F with $L_\nu^\Lambda(\mathbb{R})$ used instead of $\mathcal{S}^{\nu, \Sigma}$ and Λ used instead of Σ .

Next recall $\text{Proj}(\kappa, \eta, \Sigma)$ from [9]. For the rest of this section we fix η which is an inaccessible limit of strong cardinals. Our definition here is slightly different then the one in [9].

Definition 7.3. Suppose $\kappa < \eta$ is such that κ is an inaccessible cardinal, and suppose $(\mathcal{P}, \Sigma) \in SD(\kappa)$ and $r \in W(\kappa, \Sigma)$. We let $\text{Proj}(\kappa, \eta, (r, \Sigma))$ be the following statement: for every generic $g \subseteq \text{Coll}(\omega, < \kappa)$, in $V[g]$,

1. $(\mathcal{P}, \Sigma) \in SD(\lambda)$ and $r \in W(\kappa, \Sigma)$,
2. for every $\Sigma^{r, g}$ -cmi operator F , $\mathcal{M}_1^{\#, F, \Sigma^{r, g}}$ exists and is η -iterable via an η -extendable strategy, and
3. $\mathcal{S}^{\eta, \Sigma^{r, g}}(\mathbb{R})$ is the stack of all sound self-iterable $\Sigma^{r, g}$ -mice \mathcal{M} such that $\rho(\mathcal{M}) = \mathbb{R}$ and in $V[g]$, for some $\nu < \kappa$ and $X \in V[g \cap \text{Coll}(\omega, \nu)]$, $\mathcal{M} \in OD_X$.

¹³Recall that $MC(\Lambda)$ stands for the Mouse Capturing relative to Λ which says that for $x, y \in \mathbb{R}$, x is $OD(\Lambda, y)$ iff x is in some Λ -mouse over y .

¹⁴Notice that via S -constructions, $F(X)[g]$ can be reorganized as a Σ -premouse over z .

We let $Proj(\kappa, \eta)$ be the statement that for any pair $(\mathcal{P}, \Sigma) \in SD(\kappa)$ and $r \in W(\kappa, \Sigma)$, $Proj(\kappa, \eta, (r, \Sigma))$ holds. We let $Proj(\eta)$ be the statement that for every strong cardinal $\kappa < \eta$, $Proj(\kappa, \eta)$ holds.

Just like in [9] we can now show that:

Theorem 7.4. *Suppose $\kappa < \eta$ is such that κ is an inaccessible cardinal. Suppose $(\mathcal{M}, \Sigma) \in SD(\kappa)$, $r \in W(\kappa, \Sigma)$, and suppose that $Proj(\kappa, \eta, (r, \Sigma))$ holds. Then for any V -generic $g \subseteq Coll(\omega, < \kappa)$, $\mathcal{S}^{\eta, \Sigma^{r,g}} \models AD^+ + \theta_\Sigma = \Theta$.*

We will not prove the theorem here as the proof of the theorem is very much like the proof of the core model induction theorems in [6] (see Theorem 2.4 and Theorem 2.6), [12] (see Chapter 7) and [15]. We encourage the reader to read the discussion following Theorem 3.3 of [9] and in particular, the proof of Lemma 3.4 of [9].

Notice that it follows from Lemma 5.1 that if κ is a strong cardinal then the stack on the right side of clause 3 of Definition 7.3 is an initial segment of $\mathcal{S}^{\eta, \Sigma^{r,g}}(\mathbb{R})$. The converse is also true.

Lemma 7.5. *Suppose $\kappa < \eta$ is a strong cardinal. Suppose $(\mathcal{P}, \Sigma) \in SD(\kappa)$ and $r \in W(\eta, \Sigma)$. Suppose that clause 1 and 2 of $Proj(\kappa, \eta, (r, \Sigma))$ holds, and let $g \subseteq Coll(\omega, < \kappa)$ be V -generic. Then $\mathcal{S}^{\eta, \Sigma^{r,g}}(\mathbb{R})$ is the stack of all sound self-iterable $\Sigma^{r,g}$ -mice \mathcal{M} such that $\rho(\mathcal{M}) = \mathbb{R}$ and in $V[g]$, for some $\nu < \kappa$ and $X \in V[g \cap Coll(\omega, \nu)]$, $\mathcal{M} \in OD_X$.*

Proof. We work in $V[g]$. Because of Lemma 5.1, its enough to show that whenever $\mathcal{M} \trianglelefteq \mathcal{S}^{\eta, \Sigma^{r,g}}$ is such that $\rho(\mathcal{M}) = \mathbb{R}$ then there is some self-iterable sound $\Sigma^{r,g}$ -mouse \mathcal{N} over \mathbb{R} such that $\mathcal{M} \trianglelefteq \mathcal{N}$ and for some $\nu < \kappa$ and $X \in V[g \cap Coll(\omega, \nu)]$, $\mathcal{N} \in OD_X$. We can assume that \mathcal{M} isn't self-iterable as then there is nothing to prove. Fix then $\pi : \mathcal{Q} \rightarrow \mathcal{M}$ such that \mathcal{Q} is countable and π is elementary and letting Φ be the η -strategy of \mathcal{Q} , $\Phi \notin \mathcal{M}$. Using the core model induction and clause 1 of $Proj(\kappa, \eta, (r, \Sigma))$, we can now show that $L^{\Phi, \Sigma^{r,g}}(\mathbb{R}) \models AD^{+15}$. Let $M = L^{\Phi, \Sigma^{r,g}}(\mathbb{R})$. We have that $M \models V = OD_{\Sigma^{r,g}, \mathbb{R}}$ ¹⁶. It then follows from Theorem 2.6 that $\mathcal{P}(\mathbb{R})^M = (Lp^{\Sigma^{r,g}}(\mathbb{R}))^M$. Moreover, in M , $(Lp^{\Sigma^{r,g}}(\mathbb{R}))^M$ is the

¹⁵Presenting the proof of this result is beyond the scope of this paper. The interested readers can find similar results which translate to our context almost verbatim by consulting [12]. See also Theorem 7.4

¹⁶This can be shown as follows. Work in M . Suppose $\mathcal{N} \trianglelefteq Lp^{\Sigma^{r,g}}(\mathbb{R})$ such that $\rho(\mathcal{N}) = \mathbb{R}$. It is enough to show that there is $\mathcal{S} \trianglelefteq Lp^{\Sigma^{r,g}}(\mathbb{R})$ such that $\rho(\mathcal{S}) = \mathbb{R}$, $\mathcal{N} \trianglelefteq \mathcal{S}$ and \mathcal{S} is self-iterable. Suppose there is no such \mathcal{S} . We can then fix $(\mathcal{N}_i : i < \omega)$ such that $\mathcal{N}_0 = \mathcal{N}$, $\mathcal{N}_i \trianglelefteq Lp^{\Sigma^{r,g}}(\mathbb{R})$, $\mathcal{N}_i \triangleleft \mathcal{N}_{i+1}$ and \mathcal{N}_{i+1} is the least initial segment of $Lp^{\Sigma^{r,g}}(\mathbb{R})$ with the property that every countable submodel of \mathcal{N}_i has an iteration strategy in \mathcal{N}_{i+1} . Let then $\mathcal{S} = \cup_{i < \omega} \mathcal{N}_i$. Then \mathcal{S} is self-iterable. This is because if $\pi : \bar{\mathcal{S}} \rightarrow \mathcal{S}$ is countable then, because $\rho(\mathcal{S}) = \mathbb{R}$, the only way to iterate $\bar{\mathcal{S}}$ is to drop. Thus it is enough to show that for every $\alpha < o(\bar{\mathcal{S}})$, $\bar{\mathcal{S}}|\alpha$ has an $\omega_1 + 1$ -strategy in \mathcal{S} . But then we can fix some $i < \omega$ and $\beta < o(\mathcal{N}_i)$ such that $\pi : \bar{\mathcal{S}}|\alpha \rightarrow \mathcal{N}_i|\beta$. It follows from the construction that $\bar{\mathcal{S}}|\alpha$ has an $\omega_1 + 1$ -strategy in \mathcal{N}_{i+1} .

stack of all self-iterable $\Sigma^{r,g}$ -mice. Fix now some $\nu < \kappa$ and $X \in V[g \cap \text{Coll}(\omega, \nu)]$ such that M is OD_X (X is essentially a set coding r , $\Phi \upharpoonright V[g \cap \text{Coll}(\omega, \eta)]$ and $\Sigma^{r,g \cap \text{Coll}(\omega, \nu)}$). Also fix $\mathcal{N} \trianglelefteq (Lp^{\Sigma^{r,g}})^M$ such that $\Phi \upharpoonright HC \in \mathcal{N}$ and that \mathcal{N} is self-iterable. It now follows from Lemma 5.1 that $\mathcal{N} \trianglelefteq S^{\eta, \Sigma^{r,g}}$. Because $\Phi \upharpoonright HC \in \mathcal{N}$, it must be the case that $\mathcal{M} \trianglelefteq \mathcal{N}$. This completes the proof of the lemma. \square

Notice now that if the first clause of $\text{Proj}(\kappa, \eta, (r, \Sigma))$ holds and $g \subseteq \text{Coll}(\omega, < \kappa)$ is V -generic then any $\Sigma^{r,g}$ -cmi operator F is η -uB. It follows that F is η -uB¹⁷ and that F can be extended to act on $V_\eta[g]$. More precisely, given a $\mathbb{P} \in V_\eta[g]$, a $V[g]$ -generic $h \subseteq \mathbb{P}$, $V[g * h]$ -generic $k \subseteq \text{Coll}(\omega, < \eta)$, and $< \eta$ -complementing $T, S \in V[g]$ such that $p[T] = \text{Code}(F)$, we have that $(p[T])^{V[g * h * k]} \cap \mathbb{R}^{V[g * h]} \in V[g * h]$. We then let, in $V[g * h * k]$, F^h be given by

$$\begin{aligned} \text{dom}(F^h) &= \{a : \exists(x, n, m) \in \mathbb{R}((x, n, m) \in p[T] \wedge \pi_x(n) = a)\} \text{ and} \\ F^h(a) &= b \text{ if and only if whenever } x \in \mathbb{R} \text{ is such that } a \in N_x \text{ and } n \in \omega \text{ is such that} \\ &\quad \pi_x(n) = a, F^h(a) = \{\pi_x(m) : (x, n, m) \in p[T]\}. \end{aligned}$$

It follows that $F^h \in V[g * h]$. It also follows that $F^{h * k}$ makes sense and $F^{h * k} \upharpoonright V_\eta[g * h] \in V$. We then let $F^{h, \eta} =_{\text{def}} F^{h * k} \upharpoonright V_\eta[g * h] \in V[g * h]$.

Next, we prove a lemma which can be used to show that $\text{Proj}(\kappa, \eta, (r, \Sigma))$ holds. Suppose $\kappa < \eta$ is a strong cardinal, $(\mathcal{P}, \Sigma) \in SD(\kappa)$ and $r \in W(\kappa, \Sigma)$. Fix $j : V \rightarrow M$ witnessing that κ is η^+ -strong and let $h \subseteq \text{Coll}(\omega, < j(\kappa))$ be generic. Recall that for $\nu < j(\kappa)$, we let $h_\nu = h \cap \text{Coll}(\omega, < \nu)$. We can then extend j to $j^+ : V[h_\kappa] \rightarrow M[h]$. Given a Σ^{r, h_κ} -cmi operator F and a cardinal $\nu \in [\kappa, \eta)$, we let $F^{+\nu} = j^+(F) \upharpoonright V_\eta[h_\nu]$. Clearly we have that $F^{+\nu} \in V[h_\nu]$.

Lemma 7.6. *Suppose, in $V[h_\kappa]$, for every Σ^{r, h_κ} -cmi operator F which has a symmetric name $\dot{F} \in V^{\text{Coll}(\omega, < \kappa)}$ and for every $\nu \in [\kappa, \eta)$, $K^{c, F^{+\nu}}$ construction of $V_\eta[h_\nu]$ fails. Then in $V[h_\kappa]$, whenever F is a Σ^{r, h_κ} -cmi operator, $\mathcal{N}_\omega^{\#, F}$ exists and has an η -iteration strategy which is η -extendable.*

Proof. Suppose first that, in $V[h_\kappa]$, F is a Σ^{r, h_κ} -cmi operator that has a symmetric name. Let $\alpha \in [\kappa, \eta)$. Because $K^{c, F^{+\alpha}}$ construction of $V_\eta[h_\alpha]$ fails, by the results of [1], we get that $V_\eta[h_\alpha] \models \text{“}\mathcal{N}_\omega^{\#, F^{+\alpha}} \text{ exists and is } \omega_1 + 1\text{-iterable”}$. Let then $\mathcal{M}_\alpha = (\mathcal{N}_\omega^{\#, F^{+\alpha}})^{V[h_\alpha]}$. By homogeneity of the collapse, for every $\alpha \in [\kappa, \eta)$, $\mathcal{M}_\alpha \in V[h_\kappa]$ and $V[h_\kappa] \models \text{“}\mathcal{M}_\alpha \text{ is } \omega_1 + 1\text{-iterable”}$. It then follows that $\mathcal{M}_\alpha = \mathcal{M}_\beta$ for all $\alpha, \beta \in [\kappa, \eta)$ implying that if \mathcal{M} is the common value of \mathcal{M}_α then $V[h_\kappa] \models \text{“}\mathcal{M} \text{ has an } \eta\text{-uB iteration strategy”}$ (see Lemma 7.9). The claim then follows.

¹⁷This follows from clause 1 of Definition 7.3 and the proof of Lemma 7.9 applied to the $\mathcal{M}_1^\#$ operator.

Next suppose that, in $V[h_\kappa]$, F is an arbitrary Σ^{r, h_κ} -cmi operator. It is shown in [12] and [15] that any Σ^{r, h_κ} -cmi operator F has a symmetric name $\dot{F} \in V^{Coll(\omega, < \kappa)}$ provided Σ^{r, h_κ} has a symmetric name (in particular, see Section 1.4 of [15]). If F is defined according to clause 1 of Definition 7.2 then because Σ^{r, h_κ} has a symmetric name, the above argument finishes the claim, as F then too has a symmetric name.

Suppose then F is defined by clause 2 of Definition 7.2. Let $\xi = (\kappa^+)^V$. Working in $V[h_\kappa]$, fix α such that $\rho(\mathcal{S}^{\xi, \Sigma^{r, h_\kappa}} || \alpha) = \mathbb{R}$ and some countable $\pi : \mathcal{M} \rightarrow \mathcal{S}^{\xi, \Sigma^{r, h_\kappa}} || \alpha$ such that if Λ is the unique strategy of \mathcal{M} , then clauses 1a-1c of Definition 7.2 hold for F with $L_\xi^\Lambda(\mathbb{R})$ used instead of $\mathcal{S}^{\xi, \Sigma^{r, h_\kappa}}$ and Λ used instead of Σ . If Λ had a symmetric name then we would be done as F then can be shown to have a symmetric name. Suppose then Λ doesn't have a symmetric name.

We continue working in $V[h_\kappa]$. Because κ is inaccessible, we can fix $\nu < \kappa$ and $\sigma : H \rightarrow H_{\kappa+\omega}^V$ such that $|H| = \nu$, $\text{cp}(\sigma) = \nu$ and if $\sigma^+ : H[h_\nu] \rightarrow H_{\kappa+\omega}^V[h_\kappa]$ is the lift of σ then $\pi \in \text{rng}(\sigma)$. Let then $\mathcal{N} = \sigma^{-1}(\mathcal{S}^{\xi, \Sigma} || \alpha)$ and let $\Phi \in V[h_\kappa]$ be the canonical strategy of \mathcal{N} . It follows that Φ has a symmetric name in $V^{Coll(\omega, < \kappa)}$. Therefore, it follows from the argument above that in $V[h_\kappa]$, $\mathcal{N}_\omega^{\#, \Phi}$ exists and is η -iterable via an η -extendable strategy. But we also have that $\Lambda = (\tau\text{-pullback of } \Phi)$ where $\tau = (\sigma^+)^{-1}(\pi)$. It then follows that in $V[h_\kappa]$, $\mathcal{N}_\omega^{\#, \Lambda}$ also exists and has an η -iteration strategy which is η -extendable. Because $F \in L_\xi^\Lambda(\mathbb{R})$, it follows that $\mathcal{N}_\omega^{\#, F}$ exists and has an η -iteration strategy which is η -extendable. \square

The following is an easy consequence of Lemma 7.5 and Lemma 7.6. We will use it in Section 9 to prove the *limit derived model hypothesis*.

Corollary 7.7. *Suppose no symmetric hybrid K^c construction below η converges. Then $\text{Proj}(\eta)$ holds.*

The proof of Lemma 7.6 also gives the proof of the following corollary.

Corollary 7.8. *Suppose no symmetric hybrid K^c construction below η converges. Then η is closed under hybrid \mathcal{N}_ω -operators.*

Proof. The proof of clause 1 and clause 3 of Definition 4.14 is an immediate consequence of the proof of Lemma 7.6. To show the clause 2 we use Corollary 7.7 and Theorem 7.4. Fix $(\mathcal{Q}, \Lambda) \in IS(\eta)$, r witnessing that $(\mathcal{Q}, \Lambda) \in IS(\eta)$, a strong cardinal $\mu \in [\mu_{\mathcal{Q}, \Lambda}, \eta)$, a V -generic $g \subseteq \text{Coll}(\omega, < \mu)$, an $a \in HC^{V[g]}$ and $\mathcal{M} \sqsubseteq (Lp^{\Lambda^{r, g}}(a))^{\mathcal{S}^{\eta, \Lambda^{r, g}}}$ such that $\rho(\mathcal{M}) = a$. Let $\Phi \in \mathcal{S}^{\eta, \Lambda^{r, g}}$ be the ω_1 -strategy of \mathcal{M} . It follows from Corollary 7.7 and Theorem 7.4 that in $V[g]$, $M^* =_{\text{def}} \mathcal{S}^{\eta, \Lambda^{r, g}} \models AD^+$. Let then $\alpha < \Theta^{\mathcal{M}^*}$ be such that letting $M = M^* | \alpha$, $\Phi \in \mathcal{M}$ and α ends a weak gap. We can then find a Λ^{r, h_μ} -cmi operator F as in clause 1.b of Definition 7.2. Let x be a real witnessing that $\text{Code}(\Phi) <_W \text{Code}(F)$. Notice that we

have that in $V[h_\mu]$, F is λ -uB (see the discussion after Lemma 7.5). It then follows from Lemma 7.6 that in $V[h_\mu]$, $\mathcal{N}_\omega^{\#,F}(x)$ exists and is η -iterable via an η -extendable strategy. It then follows that in V , $\mathcal{N}_\omega^{\#,F}$ exists and is η -iterable via an η -extendable strategy. \square

Next we outline the proof of the claim we used in the last line of the first paragraph of the proof of Lemma 7.6. We state it for strategies but the proof also works for mouse operators.

Lemma 7.9. *Suppose for some limit cardinal μ , (\mathcal{M}, Σ) is an μ -extendable pair as witnessed by r , $\nu \in [|\mathcal{M}|, \mu)$ is a cardinal and $k \subseteq \text{Coll}(\omega, \nu)$ is V -generic. Suppose that $V[k] \models$ “ $\mathcal{N}_\omega^{\#, \Sigma^{r,k}}$ exists and is $\omega_1 + 1$ -iterable in all $< \mu$ -generic extensions”. Then $V \models$ “ $\mathcal{N}_\omega^{\#, \Sigma}$ exists and is μ -iterable via an μ -extendable strategy”.*

Proof. The proof is standard but the author is unaware of a published account of it. Here we outline the argument. Let $\mathcal{Q} = \mathcal{N}_\omega^{\#, \Sigma^{r,k}}$ and let $\Lambda \in V[k]$ be the following μ -strategy for \mathcal{Q} : Given an iteration tree \mathcal{T} according to Λ of limit length, $\Lambda(\mathcal{T}) = b$ iff $\mathcal{Q}(b, \mathcal{T})$ exists and whenever \mathcal{N} is an active $\Sigma^{r,k}$ -mouse over $\mathcal{M}(\mathcal{T})$ that is $\omega_1 + 1$ -iterable in all $< \mu$ -generic extensions and has ω Woodin cardinals then $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{N}$. Clearly, Λ is an μ -strategy for \mathcal{M} . We claim that it is an μ -extendible strategy.

In order to prove this, we will use the following notation. Given $\alpha < \mu$ and a transitive set $M \in H_\eta[k]$, let \mathcal{U}_M be the tree on \mathcal{Q} which makes M generically generic, i.e., if \mathcal{N} is the last model of \mathcal{U} then whenever $g \subseteq \text{Coll}(\omega, |M|)$, M is generic over $\mathbb{B}_\delta^\mathcal{N}$ where δ is the least Woodin cardinal of \mathcal{N} . We let \mathcal{N}_M be the last model of \mathcal{U}_M . We let Σ_M be the strategy coded by the strategy predicate of \mathcal{N}_M .

Fix now a $< \mu$ -generic g over $V[k]$ and $\mathcal{T} \in V[k * g]$ such that $\mathcal{T} \in H_\eta[k * g]$ is a tree on \mathcal{Q} . We say \mathcal{T} is correctly guided if for every limit $\alpha < lh(\mathcal{T})$, if b is the branch of $\mathcal{T} \upharpoonright \alpha$ then $\mathcal{Q}(b, \mathcal{T} \upharpoonright \alpha)$ -exists and if $\pi : M \rightarrow H_\mu$ is elementary with M transitive and such that $\mathcal{T} \in M[k * g]$ then letting $\mathcal{Q} = (\mathcal{J}^{\vec{E}, \Sigma_M^{M,g}}[\mathcal{M}(\mathcal{T} \upharpoonright \alpha)])^{\mathcal{N}_M[M][k * g]}$, i.e., \mathcal{Q} is the output of fully backgrounded construction of $\mathcal{N}_M[\mathcal{M}(\mathcal{T} \upharpoonright \alpha)][k * g]$ done over $\mathcal{M}(\mathcal{T} \upharpoonright \alpha)$ with respect to the interpretation of $\Sigma_M^{\mathcal{N}_M}$ on $\mathcal{N}_M[\mathcal{M}(\mathcal{T} \upharpoonright \alpha)][k * g]$, then $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{Q}$.

It is then easy to see that $\mathcal{T} \in \text{dom}(\Lambda)$ iff \mathcal{T} is correctly guided. We leave it to the reader to find (ϕ, ψ, Z) witnessing that $\Lambda \upharpoonright V_\mu$ is μ -extendable. \square

We finish this section by proving some useful lemmas. Recall that we are working with a fixed η which is an inaccessible limit of strong cardinals. Let $A \subseteq \eta$ be the set of strong cardinals that are less than η . Suppose $\kappa < \eta$ is a A -reflecting strong cardinal and $j : V \rightarrow M$ is an embedding witnessing that κ is η^+ -strong and A -reflecting. Let $h \subseteq \text{Coll}(\omega, < j(\kappa))$ be generic. We then have that j lifts to $j^+ : V[h_\kappa] \rightarrow M[h]$.

Lemma 7.10. *Suppose $\text{Proj}(\eta)$ holds and η is closed under hybrid \mathcal{N}_ω -operators. Suppose in $V[h_\kappa]$, N is a transitive model of AD^+ such that $\mathbb{R}, \text{Ord} \subseteq N$ and $(\mathcal{P}, \Sigma) \in N$ is a hod pair such that $N \models$ “ Σ has branch condensation, is super fullness preserving and is correctly guided”. Suppose further that for some $X \in V$ and some real $x \in V[h_\kappa]$, in $V[h_\kappa]$, N is ordinal definable from X , and Σ is ordinal definable from (X, x) . Then there is $(\mathcal{Q}, \Lambda) \in V \cap SD(\eta)$ and $s \in W(\eta, \Lambda)$ such that (\mathcal{Q}, Λ) is a hod pair, $\mathcal{Q} \in pI(\mathcal{P}, j^+(\Sigma))$, for every strong cardinal $\mu \in [\kappa, \eta)$, $\Lambda^{s, h_\mu} = j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_\eta[h_\mu]$ and s witnesses the first four clauses of Definition 4.4.*

Proof. Let $\mathcal{Q} = (\mathcal{M}_\infty(\mathcal{P}, \Sigma))^N$. Because we are assuming that N is OD_X in $V[h_\kappa]$, we have that $\mathcal{Q} \in V$ as $\mathcal{Q} \subseteq \text{HOD}^N$ (see Theorem 2.2). Next notice that because of comparison of hod pairs (see Theorem 2.2.2 of [7]), $j^+(\Sigma)_{\mathcal{Q}}$ is independent of Σ , i.e., whenever $(\mathcal{R}, \Psi) \in N$ is such that in N , $\Gamma(\mathcal{R}, \Psi) = \Gamma(\mathcal{P}, \Sigma)$, then $\mathcal{Q} = \mathcal{M}_\infty(\mathcal{R}, \Psi)$ and $j^+(\Psi)_{\mathcal{Q}} = j^+(\Sigma)_{\mathcal{Q}}$. The following claim is an easy consequence of this observation.

Claim 1. In $M[h]$, $j^+(\Sigma)_{\mathcal{Q}} \upharpoonright HC^{V[h_\eta]}$ is ordinal definable from $j(X)$.

Proof. Indeed, first let $\Gamma = \Gamma(\mathcal{P}, \Sigma)$. In $M[h]$, consider the pairs $(\mathcal{R}, \Psi) \in j^+(N)$ such that for some $z \in \mathbb{R}^{V[h_\kappa]}$, (\mathcal{R}, Ψ) is ordinal definable from $(j(X), z)$ and $j^+(N) \models \Gamma(\mathcal{R}, \Psi) = j^+(\Gamma)$. Then $\Psi_{\mathcal{Q}} \upharpoonright HC^{V[h_\eta]} = j^+(\Sigma)_{\mathcal{Q}} \upharpoonright HC^{V[h_\eta]}$. The claim then follows. \square

It follows from the claim that for all $\nu \in [\kappa, \eta]$, $j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_\eta[h_\nu] \in V[h_\nu]$ and $\Lambda = j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_\eta \in V$. It is now easy to extract an η -extendability witness for Λ using (j, X) . Letting s be this witness, s has the property that for every $\nu \in [\kappa, \eta)$,

$$(1) \Lambda^{s, h_\nu} = j^+(\Sigma)_{\mathcal{Q}} \upharpoonright V_\eta[h_\nu].$$

Claim 2. (\mathcal{Q}, Λ) and s are as desired.

Proof. We need to show that s witnesses that

1. (\mathcal{Q}, Λ) is self-determining below η ,
2. $\mathcal{N}_\omega^{\#, \Lambda, s}$ exists and is η -iterable via an η -extendable strategy,
3. Λ has branch condensation below η ,
4. $D(\mathcal{N}_\omega^\Lambda) \models$ “ Λ is a super fullness preserving”.
5. for every $\mathcal{R} \in pB(\mathcal{Q}, \Lambda)$, $\Lambda_{\mathcal{Q}}$ is η -stable,

To see clause (1), notice that we have that in $j^+(N)$, $\Lambda^{s,h}$ is self-determining below ω_1 and in fact $\mathcal{M}_1^{\#, \Lambda^{s,h}}$ is the witness (this follows from the results of Section 2.9-2.10 of [7]). It follows from the homogeneity of the collapse that $\mathcal{M}_1^{\#, \Lambda}$ exists and is η -iterable via an η -extendable strategy.

Clause 2 follows from the fact that η is closed under hybrid \mathcal{N}_ω -operators, and clause 3 follows from (1) above and the fact that Σ has branch condensation. Clause 5 is a consequence of Theorem 7.4. It remains to show that Clause 4 holds. It follows from (1) and clause 2 above that in $V[h_\kappa]$, $\mathcal{N}_\omega^{\#, \Sigma}$ exists and is η -iterable via an η -extendable strategy. Because in $V[h_\kappa]$, $L(\Sigma, \mathbb{R}) \models$ “ Σ is super fullness reserving” it follows from (1) that $D(\mathcal{N}_\omega^\Lambda) \models$ “ Λ is a super fullness preserving”. This finishes the proof of the claim. \square

\square

The following corollary of Lemma 7.10 will be instrumental in later sections, especially in Section 10 and Section 11.

Lemma 7.11. *Suppose $\text{Proj}(\eta)$ holds and η is closed under hybrid \mathcal{N}_ω -operators. Suppose $(\mathcal{R}, \Psi) \in IS(\eta)$ is a relativized hod pair below η . Then there is a hod pair (\mathcal{Q}, Λ) below η such that in $V[h]$, $\Psi^h \in L(\Lambda^h, \mathbb{R})$.*

Proof. Let $\mu = \mu_{\mathcal{R}, \Psi}$ and let $N = L(\Psi^{h_\mu}, \mathbb{R}^{V[h_\mu]})$. Then $N \models AD^+$. We then let $(\mathcal{P}, \Sigma) \in N$ be a hod pair such that in N , $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{R}, \Psi^{h_\mu})$. Let then (\mathcal{Q}, Λ) and s be as in Lemma 7.10. We have that s witnesses that $(\mathcal{Q}, \Lambda) \in SD(\eta)$ and satisfies the first four clauses of Definition 4.4.

It follows from Lemma 4.11 that in $V[h_\mu]$, $\mathcal{N}_\omega^{\#, \Psi, \Sigma}$ exists and has an η -extendable strategy. Let $\mathcal{N} = \mathcal{N}_\omega^{\#, \Psi, \Sigma}$. Notice that because in N , $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{R}, \Psi^{h_\mu})$, we have that in N , $\Psi^{h_\mu} \in L(\Sigma, \mathbb{R})$. It follows that $D(\mathcal{N}) \models$ “ $\Psi^{h_\mu} \in L(\Sigma, \mathbb{R})$ ”. It follows from the construction of Λ that if $\mathcal{M} = \mathcal{N}_\omega^{\#, \Psi, \Lambda, s}$ then $D(\mathcal{M}) \models$ “ $\Psi^{h_\mu} \in L(\Sigma, \mathbb{R})$ ”. Since \mathcal{M} has an η -extendable strategy which it inherits from \mathcal{N} ¹⁸, we have that in $V[h]$, $\Psi^h \in L(\Lambda^h, \mathbb{R})$.

It remains to show that s witnesses that Λ is η -super fullness preserving. Fix then $\mathcal{S} \in pI(\mathcal{Q}, \Lambda)$ and let $\xi < \lambda^{\mathcal{S}}$. Let ν be a strong cardinal $\mathcal{S} \in H_\nu$. We need to show that $\Lambda_{\mathcal{S}(\xi+1)}^{s, h_\nu}$ is $\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}}$ -super fullness preserving and correctly $\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}}$ -guided. Because $D(\mathcal{M}) \models$ “ Λ is super fullness preserving and correctly guided”¹⁹, it is enough to show that in $V[h_\nu]$, $\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{h_\nu}} \subseteq L(\Psi^{h_\nu}, \Lambda^{s, h_\nu}, \mathbb{R})$.

Notice that because we have that $D(\mathcal{M}) \models$ “both Ψ and Λ are super fullness preserving and correctly guided”, we can fix some $\mathcal{W} \in pI(\mathcal{R}, \Psi^{h_\nu})$ such that for some $\beta < \lambda^{\mathcal{W}}$, in

¹⁸Notice that \mathcal{M} can be obtained from an iterate of \mathcal{N} using the procedure we used in Lemma 4.11.

¹⁹This follows from the fact that $D(\mathcal{N}) \models$ “ Σ is super fullness preserving and correctly guided”.

$L(\Psi^{h_\nu}, \Lambda^{s, h_\nu}, \mathbb{R}^{V[h_\nu]})$, $\Gamma(\mathcal{W}(\beta), \Psi_{\mathcal{W}(\beta)}^{h_\nu}) = \Gamma(\mathcal{S}(\xi), \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu})$. It then follows that $\Psi_{\mathcal{W}(\beta)}^{h_\nu} \in \mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu}}$. It then follows that in $\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu}}$, every set is ordinal definable from a real and $\Psi_{\mathcal{W}(\beta)}^{h_\nu}$. Using Theorem 2.6, we get that

$$(1) \mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu}} = L((Lp^{\Psi_{\mathcal{W}(\beta)}^{h_\nu}}(\mathbb{R}))^{\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu}}}).$$

It follows from Lemma 4.5 that in $V[h_\nu]$,

$$(2) \mathcal{S}^{\eta, \Psi_{\mathcal{W}(\beta)}^{h_\nu}} \subseteq L(\Psi^{h_\nu}, \Lambda^{s, h_\nu}, \mathbb{R}).$$

It also follows from Lemma 5.1, that in $V[h_\nu]$,

$$(3) (Lp^{\Psi_{\mathcal{W}(\beta)}^{h_\nu}}(\mathbb{R}))^{L(\Psi^{h_\nu}, \Lambda^{s, h_\nu}, \mathbb{R})} \trianglelefteq \mathcal{S}^{\eta, \Psi_{\mathcal{W}(\beta)}^{h_\nu}}.$$

It follows from (1), (2) and (3) that indeed in $V[h_\nu]$, $\mathcal{S}^{\eta, \Lambda_{\mathcal{S}(\xi)}^{s, h_\nu}} \subseteq L(\Psi^{h_\nu}, \Lambda^{s, h_\nu}, \mathbb{R})$. \square

8 The limit derived model hypothesis

Here we introduce *the limit derived model hypothesis at η* . In Section 11, we will use it to show that if this hypothesis holds at a λ satisfying the hypothesis of the UB-Covering Conjecture and if covering with lower parts fails at η then there is a transitive proper class inner model of $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. In Section 9, we will also establish that if η is as in UB-Covering Conjecture, no symmetric hybrid K^c construction below η converges and there is no transitive inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ then the limit derived model hypothesis holds at η . The combination of these two results implies the Main Theorem.

In intuitive terms, the derived model hypothesis states that the collection of η -universally Baire sets below η is very rich. If we assume that η is a limit of Woodin cardinals then Woodin cardinals themselves will give an ample collection of η -universally Baire sets. Without Woodin cardinals, however, we have to specifically say where these universally Baire sets come from.

Definition 8.1 (The limit derived model hypothesis). *Suppose η is an inaccessible limit of strong cardinals. We say the limit derived model hypothesis holds at η and write $ldmh(\eta)$ if the following conditions are satisfied: For every $(\mathcal{Q}, \Lambda) \in IS(\eta)$ the following holds:*

1. (\mathcal{Q}, Λ) is η -stable.

2. If $(\mathcal{R}, \Psi) \in IS(\eta)$ is a Λ -hod pair below η then there is $(\mathcal{W}, \Pi) \in IS(\eta)$ such that (\mathcal{W}, Π) is a Λ -hod pair below η and for some $\alpha < \lambda^{\mathcal{W}}$, $\mathcal{W}(\alpha) \in I(\mathcal{R}, \Psi)$, $\alpha + \omega = \lambda^{\mathcal{W}}$ and $\Psi_{\mathcal{W}(\alpha)} = \Pi_{\mathcal{W}(\alpha)}$.
3. η is closed under hybrid \mathcal{N}_ω -operators.
4. $Proj(\eta)$ holds.

We call it the limit derived model hypothesis because of clause 2. Intuitively clause 2 above implies that $D(\eta, h)$ is a model of $AD_{\mathbb{R}}$. Throughout this section we fix an inaccessible cardinal η which is a limit of strong cardinals and assume that $ldmh(\eta)$ holds. We now introduce the the direct limit construction below an inaccessible cardinal and show that it behaves well (see Lemma 8.2).

Suppose that $(\mathcal{R}, \Psi) \in IS(\eta)$. Let $\mu < \eta$ be an inaccessible cardinal such that $\mathcal{R} \in H_\mu$. We then let,

$$\mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta} = \{(\mathcal{Q}, \Lambda) \in IS(\eta) : \mathcal{Q} \in H_\mu \text{ and } (\mathcal{Q}, \Lambda) \text{ is a } \Psi\text{-hod pair below } \eta\}, \text{ and}$$

$$\mathcal{F}_{\mathcal{R}, \Psi}^\eta = \bigcup_{\nu < \lambda} \mathcal{F}_{\mathcal{R}, \Psi}^{\nu, \eta}.$$

We also let $\mathcal{F}^{\mu, \eta} = \mathcal{F}_{\emptyset, \emptyset, \emptyset}^{\mu, \eta}$ and $\mathcal{F}^\eta = \mathcal{F}_{\emptyset, \emptyset, \emptyset}^\eta$.

As usual, we define $\preceq_{\mathcal{R}, \Psi}^{\mu, \eta}$ on $\mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}$ by setting $(\mathcal{Q}, \Lambda) \preceq_{\mathcal{R}, \Psi}^{\mu, \eta} (\mathcal{S}, \Phi)$ if for some $\alpha \leq \lambda^{\mathcal{S}}$, $\mathcal{S}(\alpha) \in I(\mathcal{Q}, \Lambda)$ and $\Phi_{\mathcal{S}(\alpha)} = \Lambda_{\mathcal{S}(\alpha)}$. When $\mu = \eta$ then we write $\preceq_{\mathcal{R}, \Psi}^\eta$. When $(\mathcal{R}, \Psi) = (\emptyset, \emptyset)$ then we write $\preceq_{\mathcal{R}, \Psi}^\eta$.

Lemma 8.2. $\preceq_{\mathcal{R}, \Psi}^{\mu, \eta}$ is directed.

Proof. Fix $(\mathcal{Q}, \Lambda), (\mathcal{S}, \Phi) \in \mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}$. Let $r \in R(\eta, \Psi)$, $u \in R(\eta, \Lambda)$, and $v \in R(\eta, \Phi)$. Because we are assuming η is closed under hybrid \mathcal{N}_ω -operators, we have that in $V[h_\mu]$, $\mathcal{N}_\omega^{\#, \Lambda^{u, h_\mu}, \Phi^{v, h_\mu}}$ exists and has an η -extendable strategy. It then also follows from genericity iterations and the derived model theorem that in $V[h_\mu]$, if $M = L(\Lambda^{u, h_\mu}, \Phi^{v, h_\mu}, \mathbb{R})$ then $M \models AD^+$. Next it follows from Corollary 5.5 that $M \models$ “ Λ^{u, h_μ} and Φ^{v, h_μ} are fullness preserving” (notice that M is either $L(\Lambda^{u, h_\mu}, \mathbb{R})$ or $L(\Phi^{v, h_\mu}, \mathbb{R})$ depending on whether $Code(\Phi^{v, h_\mu}) \leq_W Code(\Lambda^{u, h_\mu})$ or $Code(\Lambda^{u, h_\mu}) \leq_W Code(\Phi^{v, h_\mu})$). Using the comparison theorem of [7] (see Theorem 2.2.2 of [7]), we get that in M , there is (\mathcal{W}, Π) such that either

1. $\mathcal{W} \in pI(\mathcal{Q}, \Lambda^{u, h_\mu})$, $\mathcal{W} \in pI(\mathcal{S}, \Phi^{v, h_\mu}) \cup pB(\mathcal{S}, \Phi^{v, h_\mu})$ and $\Pi = \Lambda_{\mathcal{W}}^{u, h_\mu} = \Phi_{\mathcal{W}}^{v, h_\mu}$ or
2. $\mathcal{W} \in pI(\mathcal{S}, \Phi^{v, h_\mu})$, $\mathcal{W} \in pI(\mathcal{Q}, \Lambda^{u, h_\mu}) \cup pB(\mathcal{Q}, \Lambda^{u, h_\mu})$ and $\Pi = \Phi_{\mathcal{W}}^{v, h_\mu} = \Lambda_{\mathcal{W}}^{u, h_\mu}$.

Let then $\mathcal{M} = (\mathcal{M}_\infty(\mathcal{W}, \Pi))^M$. It follows from the homogeneity of the collapse and from Theorem 2.2, that $\mathcal{M} \in V$. Assume without loss of generality that clause 1 above holds. It then follows from the proof of Lemma 7.11 that $(\mathcal{M}, \Phi_{\mathcal{M}}) \in IS(\eta)$. □

If now $(\mathcal{Q}, \Lambda) \preceq_{\mathcal{R}, \Psi}^{\mu, \eta} (\mathcal{S}, \Phi)$ then let $\pi_{\mathcal{Q}, \mathcal{S}}^{\Lambda} : \mathcal{Q} \rightarrow \mathcal{S}(\alpha)$ be the iteration embedding where α is such that $\mathcal{S}(\alpha) \in I(\mathcal{Q}, \Lambda)$. Given $(\mathcal{S}, \Phi) \in \mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}$, we let $\mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi)$ be the direct limit of all Φ^{h_μ} -iterates of \mathcal{S} which are in $HC^V[h_\mu]$. Also let, in $V[h_\mu]$, $\pi_{\mathcal{S}, \infty, \mu, \eta, h_\mu}^{\Phi} : \mathcal{S} \rightarrow \mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi)$ be the iteration embedding. It then follows that

Lemma 8.3. $\mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi) \in V$.

Notice now that it follows from the comparison that given two (\mathcal{Q}, Λ) and (\mathcal{S}, Φ) in $\mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}$, we have that either $\mathcal{M}_{\mu, \infty}(\mathcal{Q}, \Lambda) \trianglelefteq \mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi)$ or $\mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi) \trianglelefteq \mathcal{M}_{\mu, \infty}(\mathcal{Q}, \Lambda)$. It means that if

$$\mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^* = \cup_{(\mathcal{S}, \Phi) \in \mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}} \mathcal{M}_{\mu, \infty}(\mathcal{S}, \Phi),$$

and for $\alpha < \lambda^{\mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^*}$, $\Sigma_{\mu, \eta, \alpha} = \Lambda_{\mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^*}(\alpha)$ where Λ is such that for some \mathcal{Q} , $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\mathcal{R}, \Psi}^{\mu, \eta}$ and

$$\mathcal{M}_{\mu, \infty}(\mathcal{Q}, \Lambda) = \mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^*(\alpha),$$

then letting $\Sigma_{\mu, \eta} = \oplus_{\alpha < \xi} \Sigma_{\mu, \eta, \alpha}$, $(\mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^*, \Sigma)$ is a short hod pair below η (see clause 1 of Lemma 8.4). Hence, $(\mathcal{M}_{\mu, \eta, \mathcal{R}, \Psi, \infty}^*, \Sigma) \in IS(\eta)$. When $\mu = \eta$, we drop it from our notation. Also, when $(\mathcal{R}, \Psi) = (\emptyset, \emptyset)$ then we let $\mathcal{M}_{\mu, \eta, \infty}^* = \mathcal{M}_{\mu, \eta, \emptyset, \emptyset, \infty}^*$. Next we list some consequences of the limit derived model hypothesis.

Lemma 8.4 (Consequences of the limit derived model hypothesis). *Let $h \subseteq Coll(\omega, < \eta)$ be V -generic. Then the following statements are true.*

1. For every $(\mathcal{R}, \Psi) \in IS(\eta)$, $\lambda^{\mathcal{M}_{\eta, \mathcal{R}, \Psi, \infty}^*}$ is a limit ordinal.
2. In $V[h]$, for any $A, B \in D(\eta, h)$, $L(A, B, \mathbb{R}) \models AD^+$.
3. Lower part persistence holds below η .

Proof. Clause 1 is an immediate consequence of clause 2 of $ldmh(\eta)$. Clause 2 follows from clause 3 of $ldmh(\eta)$ and Corollary 4.12. Clause 3 follows easily from Corollary 5.3 since we have that η is closed under hybrid \mathcal{N}_ω -operators. \square

9 A proof of the limit derived model hypothesis

In this section, our goal is to prove that the failure of clause 1 and clause 3 of our Main Theorem implies the limit derived model hypothesis. More precisely, we prove the following theorem.

Theorem 9.1. *Suppose η is an inaccessible limit of strong cardinals which are A -reflecting where $A = \{\nu < \lambda : \nu \text{ is a strong cardinal}\}$. Suppose no symmetric hybrid K^c -construction below η converges and suppose $\neg\#_{\Theta\text{-reg}}(\eta)$ holds. Then $ldmh(\eta)$ holds.*

We spend the rest of this section proving Theorem 9.1. For the duration of this section let

(*) no symmetric hybrid K^c -construction below η converges and $\neg\#_{\Theta\text{-reg}}(\eta)$ holds.

We assume that (*) holds. It follows from Corollary 7.8 that η is closed under hybrid \mathcal{N}_ω -operators. It also follows from Corollary 7.7 that $Proj(\eta)$ holds. The following lemma then follows from Theorem 7.4.

Lemma 9.2. *Assume (*) and suppose $\kappa < \eta$ is a strong cardinal. Suppose $(\mathcal{Q}, \Lambda) \in SD(\eta)$ is such that $\mathcal{Q} \in H_\kappa$, and suppose $r \in W(\eta, \Lambda)$. Let $g \subseteq Coll(\omega, < \kappa)$ be generic. Then $\mathcal{S}^{\eta, \Lambda^{r, g}} \models AD^+$.*

It then follows from Lemma 9.2 that every $(\mathcal{Q}, \Lambda) \in IS(\eta)$ is η -stable. We thus have shown that clause 1 and clause 3 of $ldmh(\eta)$ holds. Clause 4 follows from Corollary 7.7. It remains to demonstrate that clause 2 holds. The following is a key lemma.

Lemma 9.3. *Suppose (*) holds and let (\mathcal{P}, Σ) be a hod pair below η . Then there is a hod pair $(\mathcal{Q}, \Lambda) \in IS(\eta)$ such that $\lambda^{\mathcal{Q}}$ is a successor ordinal and $(\mathcal{P}, \Sigma) \preceq^\eta (\mathcal{Q}(\lambda^{\mathcal{Q}} - 1), \Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}} - 1)})$.*

Proof. We only give an outline as the proof of our claims have appeared in [9] under the additional assumption that $\Sigma = \emptyset$. Let $\nu_0 < \nu_1 < \nu < \eta$ be three A -reflecting strong cardinals such that $\mathcal{P} \in H_{\nu_0}$, and let $j : V \rightarrow M$ witness that ν is η^+ -strong and A -reflecting. Let $h \subseteq Coll(\omega, < j(\nu))$ be V -generic, and let $j^+ : V[h_\nu] \rightarrow M[h]$ be the lift of j . It follows from Lemma 9.2 that letting $N^* = \mathcal{S}^{\eta, \Sigma^{h\nu_1}}$, $N^* \models AD^+$. Let $\mathcal{R} = (\mathcal{M}_{\infty, F_{\Sigma^{h\nu_1, od}, \Sigma^{h\nu_1}}})^{N^*}$ (see Section 1.2 and 1.3 of [9] for a clarification of the direct limit construction). The proofs of Lemma 4.4-4.6 of [9] can now be used to show that

(1) $\mathcal{R} \in V$ and in V , \mathcal{R} has a (η, η) -strategy Ψ such that for some $u \in V$, Ψ is $OD_{\Sigma, u}$ ²⁰ and u witnesses that (\mathcal{R}, Ψ) is (η, η) -extendable and η -super fullness preserving. Moreover, for any $\kappa \in (\nu_1, \eta)$, Ψ^{u, h_κ} has branch condensation in $V[h_\kappa]$ ^{21,22}.

²⁰Recall that u is essentially $k \upharpoonright H_{\nu_1^+}^V$ where $k : V \rightarrow S$ witnesses that ν_1 is η^+ -strong and A -reflecting.

²¹Having branch condensation at \mathcal{R} rather than at an iterate is the reason behind choosing three strong cardinals.

²²To see that Ψ is η -super-fullness preserving we need to use the fact that (\mathcal{P}, Σ) is η -stable. To see it, fix a strong cardinal $\mu \in [\nu, \eta)$. We have that $j^+(\Psi)$ is $j^+(\mathcal{S}^{\eta, \Sigma^{h\nu}})$ -super fullness preserving and is

Fix then u as in (1). Because we are assuming that $(*)$ fails, we have that in $V[h_{\nu_1}]$, $\mathcal{N}_\omega^{\#, \Psi, u}$ exists and is η -iterable via an η -extendable strategy (see Lemma 7.6). It follows that in $V[h_{\nu}]$, letting $N = L(\Psi^{u, h_\nu}, \mathbb{R})$, $N \models AD^+$ and $\Psi^{u, h_\nu} \upharpoonright HC^V[h_\nu]$ is N -super fullness preserving (here we use Lemma 5.2). It follows that $D(\mathcal{N}_\omega^{\#, \Psi, u}) \models \text{“}\Psi \text{ is super fullness preserving”}$.

Notice now that we have verified that (\mathcal{R}, Ψ) satisfies the five clauses of Definition 4.4 as witnessed by u implying that (\mathcal{R}, Ψ) is a Σ -hod pair below η (clause 4 of Definition 4.4 is satisfied trivially). We then have that $(\mathcal{R}, \Psi) \in IS(\eta)$. Applying Lemma 7.11 we get $(\mathcal{S}, \Phi) \in IS(\eta)$ such that (\mathcal{S}, Φ) is a hod pair below η and in $V[h_\eta]$, $\Psi \in L(\Phi^{h_\eta}, \mathbb{R})$.

It then follows that in $V[h_\eta]$, $\Sigma^{h_\eta} \in L(\Phi^{h_\eta}, \mathbb{R})$. This follows from the generic interpretability result of Section 2.5 of [7]. Given a stack $\vec{\mathcal{T}}$ on \mathcal{P} , let \mathcal{W} be a Ψ^{u, h_ν} -iterate of \mathcal{R} such that the iteration embedding $\pi : \mathcal{R} \rightarrow \mathcal{W}$ exists and $\vec{\mathcal{T}}$ is generic for $\mathbb{B}_{\delta^{\mathcal{W}}}^{\mathcal{W}}$. Let Π be the interpretation of the strategy coded by the strategy predicate of \mathcal{W} onto $\mathcal{W}[\vec{\mathcal{T}}]$. We then have that $\vec{\mathcal{T}}$ is according to Σ^{h_η} if and only if $\mathcal{W}[\vec{\mathcal{T}}] \models \text{“}\vec{\mathcal{T}} \text{ is according to } \Pi \text{”}$. It is now easy to see that Σ^{h_η} is definable over $L(\Phi^{h_\eta}, \mathbb{R})$.

Next let $\mathcal{N} = \mathcal{N}_\omega^{\#, \Psi, \Sigma}$. We have that $D(\mathcal{N}) \models \Gamma(\mathcal{P}, \Sigma) \triangleleft_W \Gamma(\mathcal{R}, \Psi)$. It then follows that if μ is a strong cardinal such that $\mathcal{P}, \mathcal{R}, \mathcal{S} \in H_\mu$ then in $V[h_\mu]$, $\Psi^{h_\mu} \in L(\Phi^{h_\mu}, \mathbb{R})$ and $L(\Phi^{h_\mu}, \mathbb{R}) \models \Gamma(\mathcal{P}, \Sigma^{h_\mu}) \triangleleft \Gamma(\mathcal{S}, \Phi^{h_\mu})$. It then follows that if $(\mathcal{Q}, \Lambda) \in IS(\eta)$ be such that $(\mathcal{S}, \Phi) \leq^\eta (\mathcal{Q}, \Lambda)$ and $(\mathcal{P}, \Sigma) \leq^\eta (\mathcal{Q}, \Lambda)$ then for some $\alpha < \lambda^{\mathcal{Q}}$, $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$. It is now easy to see that $(\mathcal{Q}(\alpha + 1), \Lambda_{\mathcal{Q}(\alpha + 1)})$ is as desired. \square

To see that clause 2 of $ldmh(\eta)$ holds fix $(\mathcal{P}, \Sigma) \in IS(\eta)$. By successively applying Lemma 9.3 we get a (\mathcal{Q}, Λ) such that it is a short hod pair below η and for some $\alpha < \lambda^{\mathcal{Q}}$, $\lambda^{\mathcal{Q}} = \alpha + \omega$, $\mathcal{Q}(\alpha) \in pI(\mathcal{P}, \Sigma)$ and $\Lambda = \Sigma_{\mathcal{Q}(\alpha)}$ (we use here that η is an inaccessible cardinal). Relativizing Lemma 9.3 to (\mathcal{Q}, Λ) , we get $(\mathcal{R}, \Phi) \in IS(\eta)$ such that (\mathcal{R}, Φ) is a Λ -hod pair below η such that $\lambda^{\mathcal{R}} = 0$. Using Lemma 7.11 we get a hod pair (\mathcal{W}, Ψ) below η such that $D(\mathcal{N}_\omega^{\#, \Psi, \Phi}) \models \Phi \in L(\Psi, \mathbb{R})$. It then follows, just like at the end of the above proof, that (\mathcal{W}, Ψ) is as desired.

Our discussion so far almost finishes the proof of clause 2 of $ldmh(\eta)$. The only missing part is that clause 2 is stated for relativized hod pairs. To see that the relativized version of clause 2 also holds repeat the argument of the previous paragraph relativizing it to some $(\mathcal{Q}, \Lambda) \in IS(\eta)$. Notice that doing so entails proving a relativized version of Lemma 9.3. However, the argument for the relativized version is exactly the same as non relativized

correctly $j^+(\mathcal{S}^{\eta, \Sigma^{h_\nu}})$ -guided. It follows from Corollary 5.3 that $j^+(\Psi) \upharpoonright HC^V[h_\mu]$ is $(\mathcal{S}^{j(\eta), \Sigma^{h_\mu}})^M$ -super fullness preserving and correctly $(\mathcal{S}^{j(\eta), \Sigma^{h_\mu}})^M$ -guided. Because μ is strong in M , it follows from Lemma 5.1 that $\mathcal{S}^{\eta, \Sigma^{h_\mu}} = (\mathcal{S}^{j(\eta), \Sigma^{h_\mu}})^M$. We then get that Ψ^{u, h_μ} is $(\mathcal{S}^{\eta, \Sigma^{h_\mu}})$ -super fullness preserving and is correctly $\mathcal{S}^{\eta, \Sigma^{h_\mu}}$ -guided.

version.

10 Embeddings with weak condensation

In this section we prove a useful application of Theorem 6.2. Suppose η is a measurable cardinal which is a limit of strong cardinals and let $j : V \rightarrow M$ be an embedding witnessing the measurability of η . In core model induction applications it is important to know that embeddings such as j have a certain *weak condensation* property which is stated in Definition 10.2. This property of the embedding is used to show that the strategy that picks j -realizable branches is fulness preserving. The idea behind this approach is originally due to Woodin and goes back to [3]. The author generalized this property in [8] but the current work is the first place where the details of the generalized notion appear.

Throughout this section, we assume $ldmh(\eta)$. Let $h \subseteq \text{Coll}(\omega, < j(\eta))$ be V -generic. Recall that for an uncountable $\nu < \eta$, we let $h_\nu = h \cap \text{Coll}(\omega, < \nu)$. We can then extend j to $j^+ : V[h_\eta] \rightarrow M[h]$. Working in M , we let $\mathcal{P}^* = \mathcal{M}_{\eta, j(\eta), \infty}^*$. We also let $\Upsilon = \Sigma_{\eta, j(\eta)}$. It follows from the discussion before Lemma 8.4 that in M , $(\mathcal{P}^*, \Upsilon)$ is a short hod pair below $j(\eta)$. Let $\Omega = j^+(\Upsilon^{h_\eta})$ and also let

$$\mathcal{P} = \begin{cases} \mathcal{V}_\omega^{j(\eta), \Upsilon}(\mathcal{P}^*) & : \text{no level of } \mathcal{V}^{j(\eta), \Upsilon}(\mathcal{P}^*) \text{ projects across } o(\mathcal{P}^*) \\ \mathcal{M} & : \mathcal{M} \text{ is the least level of } \mathcal{V}^{j(\eta), \Upsilon}(\mathcal{P}^*) \text{ such that } \rho(\mathcal{M}) < o(\mathcal{P}^*). \end{cases}$$

Lemma 10.1. *Suppose in $M[h]$, $\mathcal{R} \in HC^{M[h]}$ is such that there is $\tau : \mathcal{R} \rightarrow j(\mathcal{P}^*)$. Then in $M[h]$, there is $(\mathcal{Q}, \Lambda) \in IS(j(\eta))$ such that $\tau[\mathcal{R}|\delta^{\mathcal{R}}] \subseteq \text{rng}(\pi_{\mathcal{Q}, \infty, j(\eta)}^\Lambda)$. Hence, $(\mathcal{R}|\delta^{\mathcal{R}}, \Psi) \in IS(j(\eta))$ where Ψ is the τ -pullback of Ω .*

Proof. We work in $M[h]$. For each $\alpha < \lambda^{\mathcal{R}}$, we let $(\mathcal{Q}_\alpha, \Lambda_\alpha) \in \mathcal{F}^{j(\eta)}$ be such that

$$\mathcal{M}_{j(\eta), \infty}(\mathcal{Q}_\alpha, \Lambda_\alpha) = j(\mathcal{P}^*)(\tau(\alpha)).$$

Because $\tau \in M[h]$, we can make sure that the sequence $((\mathcal{Q}_\alpha, \Lambda_\alpha) : \alpha < \lambda^{\mathcal{R}}) \in M[h]$. Let $(\mathcal{Q}^*, \Lambda^*)$ be the result of simultaneously comparing all $(\mathcal{Q}_\alpha, \Lambda_\alpha)$. We have that in M , $(\mathcal{Q}^*, \Lambda^*)$ is a short hod pair below $j(\eta)$. Using Lemma 7.11 and clause 2 of $ldmh(j(\eta))$, we can find $(\mathcal{Q}, \Lambda) \in IS(j(\eta))$ which is a hod pair below $j(\eta)$ and such that for all $\alpha < \lambda^{\mathcal{Q}^*}$, there is $\beta < \lambda^{\mathcal{Q}}$ such that $\mathcal{Q}(\beta) \in pI(\mathcal{Q}^*(\alpha), \Lambda_{\mathcal{Q}^*(\alpha)}^*)$ and $\Lambda_{\mathcal{Q}(\beta)} = \Lambda_{\mathcal{Q}^*(\alpha)}^*$. Because $\tau[\mathcal{R}]$ is a countable set in $M[h]$ and $j(\eta) = (\omega_1)^{M[h]}$, we can find $\mathcal{S} \in pI(\mathcal{Q}, \Lambda)$ such that $\tau[\mathcal{R}|\delta^{\mathcal{R}}] \subseteq \cup_{\alpha < \lambda^{\mathcal{S}}} \text{rng}(\pi_{\mathcal{S}(\alpha), \infty, j(\eta)}^{\Lambda_{\mathcal{S}(\alpha)}^*})$. Then $(\mathcal{S}, \Lambda_{\mathcal{S}})$ is as desired. \square

We can now define what we mean by weak condensation. Below we use $\delta^{\mathcal{P}}$ to denote $o(\mathcal{P}^*)$.

Definition 10.2 (Weak condensation). *Suppose no level of \mathcal{P} projects across $\delta^{\mathcal{P}}$ and let $\sigma = j \upharpoonright \mathcal{P}$. We say j has weak condensation if whenever $\mathcal{R} \in HC^{M[h]}$ is a hod premouse such that there is $\tau : \mathcal{R} \rightarrow j(\mathcal{P})$ and $\pi : \mathcal{P} \rightarrow \mathcal{R}$ with the property that $\tau \in M[h]$ and $\sigma = \tau \circ \pi$ then letting, in $M[h]$, $\Psi = (\tau\text{-pullback of } \Omega)$, $\mathcal{R} = \mathcal{V}_{\omega}^{j(\eta), \Psi}(\mathcal{R}|\delta^{\mathcal{R}})$.*

Notice that because of Lemma 10.1 it does make sense to say that $\mathcal{R} = \mathcal{V}_{\omega}^{j(\eta), \Psi}(\mathcal{R}|\delta^{\mathcal{R}})$. Most proofs showing that j has weak condensation use the fact that j can be applied to more sets than just \mathcal{P} such as some kind of universal model. Below we give one argument for showing that j has weak condensation. Our proof exploits the universal model idea. Other ideas include thick hull constructions (for instance see [7]).

Theorem 10.3. *Suppose no level of \mathcal{P} projects across $\delta^{\mathcal{P}}$ and $|\mathcal{P}|^V < \eta^+$. Then j has weak condensation.*

Proof. To prove the theorem we first claim that if there is a counterexample then the counterexamples can be found in $HC^{M[k]}$ where $k = h \cap \text{Coll}(\omega, \eta)$. Given an embedding $\tau : \mathcal{P} \rightarrow \mathcal{R}$ we let $\delta^{\mathcal{R}} = \tau(\delta^{\mathcal{P}})$.

Lemma 10.4. *Suppose there is a hod pair $\mathcal{R} \in HC^{M[h]}$ such that there is $\tau : \mathcal{R} \rightarrow j(\mathcal{P})$ and $\pi : \mathcal{P} \rightarrow \mathcal{R}$ with the property that $\sigma = \pi \circ \tau$ and letting, in $M[h]$, $\Psi = (\tau\text{-pullback of } \Omega)$, $\mathcal{R} \triangleleft \mathcal{V}_{\omega}^{j(\eta), \Psi}(\mathcal{R}|\delta^{\mathcal{R}})$. Then there is such a $(\tau, \mathcal{R}) \in M[k]$ with the property that $\mathcal{R} \in HC^{M[k]}$.*

Proof. We work in $M[h]$. Fix $(\tau^*, \mathcal{R}^*, \pi^*)$ which satisfies the hypothesis of the lemma. Let $\Psi^* = (\tau^*\text{-pullback of } \Omega)$ and $\mathcal{M}^* \trianglelefteq \mathcal{V}^{j(\eta), \Psi^*}(\mathcal{R}^*|\delta^{\mathcal{R}^*})$ be least such that $\rho(\mathcal{M}^*) = \delta^{\mathcal{R}^*}$ but $\mathcal{M}^* \not\trianglelefteq \mathcal{R}^*$. Let $(\mathcal{S}, \Pi) \in IS(j(\eta))$ be a hod pair below $j(\eta)$ such that

1. \mathcal{M}^* has an iteration strategy in $L(\Pi^h, \mathbb{R})$ and
2. $\tau^*[\mathcal{R}^*|\delta^{\mathcal{R}^*}] \subseteq \text{rng}(\pi_{\mathcal{S}, \infty, j(\eta)}^{\Pi})$.

Let $\phi^* : \mathcal{R}^*|\delta^{\mathcal{R}^*} \rightarrow \mathcal{S}$ be given by $\phi^*(x) = y$ if $\tau^*(x) = \pi_{\mathcal{S}, \infty, j(\eta)}^{\Pi}(y)$. We then have that $\Psi^* = (\phi^*\text{-pullback of } \Pi_{\mathcal{S}|\delta^{\mathcal{S}}})$. The existence of (\mathcal{S}, Π) follows from Lemma 10.1 and the definition of $D(j(\eta), h)$.

Let now $(\dot{\tau}, \dot{R}, \dot{\pi}, \dot{\mathcal{M}}, \dot{\phi}) \in M[k]$ be names for $(\tau^*, \mathcal{R}^*, \pi^*, \mathcal{M}^*, \phi^*)$, and let $\zeta = (j(\eta)^+)^M$. Let in $M[k]$, $l : N[k] \rightarrow H_{\zeta}^{M[k]}$ be countable such that $(\dot{\tau}, \dot{R}, \dot{\pi}, \dot{\mathcal{M}}, \dot{\phi}, (\mathcal{S}, \Pi^k)) \in \text{rng}(l)$ and $N \in M$. Let $\bar{\eta} = l^{-1}(j(\eta))$, and let $m \subseteq \text{Coll}(\omega, < \bar{\eta})$ be generic over $N[k]$ such that $m \in M[k]$. We let $P = N[k][m]$, $\bar{\tau} = (l^{-1}(\dot{\tau}))_m$, $\mathcal{R} = (l^{-1}(\dot{R}))_m$, $\pi = (l^{-1}(\dot{\pi}))_m$, $\mathcal{M} = (l^{-1}(\dot{\mathcal{M}}))_m$ and $\phi = (l^{-1}(\dot{\phi}))_m$. Let $\bar{\mathcal{P}} = l^{-1}(j(\mathcal{P}))$ and let $\bar{\Upsilon} = l^{-1}(j(\Upsilon))$. Let, in P , $\bar{\Psi} = (\tau\text{-pullback of } \bar{\Upsilon}^{k**m})$.

Let now $\tau = (l \upharpoonright \bar{\mathcal{P}}) \circ \bar{\tau}$ and in $M[h]$, $\Psi = (\tau\text{-pullback of } \Omega_{j(\mathcal{P})})$. To finish the proof it is enough to show that $(\tau, \mathcal{R}, \pi, \mathcal{M})$ is such that (i) $\sigma = \tau \circ \pi$, (ii) \mathcal{M} is a Ψ -mouse with an

iteration strategy in $D(j(\eta), h)$ and (iii) $\mathcal{M} \not\triangleleft \mathcal{R}$. It is clear that (i) and (iii) follow from elementarity of l . It is then enough to show that (ii) holds. Suppose first that $\Psi \upharpoonright HC^P = \bar{\Psi}$. Using this we show that

Claim 1. \mathcal{M} has an iteration strategy in $D(j(\eta), h)$.

Proof. We have that $P \models$ “ \mathcal{M} is a $\bar{\Psi}$ -mouse”. Let $(\bar{\mathcal{S}}, \bar{\Pi}) = l^{-1}(\mathcal{S}, \Pi)$ and let in N , $\mathcal{W} = \mathcal{N}_{\omega_1, \bar{\Pi}}^{\#}$. It then follows from elementarity of l that in P , \mathcal{M} has an iteration strategy in $L(\bar{\Pi}^{k^*m}, \mathbb{R})$ (as a $\bar{\Psi}$ -mouse). Fix $x \in \mathbb{R}^P$ such that in $(L(\bar{\Pi}^{k^*m}, \mathbb{R}))^P$, $\bar{\Psi}$ is ordinal definable from x and $\bar{\Pi}^{k^*m}$. Let then $\mathcal{U} \in N$ be a tree on \mathcal{W} according to $\bar{\Pi}$ with last model \mathcal{W}^* such that $\pi^{\mathcal{U}}$ exists, \mathcal{U} is below the first Woodin cardinal of \mathcal{W} and (\mathcal{M}, x, ϕ) is generic over \mathcal{W}^* for the extender algebra associated to the first Woodin cardinal of \mathcal{W}^* (such a tree can be constructed via generic genericity iterations). It then follows that $D(\mathcal{W}^*[\mathcal{M}, x, \phi]) \models$ “ \mathcal{M} is a $\bar{\Psi}$ -mouse and $\bar{\Psi} = (\phi\text{-pullback of } \bar{\Pi})$ ”. Let $\Phi = ((l \upharpoonright \bar{\mathcal{S}})\text{-pullback of } \bar{\Pi})$. Because $l(\mathcal{W}^*)$ is $j(\eta)$ -iterable as a Π -mouse, we have that \mathcal{W}^* is $j(\eta)$ -iterable as a Φ -mouse. It then follows that in $M[h]$, $L(\Phi^h, \mathbb{R}) \models$ “ \mathcal{M} is ω_1 -iterable as a Ψ^* -mouse where $\Psi^* = (\phi\text{-pullback of } \Phi^h)$ ”. Let $i = (\pi_{\bar{\mathcal{S}}, \infty, \bar{\eta}, m}^{\bar{\Pi}} \upharpoonright (\bar{\mathcal{S}} \upharpoonright \delta^{\bar{\mathcal{S}}}))^P$. Notice now that $\Psi^* = (l \upharpoonright (\bar{\mathcal{S}} \upharpoonright \delta^{\bar{\mathcal{S}}}) \circ \phi\text{-pullback of } \Pi^h)$. It then follows that $\Psi^* = ((l(i) \circ (l \upharpoonright (\bar{\mathcal{S}} \upharpoonright \delta^{\bar{\mathcal{S}}})) \circ \phi)\text{-pullback of } \Omega)$. But $(l(i) \circ (l \upharpoonright (\bar{\mathcal{S}} \upharpoonright \delta^{\bar{\mathcal{S}}})) \circ \phi) = \tau \upharpoonright \mathcal{R} \upharpoonright \delta^{\mathcal{R}}$, implying that $\Psi^* = \Psi$. Therefore, $L(\Phi^h, \mathbb{R}) \models$ “ \mathcal{M} is ω_1 -iterable as a Ψ -mouse”. The claim now follows because $\Phi^h \in L(\Pi^h, \mathbb{R})$. \square

It follows from the claim that (ii) would follow if we show that $\Psi \upharpoonright HC^P = \bar{\Psi}$. Clearly the equality holds provided we show that, letting $\Gamma = (\bar{\Upsilon}^{k^*m})^P$, $\Delta = (l\text{-pullback of } \Omega)$ and $Q = (H_{\omega_2})^P$

Claim 2. $\Gamma = \Delta \upharpoonright Q$.

Proof. We need to show that for every $\alpha < \lambda^{\bar{P}}$, $\Gamma_{\bar{P}(\alpha)} = \Delta_{\bar{P}(\alpha)} \upharpoonright Q$. We do this by induction. Notice that we have that Γ and $\Delta \upharpoonright Q$ agree on iterations that are in N (because both have hull condensation and l is elementary).

Suppose we have shown that for every $\beta < \alpha$, $\Gamma_{\bar{P}(\beta)} = \Delta_{\bar{P}(\beta)} \upharpoonright Q$. Suppose first that α is a limit. It is enough to show that

(1) if $\vec{\mathcal{T}} \in Q$ is a stack on $\bar{P}(\alpha)$ according to both Γ and $\Delta \upharpoonright Q$ with last model \mathcal{S} such that $\pi^{\vec{\mathcal{T}}}$ exists, and $\beta + 1 < \lambda^{\mathcal{S}}$ is such that $\Gamma_{\mathcal{S}(\beta)} = \Delta_{\mathcal{S}(\beta)} \upharpoonright Q$ then $\Gamma_{\mathcal{S}(\beta+1)} = \Delta_{\mathcal{S}(\beta+1)} \upharpoonright Q$.

That (1) finishes the proof of the claim in the case α is limit follows from the fact that (1) implies that there cannot be minimal disagreements between Γ and $\Delta \upharpoonright Q$. The proof of

(1) is very much like the proof of the claim in the case α is a successor. As the proof of (1) is notationally harder, we give its proof and omit the proof of the claim in the case α is a successor.

To prove (1), fix a stack $\vec{\mathcal{T}} \in Q$ on $\bar{\mathcal{P}}(\alpha)$ according to both Γ and $\Delta \upharpoonright Q$ with last model \mathcal{S} . Let $\beta + 1 < \lambda^{\mathcal{S}}$ be such that $\Gamma_{\mathcal{S}(\beta)} = \Delta_{\mathcal{S}(\beta)} \upharpoonright Q$. It follows from Theorem 6.2 that we can find $\vec{\mathcal{U}} \in N$ on $\bar{\mathcal{P}}(\alpha)$ with last model \mathcal{Q} such that there is $i : \mathcal{S} \rightarrow \mathcal{Q}$ with the property that $i \in N[k]$ and $\pi^{\vec{\mathcal{U}}} = i \circ \pi^{\vec{\mathcal{T}}}$ (we just have to apply Theorem 6.2 in the ultrapower of N by the transitive collapse of the ultrafilter generating j). Since $\vec{\mathcal{U}} \in N$, it follows that $\vec{\mathcal{U}}$ is according to both Γ and $\Delta \upharpoonright Q$. Notice that $\Gamma_{\mathcal{S}(\beta+1)}$ and $\Delta_{\mathcal{S}(\beta+1)}$ are i -pullbacks of $\Gamma_{\mathcal{Q}(i(\beta+1))}$ and $\Delta_{\mathcal{Q}(i(\beta+1))}$. It then follows that to prove (1) its enough to prove that

(2) if $\vec{\mathcal{U}} \in N$ is a stack on $\bar{\mathcal{P}}(\alpha)$ according to both Γ and $\Delta \upharpoonright Q$ with last model \mathcal{Q} and $\beta + 1 < \lambda^{\mathcal{Q}}$ is such that $\Gamma_{\mathcal{Q}(\beta)} = \Delta_{\mathcal{Q}(\beta)} \upharpoonright Q$ then $\Gamma_{\mathcal{Q}(\beta+1)} = \Delta_{\mathcal{Q}(\beta+1)} \upharpoonright Q^{23}$.

Let then $\vec{\mathcal{U}}$ and β be as in (2). It follows from Theorem 6.2 that to prove that $\Gamma_{\mathcal{Q}(\beta+1)} = \Delta_{\mathcal{Q}(\beta+1)} \upharpoonright Q$ its enough to prove that

(3) if $\vec{\mathcal{W}} \in Q$ is a stack on $\mathcal{Q}(\beta + 1)$ according to both Γ and Δ with a last model \mathcal{R} such that $\pi^{\vec{\mathcal{W}}}$ exists then the fragment of $\Gamma_{\mathcal{R}}$ and $\Delta_{\mathcal{R}} \upharpoonright Q$ that act on stacks that are above $\pi^{\vec{\mathcal{W}}}(\delta_{\beta}^{\mathcal{Q}})$ are the same.

Using Theorem 6.2 one more time (coupled with a Skolem hull argument, see the footnote in (2)), we can reduce (3) to

(4) if $\vec{\mathcal{W}} \in N$ is a stack on $\mathcal{Q}(\beta + 1)$ according to both $\Gamma_{\mathcal{Q}(\beta+1)}$ and $\Delta_{\mathcal{Q}(\beta+1)} \upharpoonright Q$ with a last model \mathcal{R} such that $\pi^{\vec{\mathcal{W}}}$ exists then the fragment of $\Gamma_{\mathcal{R}}$ and $\Delta_{\mathcal{R}} \upharpoonright Q$ that act on stacks that are above $\pi^{\vec{\mathcal{W}}}(\delta_{\beta}^{\mathcal{Q}})$ are the same.

Using Theorem 6.2 one last time we can reduce (4) to

(5) $\Gamma_{\mathcal{Q}(\beta+1)}$ and $\Delta_{\mathcal{Q}(\beta+1)} \upharpoonright Q$ agree on normal trees that are above $\delta_{\beta}^{\mathcal{Q}}$.

The reduction of (4) to (5) will change \mathcal{Q} to some \mathcal{Q}' but without loss of generality we

²³We warn the reader that the \mathcal{Q} here isn't obtained the same way as the \mathcal{Q} of the previous paragraph. The \mathcal{Q} of (2) is obtained via a Skolem hull of size $(\bar{\eta}^+)^N$ and it is the transitive collapse of the \mathcal{Q} in the previous paragraph. We need such a Skolem hull argument because Theorem 6.2 doesn't say anything about the size of \mathcal{Q} .

can assume that $\mathcal{Q} = \mathcal{Q}'$. Fix then a tree $\mathcal{U} \in \mathcal{Q}$ on \mathcal{Q} which is according to both $\Gamma_{\mathcal{Q}(\beta+1)}$ and $\Delta_{\mathcal{Q}(\beta+1)} \upharpoonright \mathcal{Q}$ and has a limit length. Let $b = \Gamma_{\mathcal{Q}(\beta+1)}(\mathcal{U})$. Suppose then $\mathcal{M}_b^{\mathcal{U}} \models \text{“}\delta(\mathcal{U}) \text{ is Woodin”}$. It then follows from Theorem 6.2 that there is $\mathcal{U}^* \in N \cap \mathcal{Q}$ on $\mathcal{Q}(\beta+1)$ with a last model \mathcal{N} which is according to $\Gamma_{\mathcal{Q}}$ and there is $i : \mathcal{M}_b^{\mathcal{U}} \rightarrow \mathcal{N}$ such that $\pi^{\mathcal{U}^*} = i \circ \pi_b^{\mathcal{U}}$. Because $\Gamma_{\mathcal{Q}(\beta+1)}$ and $\Delta_{\mathcal{Q}(\beta+1)} \upharpoonright \mathcal{Q}$ agree on stacks in N and because they both have branch condensation²⁴, we have that b is according to $\Delta_{\mathcal{Q}} \upharpoonright \mathcal{Q}$.

Suppose then $\mathcal{Q}(b, \mathcal{U})$ exists. Let $\mathcal{Q}^+ = \mathcal{Q}(\beta+\omega)$. The argument that shows $\Delta_{\mathcal{Q}(\beta+1)}(\mathcal{U}) = b$ is very similar when \mathcal{U} has a fatal drop and when it doesn't. The proof when \mathcal{U} doesn't have a fatal drop is notationally less demanding and we present it leaving the other one to the reader. Thus, we assume \mathcal{U} has no fatal drops. It follows from the proof of Theorem 6.2 that there is a tree $\mathcal{U}^* \in N$ on \mathcal{Q}^+ with a last model \mathcal{N} which is according to $\Gamma_{\mathcal{Q}^+}$ such that $\pi^{\mathcal{U}}$ exists, \mathcal{U}^* is based on $\mathcal{Q}(\beta+1)$, is above $\delta_{\beta}^{\mathcal{Q}}$ and \mathcal{U} is generic over the extender algebra $\mathbb{B}_{\delta_{\beta+1}^{\mathcal{N}}}$. Let $\tau = \pi^{\mathcal{U}}(\tau_{U_{\mathcal{Q}(\beta), \Gamma_{\mathcal{Q}(\beta)}}}^{\mathcal{Q}(\beta+1)})$ (recall the set U from Section 2). We then have that

(6) $D(\mathcal{N}) \models \text{“}\mathcal{Q}(b, \mathcal{U}) \text{ is a sound } \omega_1\text{-iterable } \Gamma_{\mathcal{Q}(\beta)}\text{-mouse over } \mathcal{M}(\mathcal{U})\text{”}$

Working in $M[h]$, fix some M -strong cardinal $\kappa \in (\eta, j(\eta))$. Let C be the derived model of \mathcal{N} as computed in $M[h_{\kappa}]$ by $\Delta_{\mathcal{N}} \upharpoonright HC^{M[h_{\kappa}]}$ using genericity iterations above $\delta_{\beta+1}^{\mathcal{N}}$. Then it follows from (6) and the fact that $\Gamma_{\mathcal{Q}(\beta)} = \Delta_{\mathcal{Q}(\beta)} \upharpoonright \mathcal{Q}$ that

(7) $C \models \text{“}\mathcal{Q}(b, \mathcal{U}) \text{ is a sound } \omega_1\text{-iterable } \Delta_{\mathcal{Q}(\beta)}\text{-premouse over } \mathcal{M}(\mathcal{U})\text{”}$.

Notice that it follows from Theorem 2.3 that $\Delta \upharpoonright HC^{V[h_{\kappa}]}$ is C -super fullness preserving. Let now $c = \Delta_{\mathcal{Q}(\beta+1)}(\mathcal{U})$. It then follows from (7) that if $\mathcal{Q}(c, \mathcal{U})$ exists then $b = c$. Because $\Delta \upharpoonright HC^{V[h_{\kappa}]}$ is C -super fullness preserving, it follows from (7) that $\mathcal{Q}(c, \mathcal{U})$ exists. This finishes the proof of the claim. \square

As we mentioned above, Claim 2 finishes the proof of the lemma. \square

Towards a contradiction, assume that the conclusion of the theorem is false. Unless otherwise specified, we work in $M[h]$. It follows from Lemma 10.4 that we can find a counterexample $(\tau, \mathcal{R}) \in M[k]$ such that $\mathcal{R} \in HC^{M[k]}$. We let in $M[h]$, $\Psi = (\tau\text{-pullback of } \Omega)$. Because $\tau \in M[h]$, it follows from Lemma 10.1 that we can fix $(\mathcal{S}^*, \Lambda^*) \in IS(j(\eta))$ such that $\tau[\mathcal{R}|\delta^{\mathcal{R}}] \subseteq \text{rng}(\pi_{\mathcal{S}^*, j(\eta), \infty}^{\Lambda^*})$. Let then $\xi \leq \lambda^{\mathcal{S}^*}$ be the least such that $\tau[\mathcal{R}|\delta^{\mathcal{R}}] \subseteq \text{rng}(\pi_{\mathcal{S}^*(\xi), \infty}^{\Lambda^*})$. Let $(\mathcal{S}, \Lambda) \in \mathcal{F}^{j(\eta)}$ be such that $\lambda^{\mathcal{S}}$ is a limit ordinal and for some $\gamma < \lambda^{\mathcal{S}}$,

²⁴It follows from Theorem 2.3 that $\Delta_{\mathcal{Q}(\beta+1)}$ has branch condensation.

$\mathcal{S}(\gamma) \in pI(\mathcal{S}^*(\xi), \Lambda^*)$ and $\Lambda_{\mathcal{S}(\gamma)} = (\Lambda^*)_{\mathcal{S}(\gamma)}$ (the existence of such a pair follows from clause 2 of $ldmh(\eta)$).

Let then $\sigma : \mathcal{R}|\delta^{\mathcal{R}} \rightarrow \mathcal{S}|\delta^{\mathcal{S}}$ be given by $\sigma(x) = y$ if and only if $\tau(x) = \pi_{\mathcal{S}(\gamma), \infty, j(\eta)}^{\Lambda}(y)$. It follows that $\Psi = (\sigma\text{-pullback of } \oplus_{\beta < \gamma} \Lambda_{\mathcal{S}(\beta)}^h)$. Next, we let $\mathcal{M} \trianglelefteq \mathcal{V}^{j(\eta), \Psi}(\mathcal{R}|\delta^{\mathcal{R}})$ be least such that $\rho(\mathcal{M}) = \delta^{\mathcal{R}}$ but $\mathcal{M} \not\trianglelefteq \mathcal{R}$. We can then fix $(\mathcal{S}', \Lambda') \in \mathcal{F}^{j(\eta)}$ such that $L((\Lambda')^h, \mathbb{R}) \models \text{“}\mathcal{M} \text{ is a } \Psi\text{-mouse”}$. The fact that we can find such a pair inside $\mathcal{F}^{j(\eta)}$ follows from Lemma 7.11. Without loss of generality, using comparison, we can assume that $\mathcal{S} = \mathcal{S}'$ and $\Lambda = \Lambda'$.

Claim. There is $(\mathcal{Q}, \Phi) \in \mathcal{F}_{\mathcal{P}, \Upsilon}^{j(\eta)}$ such that $\mathcal{Q} \in M$, $\lambda^{\mathcal{Q}}$ is a limit ordinal and $L(\Gamma(\mathcal{Q}, \Phi^h), \mathbb{R}) \models \text{“}\mathcal{M} \text{ is a } \Psi\text{-mouse”}$.

Proof. Let $\mu \in [\mu_{\mathcal{S}, \Lambda}, \eta)$ be a strong cardinal such that $L(\Lambda^{h_\mu}, \mathbb{R}^{M[h_\mu]}) \models \text{“}\mathcal{M} \text{ is a } \Psi \upharpoonright HC^{M[h_\mu]}\text{-mouse”}$. Using Theorem 2.2, we can find $(\mathcal{Q}^*, \Phi^*) \in L(\Lambda^{h_\mu}, \mathbb{R}^{M[h_\mu]})$ such that in $M[h_\mu]$,

1. (\mathcal{Q}^*, Φ^*) is a Υ^{h_μ} -hod pair such that $\lambda^{\mathcal{Q}^*}$ is a limit ordinal,
2. $\Gamma(\mathcal{S}, \Lambda^{h_\mu}) = \Gamma(\mathcal{Q}^*, \Phi^*)$,
3. (this follows from clause 2) $L(\Gamma(\mathcal{Q}^*, \Phi^*), \mathbb{R}) \models \text{“}\mathcal{M} \text{ is a } \Psi \upharpoonright HC^{M[h_\mu]}\text{-mouse”}$.

We can then find the desired (\mathcal{Q}, Φ) using the proof of Lemma 7.11 (one just needs to relativize that proof to (\mathcal{P}, Υ)). \square

Fix then (\mathcal{Q}, Φ) as in the claim. Because we have that $j(\eta)$ is closed under hybrid \mathcal{N}_ω -operators, we have that in $M[k]$, $\mathcal{N}_\omega^{\#, \Psi, \Phi^k}(\mathcal{M})$ exists and has a $j(\eta)$ -extendable iteration strategy. Let $\mathcal{N} = \mathcal{N}_\omega^{\#, \Psi, \Phi^k}(\mathcal{M})$. We now have that the following holds:

- (1) $D(\mathcal{N}) \models \text{“}L(\Gamma(\mathcal{Q}, \Phi^k), \mathbb{R}) \models \text{“}\mathcal{M} \text{ is a } \Psi\text{-mouse such that } \mathcal{M} \not\trianglelefteq \mathcal{R}\text{”}$.

Working in $M[k]$, let $l : N[k] \rightarrow (H_{j(\eta)^+})^{M[k]}$ be such that $N[k]$ is countable, $N \in M$, $\pi, \mathcal{R}, \mathcal{M} \in N[k]$, l is the identity on $(\mathcal{R}, \mathcal{M})$, and $(\tau, \mathcal{Q}, \Phi, \Psi, \mathcal{N}) \in rng(l)$. Let $(\bar{\tau}, \bar{\mathcal{Q}}, \bar{\Phi}, \bar{\Psi}, \bar{\mathcal{N}}) = l^{-1}(\tau, \mathcal{Q}, \Phi, \Psi, \mathcal{N})$. Notice that we have that $\bar{\Psi} = \Psi \upharpoonright N[k]$. It follows from (1) that

- (2) $D(\bar{\mathcal{N}}) \models \text{“}L(\Gamma(\bar{\mathcal{Q}}, \bar{\Phi}), \mathbb{R}) \models \text{“}\mathcal{M} \text{ is a } \bar{\Psi}\text{-mouse such that } \mathcal{M} \not\trianglelefteq \mathcal{R}\text{”}$.

Because $l : \bar{\mathcal{N}} \rightarrow \mathcal{N}$ it follows that if $\Gamma \in M[h]$ is the strategy of \mathcal{N} and $\Pi = (l\text{-pullback of } \Gamma)$ then letting C be the derived model (in $M[h]$) as computed by Π

(3) $C \models "L(\Gamma(\bar{\mathcal{Q}}, \bar{\Phi}^h), \mathbb{R}) \models \text{"}\mathcal{M} \text{ is a } \Psi\text{-mouse such that } \mathcal{M} \not\leq \mathcal{R}\text{"}$.

Notice that because $\bar{\mathcal{Q}} \in N$, $j(\bar{\mathcal{Q}})$ makes sense²⁵. Let now E be the $(\delta^{\mathcal{P}}, \delta^{\mathcal{R}})$ -extender derived from π . Let $\mathcal{R}^+ = \text{Ult}(\bar{\mathcal{Q}}, E)$ and let $\pi^+ : \bar{\mathcal{Q}} \rightarrow \mathcal{R}^+$ be the ultrapower map. Let also $\sigma^+ = j \upharpoonright \bar{\mathcal{Q}}$ and define $\tau^+ : \mathcal{R}^+ \rightarrow j(\bar{\mathcal{Q}})$ by $\tau^+(\pi^+(f)(a)) = \sigma^+(f)(\tau(a))$ where $f \in \bar{\mathcal{Q}}$ and $a \in [\delta^{\mathcal{R}}]^{<\omega}$. It is easy to check that $\sigma^+ = \tau^+ \circ \pi^+$ and that σ^+ , π^+ and τ^+ extend σ , π and τ respectively.

Let $\Phi^* = j^+(\bar{\Phi}^{h_\eta})$. We then have that $\bar{\Phi}^h = (\sigma^+ \text{-pullback of } \Phi^*)$. Let $\Phi^{**} = (\tau^+ \text{-pullback of } \Phi^*)$. It then again follows that $\bar{\Phi}^h = (\pi^+ \text{-pullback of } \Phi^{**})$. The later equality implies if D is the derived model of \mathcal{R}^+ as computed by Φ^{**} in $M[h]$ then

(4) $C \subseteq D$.

It then follows from (3) and (4) that $L(\Gamma(\mathcal{R}^+, \Phi^{**}), \mathbb{R}) \models \text{"}\mathcal{M} \text{ is a } \Psi\text{-mouse such that } \mathcal{M} \not\leq \mathcal{R}\text{"}$. Because \mathcal{R}^+ is a Ψ -mouse over $\mathcal{R} \upharpoonright \delta^{\mathcal{R}}$, it follows that $D(\mathcal{R}^+) \models \text{"}\mathcal{M} \text{ is a } \Psi\text{-mouse such that } \mathcal{M} \not\leq \mathcal{R}\text{"}$. This then implies that $\mathcal{M} \in \mathcal{R}^+$ implying that $\mathcal{M} \leq \mathcal{R}$, contradiction. \square

11 The limit derived model hypothesis and covering with lower parts

In this section our goal is to show that the derived model hypothesis implies covering with lower parts provided there is no transitive inner model containing the reals and satisfying $AD_{\mathbb{R}} + \text{"}\Theta \text{ is regular"}$. More precisely, we prove the following theorem.

Theorem 11.1. *Suppose η is a measurable limit of strong cardinals such that $\text{lcm}h(\eta)$ holds. Assume that there is no transitive inner model containing the reals and ordinals and satisfying $AD_{\mathbb{R}} + \text{"}\Theta \text{ is regular"}$. Then covering with lower parts holds at η .*

We spend the rest of this section proving the theorem. Throughout this section we assume that

(*) Covering with lower parts fails at η and $\neg \#_{\Theta\text{-reg}}(\eta)$.

Our goal is of course to show that (*) is inconsistent. Let $j : V \rightarrow M$ be an embedding witnessing that η is a measurable. Fix a V -generic $h \subseteq \text{Coll}(\omega, < j(\eta))$. We can then extend

²⁵This is what we meant by “ j applies to a universal model”.

j to $j^+ : V[h_\eta] \rightarrow M[h]$. Working in $M[h]$, we let $\mathcal{P}^* = \mathcal{M}_{\eta, j(\eta), \infty}^*$. We also let $\Sigma = \Sigma_{\eta, j(\eta)}$. Notice that in $M[h]$, (\mathcal{P}^*, Σ) is a short hod pair below $j(\eta)$. Let

$$\mathcal{P} = \begin{cases} \mathcal{V}_\omega^{j(\eta), \Sigma}(\mathcal{P}^*) & : \text{no level of } \mathcal{V}_\omega^{j(\eta), \Sigma}(\mathcal{P}^*) \text{ projects across } \delta^{\mathcal{P}^*} \\ \mathcal{M} & : \mathcal{M} \text{ is the least level of } \mathcal{V}_\omega^{j(\eta), \Sigma}(\mathcal{P}^*) \text{ such that } \rho(\mathcal{M}) < \delta^{\mathcal{P}^*}. \end{cases}$$

Our main lemma is the following whose proof will take the entire section.

Lemma 11.2. *Assume (*). There is a hod pair (\mathcal{Q}, Λ) below η such that $\lambda^{\mathcal{Q}}$ is a limit ordinal and $\mathcal{Q} \models \text{“}\delta^{\mathcal{Q}} \text{ is a regular cardinal”}$.*

Because of Lemma 1.2, the lemma immediately gives a contradiction. We spend the rest of this section proving Lemma 11.2. Our first lemma shows that \mathcal{P} has a small size.

Lemma 11.3. $|\mathcal{P}|^V = \eta$.

Proof. This follows from the failure of covering with lower parts at η . We have that $\mathcal{P} \leq Lp_\omega^\Sigma(\mathcal{P}^*)$. Since (*) holds we get that $\mathcal{P} \in H_{\eta^+}$. \square

Our goal now is to describe, in $M[h]$, a strategy Λ for \mathcal{P} which extends Σ^h . Later we will show that $\mathcal{P} = \mathcal{V}_\omega^{\eta, \Sigma}(\mathcal{P}^*)$. It then will follow from our construction and from Theorem 10.3 that in $M[h]$, Λ is ω_1 -fulness preserving. The crucial fact that we will explore is that because of Lemma 11.3, letting $\sigma = j \upharpoonright \mathcal{P}$, $\sigma \in M$. The idea behind the construction of Λ goes back to [8]. We give the construction below. Notice that describing an iteration strategy is tantamount to specifying when a given iteration is according to the strategy.

Definition 11.4 (j -realizable iterations). *Suppose $\vec{\mathcal{T}} \in HC^{M[h]}$ is a stack on \mathcal{P} . Working in $M[h]$, we say $\vec{\mathcal{T}}$ is j -realizable if there is a sequence $(\sigma_{\mathcal{R}} : \mathcal{R} \in \text{tn}(\vec{\mathcal{T}}))$ and a sequence $(\nu_{\mathcal{R}}, \gamma_{\mathcal{R}} : \mathcal{R} \in \text{tn}(\vec{\mathcal{T}}))$ such that*

1. $\sigma_{\mathcal{P}} = \sigma$, for all terminal nodes \mathcal{R} of $\vec{\mathcal{T}}$, $\sigma_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P})$ and whenever $\mathcal{R} \prec_{\vec{\mathcal{T}}, s} \mathcal{Q}$, $\sigma_{\mathcal{R}} = \sigma_{\mathcal{Q}} \circ \pi_{\mathcal{R}, \mathcal{Q}}^{\vec{\mathcal{T}}}$.
2. For every non-trivial terminal node \mathcal{R} of $\vec{\mathcal{T}}$, $\nu_{\mathcal{R}} < j(\eta)$ is the least M -strong cardinal κ such that for some $\gamma < \lambda^{\mathcal{M}_{\kappa, j(\eta), \infty}(\gamma)}$, letting $\mathcal{S}_{\mathcal{R}} = \mathcal{M}_{\kappa, j(\eta), \infty}(\gamma)$ and $\Lambda_{\mathcal{R}} = \Sigma_{\kappa, j(\eta), \gamma}$, we have that $\sigma_{\mathcal{R}}[\mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1)] \subseteq \text{rng}(\pi_{\mathcal{S}_{\mathcal{R}}, \infty, j(\eta)}^{\Lambda_{\mathcal{R}}^h})$. The least such γ is $\gamma_{\mathcal{R}}$.
3. For every non-trivial terminal node \mathcal{R} , letting $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}})$ be as above and letting $k_{\mathcal{R}} : \mathcal{R}(\xi^{\vec{\mathcal{T}}, \mathcal{R}} + 1) \rightarrow \mathcal{S}_{\mathcal{R}}$ be given by $k_{\mathcal{R}}(x) = y$ if and only if $k_{\mathcal{R}}(x) = \pi_{\mathcal{S}_{\mathcal{R}}, \infty, j(\eta)}^{\Lambda_{\mathcal{R}}^h}(y)$, $k_{\mathcal{R}} \vec{\mathcal{T}}_{\mathcal{R}}$ is according to $\Lambda_{\mathcal{R}}^h$.

4. Suppose \mathcal{R} is a non-trivial terminal node of $\vec{\mathcal{T}}$. Let $\mathcal{S}_{\mathcal{R}}^*$ be the last model of $k_{\mathcal{R}}\vec{\mathcal{T}}_{\mathcal{R}}$. Suppose $\vec{\mathcal{T}}_{\mathcal{R}}$ has a last model $\mathcal{Q}_{\mathcal{R}}$ and that $\pi^{\vec{\mathcal{T}}_{\mathcal{R}}}$ is defined. It then follows that $\mathcal{Q}_{\mathcal{R}} \in \text{tn}(\vec{\mathcal{T}})$ and $\mathcal{R} \prec^{\vec{\mathcal{T}},s} \mathcal{Q}_{\mathcal{R}}$. Let $k_{\mathcal{R}}^* : \mathcal{Q}_{\mathcal{R}} \rightarrow \mathcal{S}_{\mathcal{R}}^*$ come from the copying construction. Then for all $x \in \mathcal{Q}_{\mathcal{R}}$, $\sigma_{\mathcal{Q}_{\mathcal{R}}}(x) = \sigma_{\mathcal{R}}(f)(\pi_{\mathcal{S}_{\mathcal{R}}^*, \infty, j(\eta)}^{\Lambda_{\mathcal{R}}^h}(k_{\mathcal{R}}^*(a)))$ where $f \in \mathcal{R}$ and $a \in [\mathcal{Q}_{\mathcal{R}}(\pi_{\mathcal{R}, \mathcal{Q}_{\mathcal{R}}}^{\vec{\mathcal{T}}}(\xi^{\vec{\mathcal{T}}, \mathcal{R}}) + 1)]^{<\omega}$ are such that $x = \pi_{\mathcal{R}, \mathcal{Q}_{\mathcal{R}}}^{\vec{\mathcal{T}}}(f)(a)$.
5. Suppose \mathcal{R} is a trivial terminal node of $\vec{\mathcal{T}}$. Then for every $\xi < \lambda^{\mathcal{R}}$, there is an M -strong cardinal $\nu < j(\eta)$ and $\xi < \lambda^{\mathcal{M}_{\nu, j(\eta), \infty}}$ such that letting $\mathcal{S} = \mathcal{M}_{\nu, j(\eta), \infty}(\xi + 1)$, $\sigma_{\mathcal{R}}[\mathcal{R}(\xi + 1)] \subseteq \text{rng}(\pi_{\mathcal{S}, \infty, j(\eta)}^{\Sigma_{\nu, j(\eta), \infty}^{\xi+1}})$.

We say that $(\sigma_{\mathcal{R}}^{\vec{\mathcal{T}}} : \mathcal{R} \in \text{tn}(\vec{\mathcal{T}}))$ are the j -realizable embeddings of $\vec{\mathcal{T}}$ and $(\nu_{\mathcal{R}}^{\vec{\mathcal{T}}}, \gamma_{\mathcal{R}}^{\vec{\mathcal{T}}} : \mathcal{R} \in \text{tn}(\vec{\mathcal{T}}))$ are the j -realizable pairs of $\vec{\mathcal{T}}$.

Definition 11.5 (The definition of $\text{dom}(\Lambda)$). Suppose $\vec{\mathcal{T}} \in \text{HC}^{M[h]}$ is a stack on \mathcal{P} such that either there is a strongly linear closed and cofinal set $C \subseteq \text{tn}(\vec{\mathcal{T}})$ or $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is of limit length. Working in $M[h]$, we let $\vec{\mathcal{T}} \in \text{dom}(\Lambda)$ if $\vec{\mathcal{T}}$ is j -realizable. We let $\Lambda(\vec{\mathcal{T}}) = b$ if $\vec{\mathcal{T}} \widehat{\ } \{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is j -realizable.

To show that Λ is an iteration strategy we need to show that

Lemma 11.6. Whenever $\vec{\mathcal{T}} \in \text{dom}(\Lambda)$, $\Lambda(\vec{\mathcal{T}})$ is defined.

Proof. Suppose first that there is a strongly linear closed and cofinal set $C \subseteq \text{tn}(\vec{\mathcal{T}})$. In this case, $\vec{\mathcal{T}}$ has a unique branch $b = b_C$. It is then enough to show that $\vec{\mathcal{T}} \widehat{\ } \{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is j -realizable. Notice that there are no drops along b . We let $\mathcal{Q} = \mathcal{M}_b^{\vec{\mathcal{T}}}$ and define $\sigma_{\mathcal{Q}} : \mathcal{Q} \rightarrow j(\mathcal{P})$ by letting $\sigma_{\mathcal{Q}}^{\vec{\mathcal{T}}}(x) = y$ if and only if whenever $\mathcal{R} \in C$ is such that there is $\bar{x} \in \mathcal{R}$ with the property that $\pi_{\mathcal{R}, \mathcal{Q}}^{\vec{\mathcal{T}}}(\bar{x}) = x$ then $\sigma_{\mathcal{R}}^{\vec{\mathcal{T}}}(\bar{x}) = y$. Because $\sigma_{\mathcal{R}}^{\vec{\mathcal{T}}} \in M[h]$, it is easy to see that Lemma 10.1 implies that clause 5 of Definition 11.4 is satisfied.

Suppose then there is no strongly linear closed and cofinal set $C \subseteq \text{tn}(\vec{\mathcal{T}})$. It follows that the branch we are looking for is a branch for $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$. Let $\mathcal{R} = \mathcal{S}_{\vec{\mathcal{T}}}$ and $\mathcal{T} = \vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$. We have that $\sigma_{\mathcal{R}}^{\vec{\mathcal{T}}}$ and $(\nu_{\mathcal{R}}^{\vec{\mathcal{T}}}, \gamma_{\mathcal{R}}^{\vec{\mathcal{T}}})$ are defined. Let then $b = \Lambda_{\mathcal{R}}^h(k_{\mathcal{R}}^{\vec{\mathcal{T}}}\mathcal{T})$. If $\mathcal{M}_b^{\vec{\mathcal{T}}}$ is not a terminal node of $\vec{\mathcal{T}} \widehat{\ } \{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ then we are done. Suppose then that $\mathcal{M}_b^{\vec{\mathcal{T}}}$ is a terminal node of $\vec{\mathcal{T}} \widehat{\ } \{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$. Let $\mathcal{Q} = \mathcal{M}_b^{\vec{\mathcal{T}}}$, $\mathcal{S}^* = \mathcal{M}_b^{k_{\mathcal{R}}^{\vec{\mathcal{T}}}\mathcal{T}}$ and let $k : \mathcal{Q} \rightarrow \mathcal{S}^*$ come from the copying construction. We need to show that clauses 1-5 of Definition 11.4 hold for $\vec{\mathcal{T}} \widehat{\ } \{\mathcal{Q}\}$.

We let $\sigma_{\mathcal{Q}} = \pi_{\mathcal{S}^*, \infty, j(\eta)}^{\Lambda_{\mathcal{R}}^h} \circ k \in M[h]$. Notice now that clauses 1 and 4 are satisfied with our definition of $\sigma_{\mathcal{Q}}$ while clauses 2 and 3 are vacuous for \mathcal{Q} as \mathcal{Q} is a trivial terminal node. It remains to show that clause 5 of Definition 11.4 is satisfied. Fix then $\xi < \lambda^{\mathcal{Q}}$. Let $(\mathcal{W}, \Psi) \in \mathcal{F}^{j(\eta)}$ be such that $\mathcal{M}_{j(\eta), \infty}(\mathcal{W}, \Psi) = j(\mathcal{P})(\sigma_{\mathcal{Q}}(\xi))$. Because $\sigma_{\mathcal{Q}}[\mathcal{Q}(\xi + 1)] \in M[h]$, we can find $\mathcal{S} \in pI(\mathcal{W}, \Psi)$ such that $\sigma_{\mathcal{Q}}[\mathcal{Q}(\xi + 1)] \subseteq \text{rng}(\pi_{\mathcal{S}, \infty, j(\eta)}^{\Psi_{\mathcal{S}}})$. Let now $\kappa < j(\eta)$ be

such that $\mathcal{S} \in H_\kappa^M$ and let $\Phi = \Psi_{\mathcal{S}}$. Let $\nu = \mu_{\mathcal{S}, \Phi}$ ($\mu_{\mathcal{S}, \Phi}$ was defined after Definition 4.8). Then clearly ν witnesses clause 5 of Definition 11.4. \square

Next we show that

Lemma 11.7. $\mathcal{P} \models \text{“cf}(\delta^{\mathcal{P}}) \text{ is measurable”}$ or $\mathcal{J}_1(\mathcal{P}) \models \text{“cf}(\delta^{\mathcal{P}}) \text{ is measurable”}$.

Proof. We first show that $\text{cf}(\delta^{\mathcal{P}}) = \eta$. Assume not and let $\kappa < \eta$ be the cofinality of $\delta^{\mathcal{P}}$ in V . Let $f : \kappa \rightarrow \lambda^{\mathcal{P}}$ be an increasing cofinal function. We can then fix $(\mathcal{Q}_\alpha, \Phi_\alpha : \alpha < \kappa) \subseteq \mathcal{F}^\eta$ such that for $\alpha < \kappa$, $\mathcal{P}(f(\alpha)) = \mathcal{M}_{\eta, \infty}(\mathcal{Q}_\alpha, \Phi_\alpha)$. We can then using the regularity of η , comparison, clause 2 of $\text{ldmh}(\eta)$ and Lemma 7.11, find a hod pair (\mathcal{Q}, Φ) below η such that for all $\alpha < \kappa$, $(\mathcal{Q}_\alpha, \Phi_\alpha) \prec^\eta (\mathcal{Q}, \Phi)$. It then follows that for every $\alpha < \kappa$, $\mathcal{P}(f(\alpha)) \triangleleft_{\text{hod}} \mathcal{M}_{\eta, \infty}(\mathcal{Q}, \Phi) \triangleleft_{\text{hod}} \mathcal{P}$. Since $\mathcal{P} = \cup_{\alpha < \kappa} \mathcal{P}(f(\alpha))$, we get a contradiction.

Suppose then neither $\text{cf}^{\mathcal{P}}(\delta^{\mathcal{P}})$ nor $\text{cf}^{\mathcal{J}_1(\mathcal{P})}(\delta^{\mathcal{P}})$ is measurable. It then follows that $\delta^{\mathcal{P}}$ is regular both in \mathcal{P} and in $\mathcal{J}_1(\mathcal{P})$. To see this assume not and let $\kappa = \text{cf}^{\mathcal{P}}(\delta^{\mathcal{P}})$. Let $(\mathcal{Q}, \Lambda) \in \mathcal{F}^\eta$ be such that $\pi_{\mathcal{Q}, \infty, \eta}^\Lambda(\kappa^*) = \kappa$ for some $\kappa^* \in \mathcal{Q}$. If now κ^* isn't measurable in \mathcal{Q} then $\pi_{\mathcal{Q}, \infty, \eta}^\Lambda[\kappa^*]$ is cofinal in κ . It then follows that $\text{cf}(\kappa) < \eta$ implying that $\text{cf}(\lambda^{\mathcal{P}}) < \eta$. We then must have that κ is a measurable cardinal.

Notice that we must have that $\rho(\mathcal{P}) < \delta^{\mathcal{P}}$ (otherwise it is not hard to see that $j(\mathcal{P})$ is a Σ^h -iterate of \mathcal{P}). We now work in $M[h]$. We can fix $(\mathcal{S}, \Phi) \in \mathcal{F}^{j(\eta)}$ such that $\Sigma^h \in L(\Phi^h, \mathbb{R})$ and $L(\Phi^h, \mathbb{R}) \models \text{“}\mathcal{P} \trianglelefteq Lp^{\Sigma^h}(\mathcal{P}|\delta^{\mathcal{P}})\text{”}$ (this follows from Lemma 7.11). We then have that in $L(\Phi^h, \mathbb{R})$, Σ^h is a fullness preserving iteration strategy (see Corollary 5.5). It follows that in $L(\Phi^h, \mathbb{R})$ for some α , $\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \text{HOD}$ and that $\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\delta^{\mathcal{M}_\infty(\mathcal{P}, \Sigma)} = V_{\theta_\alpha}^{\text{HOD}}$. However, $\rho(\mathcal{M}_\infty(\mathcal{P}, \Sigma)) < \delta^{\mathcal{M}_\infty(\mathcal{P}, \Sigma)}$. \square

Lemma 11.8. $\rho(\mathcal{P}) \geq \delta^{\mathcal{P}}$.

Proof. Suppose not. Then we have that $\rho(\mathcal{P}) < \delta^{\mathcal{P}}$. Let $\xi = \lambda^{\mathcal{P}}$, $\kappa = \text{cf}^{\mathcal{P}}(\delta^{\mathcal{P}})$ and let μ be the normal Mitchell order 0 measure on κ on the \mathcal{P} -sequence. Let $\mathcal{R} = \text{Ult}_n(\mathcal{P}, \mu)$ where n is the largest such that $\rho_n(\mathcal{P}) > \delta^{\mathcal{P}}$. We then have that $\rho(\mathcal{R}) < \delta_\xi^{\mathcal{R}}$ and \mathcal{R} is $\delta_\xi^{\mathcal{R}}$ -sound. Let $\Omega = \oplus_{\alpha < \xi} \Sigma_{\mathcal{R}(\alpha)}$. Given $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{R}, \Lambda_{\mathcal{R}})$, we let, in $M[h]$, $\Phi^{\mathcal{S}, \vec{\mathcal{T}}} = (\pi^{\vec{\mathcal{T}}}$ -pullback of $\Lambda_{\mathcal{S}, \vec{\mathcal{T}}}$) and $\Gamma_{\vec{\mathcal{T}}, \mathcal{S}} = \Gamma(\mathcal{R}, \Phi^{\mathcal{S}, \vec{\mathcal{T}}})$.

Claim 1. For any two $(\vec{\mathcal{T}}_0, \mathcal{S}_0), (\vec{\mathcal{T}}_1, \mathcal{S}_1) \in I(\mathcal{R}, \Lambda_{\mathcal{R}})$, either $\Gamma_{\vec{\mathcal{T}}_0, \mathcal{S}_0} \leq_w \Gamma_{\vec{\mathcal{T}}_1, \mathcal{S}_1}$ or $\Gamma_{\vec{\mathcal{T}}_1, \mathcal{S}_1} \leq_w \Gamma_{\vec{\mathcal{T}}_0, \mathcal{S}_0}$.

Proof. Notice that it is enough to show that whenever $i = 0, 1$, $(\vec{\mathcal{U}}_i, \mathcal{W}_i) \in B(\mathcal{R}, \Phi^{\mathcal{S}_i, \vec{\mathcal{T}}_i})$ then in $M[h]$, letting for $i \in 2$, $\Phi_i = (\Phi^{\mathcal{S}_i, \vec{\mathcal{T}}_i})_{\mathcal{W}_i}$, $\mathcal{N}_\omega^{\#, \Phi_0, \Phi_1}$ exists and is ω_1 -iterable. Fix then such $(\vec{\mathcal{U}}_i, \mathcal{W}_i)$ for $i = 0, 1$ and let Φ_i be as above. Notice that by the definition of Λ and $\Phi^{\mathcal{S}, \vec{\mathcal{T}}}$,

for $i = 0, 1$, $(\mathcal{W}_i, \Phi_i) \in D(j(\eta), h)$ (see Lemma 10.1). We can now apply clause 3 of $ldmh(\eta)$ and Lemma 4.11. \square

It follows from the claim that there is $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{R}, \Lambda_{\mathcal{R}})$ such that, in $M[h]$, $\Gamma_{\vec{\mathcal{T}}, \mathcal{S}}$ is minimal in the Wadge order. Fix then such a $(\vec{\mathcal{T}}, \mathcal{S})$ and let $\Phi = \Phi^{\mathcal{S}, \vec{\mathcal{T}}}$. Notice that it follows from the copying construction that for any $(\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{R}, \Phi)$ there is $(\vec{\mathcal{W}}, \mathcal{S}^*) \in I(\mathcal{R}, \Lambda_{\mathcal{R}})$ such that $(\pi^{\vec{\mathcal{U}}}$ -pullback of $\Phi_{\mathcal{Q}, \vec{\mathcal{U}}}) = \Phi^{\mathcal{S}^*, \vec{\mathcal{W}}}$. It then follows from the minimality of $\Gamma(\mathcal{R}, \Phi)$ that for any $(\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{R}, \Phi)$, $\Gamma(\mathcal{Q}, \Phi_{\mathcal{Q}, \vec{\mathcal{U}}}) = \Gamma(\mathcal{R}, \Phi)$ (this is because we already have that $\Gamma(\mathcal{Q}, \Phi_{\mathcal{Q}, \vec{\mathcal{U}}}) \leq_W \Gamma(\mathcal{R}, \Phi)$). It now follows from Theorem 2.3 that, in $M[h]$, Φ is $\Gamma(\mathcal{R}, \Phi)$ -fullness preserving. Next we show that \mathcal{R} isn't full.

Claim 2. Suppose $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{R}(\xi), \Omega)$. Then $\mathcal{S} \neq \mathcal{V}_{\omega}^{j(\eta), \Omega_{\mathcal{S}|\delta^{\mathcal{S}}}}(\mathcal{S}|\delta^{\mathcal{S}})$.

Proof. Towards a contradiction, suppose not. Let \mathcal{S}^* be such that $(\vec{\mathcal{T}}, \mathcal{S}^*) \in I(\mathcal{R}, \Lambda)$. We can view \mathcal{S}^* as an $\Omega_{\mathcal{S}}$ -hod premouse. Let Φ^* be the fragment of $\Lambda_{\mathcal{S}^*, \vec{\mathcal{T}}}$ which acts on stacks above $\delta^{\mathcal{S}}$ and let in $M[h]$, $\Gamma = \Gamma(\mathcal{S}^*, \Phi^*)$. Notice that iterations according to Φ^* are continuous at $\delta^{\mathcal{S}^*}$ (this is because $\text{cf}^{\mathcal{S}^*}(\delta^{\mathcal{S}^*}) < \delta^{\mathcal{S}}$). It then follows from Theorem 2.3 that Φ^* has branch condensation and is Γ fullness preserving.

Let now in $M[h]$, $N = L(\Phi^*, \mathbb{R})$. Notice that it follows from Lemma 10.1 that $\Phi^* \in D(j(\eta), h)$. It then follows from Lemma 4.11 that $N \models AD^+$. We then have that in N , Φ^* is Γ -fullness preserving and \mathcal{S} is the unique sound $\Omega_{\mathcal{S}|\delta^{\mathcal{S}}}$ -hod premouse \mathcal{M} over $\mathcal{S}|\delta^{\mathcal{S}}$ such that $\rho(\mathcal{M}) < \delta^{\mathcal{S}}$ and \mathcal{M} has an iteration strategy Ψ with branch condensation such that $\Gamma(\mathcal{M}, \Psi) = \Gamma$. Thus, in N , \mathcal{S} is ordinal definable from $\mathcal{S}|\delta^{\mathcal{S}}$ and $\Omega_{\mathcal{S}|\delta^{\mathcal{S}}}$. But we have that $(Lp^{\Omega_{\mathcal{S}|\delta^{\mathcal{S}}}}(\mathcal{S}|\delta^{\mathcal{S}}))^N \trianglelefteq \mathcal{V}_{\omega}^{j(\eta), \Omega_{\mathcal{S}|\delta^{\mathcal{S}}}}(\mathcal{S}|\delta^{\mathcal{S}})$. It follows from strong mouse capturing applied in N and our assumption that \mathcal{S} is full (i.e., $\mathcal{S} = \mathcal{V}_{\omega}^{j(\eta), \Omega_{\mathcal{S}|\delta^{\mathcal{S}}}}(\mathcal{S}|\delta^{\mathcal{S}})$), that $\mathcal{S} \in \mathcal{S}$. This is a contradiction! \square

Now, using Theorem 2.4 we can find $(\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{R}, \Phi)$ such that $\Phi_{\mathcal{Q}, \vec{\mathcal{U}}}$ has branch condensation. Let in $M[h]$, $N = L(\Phi_{\mathcal{Q}, \vec{\mathcal{U}}}, \mathbb{R})$, $\Gamma = \Gamma(\mathcal{R}(\xi), \Omega)$ and $\Gamma^+ = \Gamma(\mathcal{Q}, \Phi_{\mathcal{Q}, \vec{\mathcal{U}}})$. We then have that $N \models AD^+$ (see Lemma 10.1). Because in M , (\mathcal{P}^*, Σ) is a short hod pair below $j(\eta)$, we have that Γ is a Solovay pointclass of N , i.e., there is some α such that $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_{\alpha}^N\}$. Because $\rho(\mathcal{Q}) < \delta_{\pi^{\vec{\mathcal{U}}}(\xi)}^{\mathcal{Q}}$, we get a contradiction as follows. Let, in N , (\mathcal{W}, Ψ) be a hod pair such that $\Gamma(\mathcal{W}, \Psi) = \Gamma^+$. It then follows that in N , $\rho(\mathcal{M}_{\infty}(\mathcal{W}, \Psi)) < \theta_{\alpha}^N$. But by Theorem 2.2, $N \models V_{\theta_{\alpha}}^{\text{HOD}^N} = \mathcal{M}_{\infty}(\mathcal{W}, \Psi)|\theta_{\alpha}$. This is a contradiction. \square

It follows from Lemma 11.8 and Theorem 10.3 that j has weak condensation. This implies that in $M[h]$, Λ is $D(j(\eta), h)$ -fullness preserving. We now would like to get a set of

M -inaccessible cardinals $\nu < j(\eta)$ such that letting $\mathcal{R}^* = \mathcal{M}_{\nu, j(\eta), \infty}^*$, $\mathcal{R}_\nu = \mathcal{V}_\omega^{\nu, \Sigma_{\nu, j(\eta)}^h}(\mathcal{R}^*)$ and $Y_\nu = \cup_{\alpha < \lambda^{\mathcal{R}_\nu}} \pi_{\mathcal{R}_\nu(\alpha), \infty, j(\eta)}^{\Sigma_{\alpha, \nu, j(\eta)}}[\mathcal{R}(\alpha)]$, $\mathcal{R}_\nu = \text{Hull}_1^{j(\mathcal{P})}(Y_\nu \cup \sigma[\mathcal{P}])$. We say ν is *good* if it has the above properties. If ν is good we let $\sigma_\nu : \mathcal{R}_\nu \rightarrow j(\mathcal{P})$ be the uncollapse map and $\pi_\nu = \sigma_\nu^{-1} \circ \sigma$.

Lemma 11.9. *Work in $M[h]$. Suppose $X \subseteq j(\mathcal{P})$ is a countable set. Then there is a good ν such that $X \subseteq \sigma_\nu[\mathcal{R}_\nu]$.*

Proof. The lemma follows by taking an ultrapower of M . Indeed, let U be the ultrafilter on η such that $M = \text{Ult}(V, U)$. Let $N = \text{Ult}(M, j(U))$ and let $i : M \rightarrow N$. Let $k \subseteq \text{Coll}(\omega, < i(j(\eta)))$ be a V -generic such that $h = k \cap \text{Coll}(\omega, < j(\eta))$. We then have that i lifts to $i^+ : M[h] \rightarrow N[k]$. Notice then we have that $i(X) = i[X]$. Its then not hard to see that in N , $j(\eta)$ is as desired. \square

Notice that η is good, $\mathcal{P} = \mathcal{R}_\eta$ and $\sigma = \sigma_\eta$. Fix now a good ν and let $\Sigma_\nu = \Sigma_{\nu, j(\eta)}$. Just like we defined Λ for \mathcal{P} we can define Λ_ν for \mathcal{R}_ν . It is defined using the definition of Λ with $(\mathcal{P}, \sigma, \Sigma)$ changed to $(\mathcal{R}_\nu, \sigma_\nu, \Sigma_\nu)$. It then follows from Lemma 11.8 and Lemma 10.3 that Λ_ν is $D(j(\eta), h)$ -fullness preserving.

Let now $\mu < j(\eta)$ be good. We can then apply Theorem 2.3 and Theorem 2.4 to conclude that some tail of Λ_ν has branch condensation, i.e, for some $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{R}_\mu, \Lambda_\mu)$, $(\Lambda_\mu)_{\mathcal{S}, \vec{\mathcal{T}}}$ has branch condensation. Fix such a $\vec{\mathcal{T}}$ and let $\Phi = (\Lambda_\mu)_{\mathcal{S}, \vec{\mathcal{T}}}$. Fix then an M -strong cardinal ν such that $\mathcal{S} \in HC^{M[h_\nu]}$. Let in $M[h_\nu]$, $\mathcal{Q} = \mathcal{M}_\infty(\mathcal{S}, \Phi \upharpoonright HC^{M[h_\nu]})$. Let $\Pi = \Phi_{\mathcal{Q}} \upharpoonright H_{j(\eta)}^M$. We then have that $(\mathcal{Q}, \Pi) \in M$. Let s be the $j(\eta)$ -extendability witness we get for Π using Theorem 6.2 (the theorem applies because Φ is $D(j(\eta), h)$ -super fullness preserving and is correctly $D(j(\eta), h)$ -guided, see Remark 6.3). We have that $\Pi^{s, h} = \Phi$.

Lemma 11.10. *In M , (\mathcal{Q}, Π) is a hod pair below $j(\eta)$ as witnessed by s .*

Proof. Work in M . We need to show that s witnesses that

1. $(\mathcal{Q}, \Pi) \in SD(j(\eta))$,
2. $\mathcal{N}_\omega^{\#, \Pi, s}$ exists and is $j(\eta)$ -iterable via a $j(\eta)$ -extendable strategy,
3. Π has branch condensation below $j(\eta)$,
4. $D(\mathcal{N}_\omega^\Pi) \models$ “ Π is super fullness preserving”.
5. for every $\mathcal{Q} \in pB(\mathcal{P}, \Pi)$, $\Pi_{\mathcal{Q}}$ is $j(\eta)$ -stable,
6. Π is $j(\eta)$ -super fullness preserving.

Clause 2 follows from the fact that $j(\eta)$ is closed under hybrid \mathcal{N}_ω -operators. Clause 1 then follows from clause 2 and the generic interpretability results of [7] (see Theorem 2.5.10 of [7]). Clause 3 is a consequence of the fact that $\Pi^{s,h} = \Phi$ and Φ has branch condensation in $M[h]$. Clause 4 follows because Φ is $D(\eta, h)$ -super fullness preserving. Clause 5 follows from the fact that $Proj(j(\eta))$ holds.

It remains to show that Π is $j(\eta)$ -super fullness preserving as witnessed by s . Fix then $\mathcal{R} \in pI(\mathcal{Q}, \Pi)$ and let $\xi < \lambda^{\mathcal{R}}$. Organizing the proof as an induction, we can assume that clause 6 holds for all $\zeta < \xi$. Hence, we have that $(\mathcal{R}(\xi), \Pi_{\mathcal{R}(\xi)})$ is a hod pair below $j(\eta)$. Fix an M -strong cardinal $\nu < j(\eta)$ such that $\mathcal{R} \in H_\nu^M$. We need to see that $\Pi_{\mathcal{R}(\xi+1)}^{s,h_\nu}$ is $\mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$ -super fullness preserving and is correctly $\mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$ -guided. Notice that because for every $\alpha < \lambda^{\mathcal{P}}$, $D(\mathcal{P}(\alpha + \omega)) \models \text{“}\Sigma_{\mathcal{P}(\alpha)} \text{ is super-fullness preserving and is correctly guided”}$ we have that in $M[h_\nu]$, $\Pi_{\mathcal{R}(\xi+1)}^{s,h_\nu} \upharpoonright HC$ is $\Gamma(\mathcal{R}(\xi + \omega), \Pi_{\mathcal{R}(\xi+\omega)}^{s,h_\nu})$ -super fullness preserving and is correctly $\Gamma(\mathcal{R}(\xi + \omega), \Pi_{\mathcal{R}(\xi+\omega)}^{s,h_\nu})$ -guided. Let then in $M[h_\nu]$, $\mathcal{N} = (Lp^{\Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(\mathbb{R}))^{\Gamma(\mathcal{R}(\xi+\omega), \Pi_{\mathcal{R}(\xi+\omega)}^{s,h_\nu})}$. It follows from Lemma 5.1 that $\mathcal{N} \trianglelefteq \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$.

It is then enough to show that $\mathcal{N} = \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$. Suppose not. Fix now $(B_i : i < \omega) \subseteq (\mathbb{B}[\mathcal{R}(\xi), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}])^{\Gamma(\mathcal{R}(\xi+\omega), \Pi_{\mathcal{R}(\xi+\omega)}^{s,h_\nu})}$ that guide $\Pi_{\mathcal{R}(\xi+1)}^{s,h_\nu} \upharpoonright HC^V[h_\nu]$. We then have that $(B_i : i < \omega) \subseteq \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$. Because $\mathcal{N} \triangleleft \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$, it follows that $(B_i : i < \omega) \in \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$. It then follows that $\Pi_{\mathcal{R}(\xi+1)}^{s,h_\nu} \in \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$. Since Π^h is $D(j(\eta), h)$ -fullness preserving, we have that for some $a \in HC^V[h_\nu]$,

$$\mathcal{V}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(a) \triangleleft (Lp^{\Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(a))^{\mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}}.$$

Let $\mathcal{W} \trianglelefteq (Lp^{\Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(a))^{\mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}}$ be such that $\rho(\mathcal{W}) = a$ and $\mathcal{W} \not\trianglelefteq \mathcal{V}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(a)$. Let $\Phi \in \mathcal{S}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}$ be the strategy of \mathcal{W} . Using the second clause of $ldmh(j(\eta))$ in $M[h]$, we can fix a hod pair (\mathcal{R}^*, Ω) below $j(\eta)$ such that for some $\gamma < \lambda^{\mathcal{R}^*}$, $\mathcal{R}^*(\gamma) \in pI(\mathcal{R}(\xi), \Pi_{\mathcal{R}(\xi)})$ and $\Omega_{\mathcal{R}^*(\gamma)} = \Pi_{\mathcal{R}^*(\gamma)}$. Let $\mu < j(\eta)$ be an M -strong cardinal such that $\mathcal{R}^*, \mathcal{R} \in H_\mu^M$. It then follows that

(1) in $M[h_\mu]$, $\Phi^{h_\mu} \in L(\Omega^{h_\mu}, \mathbb{R})$ (see Lemma 4.5).

It then follows that in fact $\mathcal{W} \trianglelefteq \mathcal{V}^{j(\eta), \Pi_{\mathcal{R}(\xi)}^{s,h_\nu}}(a)$, contradiction. \square

It follows from the lemma and clause 2 of $ldmh(j(\eta))$ that

Corollary 11.11. $\mathcal{M}_{j(\eta), \infty}(\mathcal{Q}, \Pi) \triangleleft_{hod} j(\mathcal{P})$.

We then let $\mathcal{M}_\mu = \mathcal{M}_{j(\eta), \infty}(\mathcal{Q}, \Pi)$. Because \mathcal{Q} is a Λ_μ iterate of \mathcal{P} and Π^h is the corresponding tail, it follows from the definition of Λ_μ that we have $m_\mu : \mathcal{M}_\mu \rightarrow j(\mathcal{P})$ such

that

$$\sigma_\mu = m_\mu \circ \sigma_{\mathcal{Q}, \infty, j(\eta)}^\Pi \circ \pi_{\mathcal{S}, \infty, \nu, j(\eta)}^\Phi \circ \pi^{\vec{T}}.$$

Indeed, given $x \in \mathcal{M}_{j(\eta), \infty}(\mathcal{Q}, \Pi)$ let $\mathcal{R} \in pI(\mathcal{Q}, \Pi^h)$ be such that $\pi_{\mathcal{R}, \infty, j(\eta)}^\Pi(y) = x$ for some $y \in \mathcal{R}$. Let $\sigma_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P})$ be given by the construction of Λ_μ . Then set $m_\mu(x) =_{def} \sigma_{\mathcal{R}}(y)$. Its not hard to show, using the construction of Λ_μ , that m_μ is as desired. The following lemma is also easy.

Lemma 11.12. $\text{cp}(m_\mu) > \sup \sigma_\mu[\delta^{\mathcal{R}_\mu}]$.

Proof. The claim follows immediately because whenever $\mathcal{S} \in pI(\mathcal{R}_\mu, \Lambda_\mu)$, $i : \mathcal{R}_\mu \rightarrow \mathcal{S}$ is the iteration embedding and $\sigma_{\mathcal{S}} : \mathcal{S} \rightarrow j(\mathcal{P})$ is the realizability map given by the construction of Λ_μ then for all $\alpha < \lambda^{\mathcal{R}_\mu}$, $\sigma_{\mathcal{S}} \upharpoonright \mathcal{S}(i(\alpha)) = \pi_{\mathcal{S}(i(\alpha)), \infty, j(\eta)}^{\Sigma_{\mu, j(\eta)}}$. \square

We now want to prove that $\mathcal{P} \models \text{“}\delta^{\mathcal{P}} \text{ is regular”}$. To do this we will show that j has *condensation*, which naturally is a stronger property than weak condensation.

11.1 Embeddings with condensation

We continue with the set up of the previous section. We start with an easy lemma.

Lemma 11.13. *Suppose ν is good and $A \in \mathcal{R}_\nu \cap \mathcal{P}(\delta^{\mathcal{R}_\nu})$. Then for every formula ϕ and $s \in [\delta^{\mathcal{R}_\nu}]^{<\omega}$,*

$$\mathcal{R}_\nu \models \phi[s, A] \iff j(\mathcal{P}) \models \phi[\pi_{\mathcal{R}_\nu(\alpha), \infty, j(\eta)}^{\Sigma_\nu}(s), \sigma_\nu(A)]$$

where $\alpha < \lambda^{\mathcal{R}}$ is such $s \in [\delta_\alpha^{\mathcal{R}_\nu}]^{<\omega}$.

Proof. The proof is an easy consequence of the fact that $\sigma_\nu \upharpoonright (\mathcal{R}_\nu \upharpoonright \delta_\alpha^{\mathcal{R}_\nu}) = \pi_{\mathcal{R}_\nu(\alpha), \infty, j(\eta)}^{\Sigma_\nu}$. \square

Suppose now that \mathcal{R} is a hod premouse such that there is $\tau : \mathcal{R} \rightarrow j(\mathcal{P})$. We then say \mathcal{Q} is τ -realizable if there are $\pi : \mathcal{R} \rightarrow \mathcal{Q}$ and $\gamma : \mathcal{Q} \rightarrow j(\mathcal{P})$ such that $\tau = \gamma \circ \pi$. We say that (π, γ) witness that \mathcal{Q} is τ -realizable. Now suppose $A \in \mathcal{P} \cap \mathcal{P}(\delta^{\mathcal{P}})$ and in $M[h]$, \mathcal{R} is σ -realizable as witnessed by (π, τ) . Then we let $\Sigma(\mathcal{R}, \tau) = (\tau\text{-pullback of } j(\Sigma)^h)$ and $A_{\mathcal{R}} \subseteq \omega \times [\delta^{\mathcal{R}}]^{<\omega}$ be the set given by

$$(\phi, s) \in A_{\mathcal{R}, \tau} \iff j(\mathcal{P}) \models \phi[\pi_{\mathcal{R}(\alpha), \infty, j(\eta)}^{\Sigma(\mathcal{R}, \tau)}(s), \sigma(A)].$$

where α is such that $s \in [\delta_\alpha^{\mathcal{R}}]^{<\omega}$. Let also $T_A^{\mathcal{P}} = \{(\phi, s) : \phi \text{ is a formula, } s \in [\delta^{\mathcal{P}}]^{<\omega} \text{ and } \mathcal{P} \models \phi[s, A]\}$.

Definition 11.14 (Condensation). *Suppose $A \in \mathcal{P} \cap \mathcal{P}(\delta^{\mathcal{P}})$ and \mathcal{R} is σ -realizable as witnessed by (π, τ) . We then say τ has A -condensation if whenever \mathcal{Q} is τ -realizable as witnessed by (π^*, τ^*) , $\pi^*(\pi(T_A^{\mathcal{P}})) = A_{\mathcal{Q}, \tau^*}$. We say τ has condensation if τ has A -condensation for every $A \in \mathcal{P} \cap \mathcal{P}(\delta^{\mathcal{P}})$.*

Notice that if \mathcal{R} is σ -realizable as witnessed by (π, τ) and \mathcal{Q} is τ -realizable as witnessed by (π^*, τ^*) then if τ has A -condensation then τ^* too has A -condensation. This is an immediate consequence of the fact that τ^* -realizability implies τ -realizability.

Lemma 11.15. *σ has condensation.*

Proof. Suppose not. Fix $A \in \mathcal{P} \cap \mathcal{P}(\delta^{\mathcal{P}})$. We first assume that

- (1) there is a good ν such that σ_ν has A -condensation.

We claim that (1) implies that σ has A -condensation. Fix ν as in (1). Let then U be the ultrafilter of η such that $M = Ult(V, U)$ and let $i : M \rightarrow N = Ult(M, j(U))$. Let $g \subseteq Coll(\omega, < i(j(\eta)))$ be V -generic such that $h = g \cap Coll(\omega, < j(\eta))$. Then i lifts to $i^+ : M[h] \rightarrow N[g]$. Notice that $i^+(\sigma_\nu) = i(\sigma) \circ \sigma_\nu$. It follows that $j(\mathcal{P})$ is $i^+(\sigma_\nu)$ -realizable as witnessed by $(\sigma_\nu, i(\sigma))$. Hence, $i(\sigma)$ has $j(A)$ -condensation. By elementarity σ has A -condensation. It is then enough to prove that (1) holds.

Suppose (1) fails. Using Lemma 11.9, we can then find a sequence $(\mathcal{Q}_i, \pi_i, \tau_i, k_i, \psi_i, \nu_i : i < \omega) \in M[h]$ such that

1. $\nu_0 = \eta$ and $(\nu_i : i < \omega)$ is an increasing sequence of good points,
2. for $i < \omega$, \mathcal{Q}_i is σ_{ν_i} -realizable as witnessed by (π_i, τ_i) and $k_i : \mathcal{Q}_i \rightarrow \mathcal{R}_{\nu_{i+1}}$ is given by $k_i = \sigma_{\nu_i}^{-1} \circ \tau_i$
3. for $i < \omega$, $\sigma_{\nu_i}[\mathcal{R}_{\nu_i}] \subseteq rng(\sigma_{\nu_{i+1}})$, $\psi_i = \sigma_{\nu_{i+1}}^{-1} \circ \sigma_{\nu_i}$ and for $i < m$, letting $\psi_{i,m} = \sigma_{\nu_m}^{-1} \circ \sigma_{\nu_i}$ and $A_i = \psi_{0,i}(A)$, $\pi_i(T_{A_i}^{\mathcal{R}_i}) \neq A_{\mathcal{Q}_i, \tau_i}$ (i.e., $(\mathcal{Q}_i, \pi_i, \tau_i)$ witnesses that σ_{ν_i} doesn't have A_i -condensation).

Let now ν be a good point such that $\sup_{i < \omega} \nu_i < \nu$ and letting $X = \cup_{i < \omega} (\tau_i[\mathcal{Q}_i] \cup \sigma_{\nu_i}[\mathcal{R}_{\nu_i}])$, $X \subseteq rng(\sigma_\nu)$. Let (\mathcal{S}^*, Φ^*) be a hod pair below $j(\eta)$ such that $\mathcal{M}_\nu \triangleleft \mathcal{M}_{j(\eta), \infty}(\mathcal{S}^*, \Phi^*)$ and $\lambda^{\mathcal{S}}$ is limit but not divisible by ω^2 . Let $B = m_\nu^{-1}(j(A))$. Let now $\sigma_i = m_\nu^{-1} \circ \sigma_{\nu_i}$ and $\tau_i^* = m_\nu^{-1} \circ \tau_i$. Notice now that we can define the notion of A -condensation also for the embeddings σ_i . Assuming that this is already done, notice that we have that

1. for $i < \omega$, \mathcal{Q}_i is σ_i -realizable as witnessed by (π_i, τ_i^*) and $k_i : \mathcal{Q}_i \rightarrow \mathcal{R}_{\nu_{i+1}}$ is given by $k_i = \sigma_i^{-1} \circ \tau_i^*$

2. $(\mathcal{Q}_i, \pi_i, \tau_i^*)$ witnesses that σ_i doesn't have A_i -condensation.

The importance of this move is that letting $\mathcal{R}_i = \mathcal{R}_{\nu_i}$ the badness of $(\mathcal{Q}_i, \mathcal{R}_i, \pi_i, \tau_i^*, k_i, \psi_i, \sigma_i : i < \omega)$ can now be witnessed in the derived model of \mathcal{S}^* as computed by Φ^* . More precisely, letting $\Sigma_i = \Sigma_{\nu_i}$ and $\Psi_i = (\tau_i$ -pullback of $j(\Sigma)^h$),

(2) in $M[h]$, letting $N = D(\mathcal{S}^*, (\Phi^*)^h) = L(\Gamma(\mathcal{S}^*, (\Phi^*)^h), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that for every i , $(\phi, s) \in T_{A_i}^{\mathcal{R}_i}$ if and only if $\theta[\pi_{\mathcal{R}_i(\alpha), \infty}^{\Sigma_i^h}(s), t]$ where α is the least such that $s \in [\delta_\alpha^{\mathcal{R}_i}]^{<\omega}$. However, in N , for each i , there is a pair $(\phi_i, s_i) \in T_{\pi_i(A_i)}^{\mathcal{Q}_i}$ such that $\neg\theta[\pi_{\mathcal{Q}_i(\alpha), \infty}^{\Psi_i^h}(s_i), t]$ where α is the least such that $s_i \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega}$.

To continue, we set up a notation. Suppose K is a transitive model of AD^+ and $b = ((\mathcal{M}_i, \Sigma_i), \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in K$ is such that $(\mathcal{M}_i, \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in HC^K$. Suppose $\theta(u, v)$ is a formula and t is a finite sequence of ordinals. We write $K \models "(b, \theta(u, v), t)$ is bad" if in K , letting $K^* = L(\{D \subseteq \mathbb{R} : w(D) \leq t(0)\})$ then $b \in K^*$ and in K^*

1. for every $i < \omega$, \mathcal{M}_i is a hod premouse such that $\lambda^{\mathcal{M}_i}$ is limit and Σ_i is an ω_1 -iteration strategy for $\mathcal{M}_i|\delta^{\mathcal{M}_i}$ with the property that for every $\alpha < \lambda^{\mathcal{M}_i}$, $(\Sigma_i)_{\mathcal{M}_i(\alpha)}$ has branch condensation and is fullness preserving,
2. for every i , $\xi_i : \mathcal{M}_i \rightarrow \mathcal{M}_{i+1}$,
3. for every i , \mathcal{N}_i is ξ_i -realizable as witnessed by (γ_i, l_i) ,
4. for every $\alpha < \lambda^{\mathcal{N}_i}$, letting $\Psi_i = (l_i$ -pullback of $\Sigma_i)$, $(\Psi_i)_{\mathcal{N}_i(\alpha)}$ has branch condensation and is fullness preserving,
5. $C \in M_0 \cap \mathcal{P}(\delta^{\mathcal{M}_i})$ and letting $C_0 = C$ and $C_{i+1} = \xi_i(C_i)$, for every i ,

$$(\phi, s) \in T_{C_i}^{\mathcal{M}_i} \text{ if and only if } \theta[\pi_{\mathcal{M}_i(\alpha), \infty}^{\Sigma_i}(s), t]$$

where α is least such that $s \in [\delta_\alpha^{\mathcal{M}_i}]^{<\omega}$ but for every i , there is $(\phi_i, s_i) \in T_{\gamma_i(C_i)}^{\mathcal{N}_i}$ such that $\neg\theta[\pi_{\mathcal{N}_i(\alpha), \infty}^{\Psi_i}(s), t]$ where α is least such that $s_i \in [\delta_\alpha^{\mathcal{N}_i}]^{<\omega}$.

Let now $\kappa = \mu_{\mathcal{S}^*, (\Phi^*)^h}$. Working in $M[h_\kappa]$, let $(\mathcal{W}^*, \Pi^*) \in \mathcal{F}_{\mathcal{P}, \Sigma}^{\kappa, j(\eta)}$ be such that $\Gamma(\mathcal{W}^*, (\Pi^*)^h) = \Gamma(\mathcal{S}^*, (\Phi^*)^h)$. Let $b = ((\mathcal{R}_i, \Sigma_i), \mathcal{Q}_i, \pi_i, k_i, \psi_i, A_0 : i < \omega)$. We can then rewrite (2) in terms of $(\mathcal{W}^*, (\Pi^*)^h)$ and get that

(3) in $M[h]$, letting $N = D(\mathcal{W}^*, (\Pi^*)^h) = L(\Gamma(\mathcal{W}^*, (\Pi^*)^h), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad.

Let then $\mathcal{N}^* = \mathcal{N}_{\omega}^{\#, \Pi^*, \oplus_{i < \omega} \Sigma_i}$. Let \mathcal{N} be an iterate of \mathcal{N}^* via the canonical iteration strategy of \mathcal{N}^* such that H_{κ}^M is generically generic over the extender algebra of \mathcal{N} at its bottom Woodin cardinal. We can now witness (3) inside $\mathcal{N}[H_{\kappa}^M][h_{\kappa}]$ as follows:

(4) $D(\mathcal{N}[H_{\kappa}^M][h_{\kappa}]) \models$ “letting $N = D(\mathcal{W}^*, \Pi^*) = L(\Gamma(\mathcal{W}^*, \Pi^*), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad”.

We will get a contradiction using (4). Notice that the sequence $a = (\mathcal{R}_i, \psi_i, \Sigma_i, A_i : i < \omega) \in M$. However, the sequence $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$ may not be in M . Let then $d \in M^{Coll(\omega, < \kappa)}$ be a name for $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$. Let $\zeta = (j(\eta)^+)^M$, $g = h \cap Coll(\omega, < \eta)$ and let $\pi : P[g] \rightarrow (H_{\zeta}^M)[h]$ be such that $P \in V$, $cp(\pi) > \eta$, $|P|^V = \eta$, $\{\mathcal{N}, d, a, (\mathcal{W}^*, \Pi^*)\} \in rng(\pi)$. Let $\mathcal{M} = \pi^{-1}(\mathcal{N})$, $e = \pi^{-1}(a)$ and $c = \pi^{-1}(d)$. Let for $i < \omega$, $e(i) = (\mathcal{K}_i, \xi_i, \bar{\Sigma}_i, B_i : i < \omega)$ and $(\mathcal{W}, \Pi) = \pi^{-1}(\mathcal{W}^*, \Pi^*)$. Also we let $\bar{\kappa} = \pi^{-1}(\kappa)$. By elementarity, (4) gives that

(5) whenever $m \subseteq Coll(\omega, < \pi^{-1}(j(\eta)))$ is $P[g]$ -generic then in $P[g][m]$, letting $d = d_{g * m_{\bar{\kappa}}}$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $f = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, B_i : i < \omega)$, $D(\mathcal{M}[H_{\bar{\kappa}}^P][g * m_{\bar{\kappa}}]) \models$ “letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(f, \theta(u, v), t)$ is bad”.

Using genericity iterations we can completely internalize (5) to $\mathcal{M}^* = \mathcal{M}[H_{\bar{\kappa}}^P]$ and get that

(6) in \mathcal{M}^* , there is a name $d^* \in (\mathcal{M}^*)^{Coll(\omega, < \bar{\kappa})}$ such that whenever $m \subseteq Coll(\omega, < \bar{\kappa})$ is \mathcal{M}^* -generic then letting $d = d_{m_{\bar{\kappa}}}^*$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $f = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, B_i : i < \omega)$, $D(\mathcal{M}^*[m]) \models$ “letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(f, \theta(u, v), t)$ is bad”.

Work now in $M[h]$. Notice that for every i , $\bar{\Sigma}_i = ((\pi\text{-pullback of } \Sigma_i)) \upharpoonright P$ and $\Pi = ((\pi\text{-pullback of } \Pi^*)) \upharpoonright P$. In what follows, we abuse our notation and let for every i , $\bar{\Sigma}_i = (\pi\text{-pullback of } \Sigma_i)$ and $\Pi = (\pi\text{-pullback of } \Pi^*)$. It then follows that in $M[h]$, \mathcal{M} is a $\Pi^h \oplus (\oplus_{i < \omega} \bar{\Sigma}_i^h)$ -mouse which has an η -extendable strategy. Let now $C = D(\mathcal{W}, \Pi^h)$. It is easy to see that (6) gives $(\mathcal{S}_i, \gamma_i, l_i : i < \omega)$ such that if $f = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, B_i : i < \omega)$

(7) in C , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(f, \theta(u, v), t)$ is bad.

Fix then $\theta(u, v)$ and t as in (6). Let E_i be the $(\delta^{\mathcal{K}_i}, \delta^{\mathcal{K}_{i+1}})$ -extender derived from ξ_i and F_i be $(\delta^{\mathcal{K}_i}, \delta^{\mathcal{S}_i})$ -extender derived from γ_i . Let $\mathcal{K}_0^+ = \mathcal{W}$, $\mathcal{S}_i^+ = \text{Ult}(\mathcal{K}_i, F_i)$ and $\mathcal{K}_{i+1}^+ = \text{Ult}(\mathcal{K}_i^+, E_i)$. Let $p_i = \sigma_{\nu_i} \circ (\pi \upharpoonright \mathcal{K}_i)$. Then we have that p_i, γ_i, ξ_i and l_i extend to $p_i^+ : \mathcal{K}_i^+ \rightarrow j(\mathcal{W}), \gamma_i^+ : \mathcal{K}_i^+ \rightarrow \mathcal{S}_i^+, \xi_i^+ : \mathcal{K}_i^+ \rightarrow \mathcal{K}_{i+1}^+$ and $l_i^+ : \mathcal{S}_i^+ \rightarrow \mathcal{K}_{i+1}^+$ such that $p_i^+ = p_{i+1}^+ \circ \xi_i^+$ and $\xi_i^+ = l_i^+ \circ \gamma_i^+$.

Using the ‘‘three dimensional argument’’ we can simultaneously iterate $(\mathcal{K}_i^+, \mathcal{S}_i^+ : i < \omega)$ using strategies $\Pi_i = (p_i^+$ -pullback of $\pi(\Pi)^h)$ and $\Omega_i = (l_i^+ \circ p_i^+$ -pullback of $\pi(\Pi)^h)$ to make $\mathbb{R}^{M[h]}$ -generic. Such genericity iterations have been used by many authors. The details of such genericity iterations are spelled out in Definition 1.35 of [6]. The outcome of this iteration is a sequence of models $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and embeddings $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ with the property that $\xi_{i,\omega} : \mathcal{K}_{i,\omega} \rightarrow \mathcal{K}_{i+1,\omega}, \gamma_{i,\omega} : \mathcal{K}_{i,\omega} \rightarrow \mathcal{S}_{i,\omega}, l_{i,\omega} : \mathcal{S}_{i,\omega} \rightarrow \mathcal{K}_{i+1,\omega}$ and for every $i < \omega$, $\xi_{i,\omega} = l_{i,\omega} \circ \xi_i$. Moreover, the iterations \mathcal{K}_i^+ -to- $\mathcal{K}_{i,\omega}$ and \mathcal{S}_i -to- $\mathcal{S}_{i,\omega}$ are above respectively $\delta^{\mathcal{K}_i}$ and $\delta^{\mathcal{S}_i}$. Let then $C_i = D(\mathcal{K}_{i,\omega})$ and $D_i = D(\mathcal{S}_{i,\omega})$. One important remark is that for every $i < \omega$, $\mathcal{K}_{i,\omega}$ is a $\bar{\Sigma}_i$ -hod premouse and $\mathcal{S}_{i,\omega}$ is a Ψ_i -premouse where $\Psi_i = (l_i$ -pullback of $\bar{\Sigma}_i)$. Another important remark is that $C_i \subseteq D_i \subseteq C_{i+1}$. The most important remark, however, is that the construction of the sequences $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ guarantees that the direct limit of $\mathcal{K}_{i,\omega}$ under $\xi_{i,\omega}$ is well-founded. Let then n be such that for every $m \geq n$, $\xi_{m,\omega}(t) = t$. It then follows from (7) and the fact that for every $i < \omega$, $C \subseteq C_i$ and $C \subseteq D_i$ that

(9) for every $i < \omega$, in C_i , for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{K}_i}]^{<\omega}$, $\mathcal{K}_i \models \phi[s, B_i]$ if and only if $\theta[\pi_{\mathcal{K}_i(\alpha), \infty}^{\bar{\Sigma}_i^h}(s), t]$ where $\alpha < \lambda^{\mathcal{K}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{K}_i}]^{<\omega}$.

(10) for every i , in D_i , there is a formula ϕ and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ such that $\mathcal{S}_i \models \phi[s, \gamma_i(B_i)]$ and $\neg \theta[\pi_{\mathcal{S}_i(\alpha), \infty}^{\Psi_i^h}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

It follows from elementarity of $\gamma_{i,\omega}$, (9) and the fact that if $i \geq n$ then $\gamma_{i,\omega}(t) = t$ that

(11) for every $i \geq n$, in D_i , for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ and $\mathcal{S}_i \models \phi[s, \gamma_i(B_i)]$ if and only if $\theta[\pi_{\mathcal{S}_i(\alpha), \infty}^{\Psi_i^h}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

Clearly (10) and (11) contradict one another. We conclude that indeed σ has A -condensation. \square

11.2 The continuation of the proof of Theorem 11.1

We can now show that there is an embedding $m : \mathcal{M}_\eta \rightarrow j(\mathcal{P})$ such that the critical point of m is $\delta^{\mathcal{P}}$. As we mentioned before, this shows that $\mathcal{P} \models \text{“}\delta^{\mathcal{P}} \text{ is regular”}$ contradicting our assumption that $\neg \#_{\Theta\text{-reg}}$ holds (see Lemma 1.1). To construct such an m , we show that iterations according to Λ are σ -realizable as witnessed by iteration embeddings according to Λ .

Suppose $(\mathcal{Q}, \vec{\mathcal{U}}) \in pI(\mathcal{P}, \Lambda)$ is such that $\Psi =_{def} \Lambda_{\mathcal{Q}, \vec{\mathcal{U}}}$ has branch condensation and $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Psi)$. Given $\mathcal{S} \in tn(\vec{\mathcal{T}})$ let $\gamma^{\mathcal{S}}$ be the sup of the generators of $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ and let $\tau_{\mathcal{S}} : \mathcal{S} \rightarrow j(\mathcal{P})$ be given by $\tau_{\mathcal{S}}(x) = j(f)(\pi_{\mathcal{S}(\alpha), \infty, j(\eta)}^{\Psi_{\mathcal{S}(\alpha)}}(s))$ where $s \in [\gamma^{\mathcal{S}}]^{<\omega}$ is such that $x = \pi^{\vec{\mathcal{T}}}(f)(s)$ and $\alpha < \lambda^{\mathcal{S}}$ is the least such that $s \in [\delta_{\alpha}^{\mathcal{S}}]^{<\omega}$. Notice that $\sigma = \tau_{\mathcal{P}}$. It follows from Lemma 11.15 that $\tau_{\mathcal{S}}$ is elementary.

Lemma 11.16. *For every $(\vec{\mathcal{T}}, \mathcal{R}) \in pI(\mathcal{Q}, \Psi)$ and $\mathcal{S} \in tn(\vec{\mathcal{T}})$ such that $\mathcal{S} \prec^{\vec{\mathcal{T}}, s} \mathcal{R}$, $\tau_{\mathcal{S}} = \tau_{\mathcal{R}} \circ \pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}$.*

Proof. Fix such a $(\vec{\mathcal{T}}, \mathcal{R})$ and \mathcal{S} . We prove this by induction. Assume that whenever $\mathcal{Q}, \mathcal{W} \in tn(\vec{\mathcal{T}})$ are such that $\mathcal{W} \neq \mathcal{R}$ and $\mathcal{Q} \prec^{\vec{\mathcal{T}}, s} \mathcal{W}$ then $\tau_{\mathcal{Q}} = \tau_{\mathcal{W}} \circ \pi_{\mathcal{Q}, \mathcal{W}}^{\vec{\mathcal{T}}}$ and that both $\tau_{\mathcal{Q}}$ and $\tau_{\mathcal{W}}$ are elementary. Suppose now $x \in \mathcal{S}$. There is then $f \in \mathcal{P}$ and $s \in [\gamma^{\mathcal{S}}]^{<\omega}$ such that $x = \pi_{\mathcal{P}, \mathcal{S}}^{\vec{\mathcal{T}}}(f)(s)$. It then follows that if α is such that $s \in [\delta_{\alpha}^{\mathcal{S}}]$ then $\tau_{\mathcal{S}}(x) = j(f)(\pi_{\mathcal{S}(\alpha), j(\eta), \infty}^{\Psi}(s))$. But $\pi_{\mathcal{S}(\alpha), \infty, j(\eta)}^{\Psi}(s) = \pi_{\mathcal{R}(\pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}(\alpha)), \infty, j(\eta)}^{\Psi}(\pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}(s))$. Because $\pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}(x) = \pi^{\vec{\mathcal{T}}}(f)(\pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}(s))$, we get that $\tau_{\mathcal{S}}(x) = \tau_{\mathcal{R}}(\pi_{\mathcal{S}, \mathcal{R}}^{\vec{\mathcal{T}}}(x))$. \square

We can then define $m : \mathcal{M}_\eta \rightarrow j(\mathcal{P})$ by $m(x) = y$ if and only if whenever $\mathcal{R} \in pI(\mathcal{Q}, \Psi)$ is such that for some $z \in \mathcal{R}$, $\pi_{\mathcal{R}, \infty, j(\eta)}^{\Psi}(z) = x$ then $y = \tau_{\mathcal{R}}(z)$. It then follows that $m : \mathcal{M}_\eta \rightarrow j(\mathcal{P})$ is elementary and that $\text{cp}(m) = \delta^{\mathcal{M}_\eta}$ which contradicts the assumption that $\neg \#_{\Theta\text{-reg}}$ holds. This then finishes the proof of Theorem 11.1.

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